1. Recall the concept of “ambient isotopy” from a previous homework. Write $D^n \subset \mathbb{R}^n$ for the closed unit disk, and $B^n$ for the open unit ball, both centred at 0. Recall that $\partial D^n = S^{n-1}$.

(a) Suppose $q$ is a point in $\mathbb{R}^n$. Verify that $H(x,t) = x - tq$ gives an ambient isotopy from $\{q\}$ to $\{0\}$, the origin, in $\mathbb{R}^n$.

(b) We may use the homeomorphism $\phi : \mathbb{R}^n \rightarrow B^n$ and its inverse $\phi^{-1}$ given by the formulas

$$\phi(x) = \frac{x}{1 + \|x\|}, \quad \phi^{-1}(y) = \frac{y}{1 - \|y\|}$$

to view $D^n$ as a compactification of $\mathbb{R}^n$. Write $p = \phi(q)$. Show that the ambient isotopy in the previous part of the question extends to an ambient isotopy $\tilde{H} : D^n \times I \rightarrow D^n$ taking $\{p\}$ to $\{0\}$, relative to $S^{n-1}$.

2. Let $D^2$ denote the closed unit disk in $\mathbb{R}^2$, that is $D^2 = \{x \in \mathbb{R}^2 | \|x\| \leq 1\}$.

(a) Prove that there does not exist a retraction for the inclusion $i : S^1 \rightarrow D^2$ (i.e., there is no map $r : D^2 \rightarrow S^1$ such that $r \circ i = \text{id}_{S^1}$).

(b) Suppose $f : D^2 \rightarrow D^2$ is a continuous map. Prove that there is some $x \in D^2$ such that $x = f(x)$. Hint: consider the rays starting at $f(x)$ and passing through $x$.

3. Consider the following three subspaces of $\mathbb{R}^3$:

- $X_1 = \{(0, 0, z) \in \mathbb{R}^3 | z \in \mathbb{R}\}$;
- $X_2 = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$;
- $X_3 = \{(0, y, z) \in \mathbb{R}^3 | (y - 1)^2 + z^2 = 1\}$.

Choose a basepoint $p \in \mathbb{R}^3 \setminus (X_1 \cup X_2 \cup X_3)$.

(a) Calculate $\pi_1(\mathbb{R}^3 \setminus (X_1 \cup X_2), p)$.

(b) Consider the picture in Figure 1. Express $[\gamma]$ in terms of $[\alpha]$ and $[\beta]$ in $\pi_1(\mathbb{R}^3 \setminus (X_1 \cup X_2), p)$. Proof-by-picture is sufficient, provided it’s clear.

(c) Calculate $\pi_1(\mathbb{R}^3 \setminus (X_2 \cup X_3), p)$.
Figure 1: Representatives of three classes in $\pi_1(R^3 \setminus (X_1 \cup X_2))$