1. By a **discrete** set in $\mathbb{R}^n$, we mean a subset $D \subset \mathbb{R}^n$ such that the subspace topology on $D$ is discrete. Throughout this question $n$ is an integer greater than 1.

(a) If $D$ is discrete in $\mathbb{R}^n$, prove $D$ is countable.

(b) Let $F$ be a countable set of points in $\mathbb{R}^n$ with the usual topology. Prove that $\mathbb{R}^n \setminus F$ is path connected.

(c) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function. Prove that there are at most two values $x \in \mathbb{R}$ such that $f^{-1}(x)$ is nonempty and countable.

(a) There is a countable base $B$ for the topology on $\mathbb{R}^n$: for instance, the set of all balls $B(q, 1/n)$ where $q \in \mathbb{Q}^n$ and $n \in \mathbb{N}$. Let $D$ be a discrete subset of $\mathbb{R}^n$. For each $d \in D$, there exists an open set $U \subset \mathbb{R}^n$ containing $d$ but no points of $D \setminus \{d\}$ (since the induced topology on $D$ is discrete). We may replace $U$ by a set from the base $B$. If we do this for all $d \in D$, we obtain an injection $D \rightarrow B$, so that $D$ must be countable.

(b) Consider two different points $p, q \in \mathbb{R}^n \setminus F$. Let $X$ denote the uncountable set of all circles in $\mathbb{R}^n$ passing through both $p$ and $q$, and let $Y \subset X$ denote the subset consisting of all circles that are not disjoint from $F$. We claim $Y$ is countable, and therefore a proper subset of $X$. For each $\Gamma \in Y$, let $r(\Gamma) \in F$ be a point that lies on $\Gamma$. Suppose $r(\Gamma) = r(\Gamma')$ for two circles, then $\Gamma$ and $\Gamma'$ are two circles meeting at 3 points, and a well-known geometry exercise shows $\Gamma = \Gamma'$. Therefore $r : Y \rightarrow F$ gives an injective function, so $Y$ is countable. In particular, there is some circle in $\mathbb{R}^n \setminus F$ passing through $p, q$. Since a circle is the surjective image of $[0, 1]$, the circle is path connected, so there is a path joining $p$ to $q$ in $\mathbb{R}^n \setminus F$, as required.

(c) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ can have at most two (global) extreme values, a maximum and a minimum. We prove that any point in $\text{im}(f)$ having a countable preimage must be an extreme value of the function. Since $\mathbb{R}^n$ is connected, the image $\text{im}(f)$ is connected, which is to say that $\text{im}(f)$ is an interval $I$. Suppose $y$ is an interior point of $I$ such that $f^{-1}(y)$ is countable. Then $I \setminus \{y\}$ is disconnected, but it is the image of the continuous function $f$ restricted to the path-connected set $\mathbb{R}^n \setminus (f^{-1}(y))$. This is a contradiction. □
2. Let \( p = (0, 1) \in \mathbb{R}^2 \) and \( q = (0, -1) \in \mathbb{R}^2 \). Let \( N = \{1, 1/2, 1/3, 1/4, \ldots \} \). Let \( X \) denote the following subset of \( \mathbb{R}^2 \):

\[
X = N \times [-1, 1] \cup \{p, q\}.
\]

(a) Determine the connected components of \( X \);

(b) Suppose \( f : X \to \{0, 1\} \) is a continuous function. Show that \( f(p) = f(q) \), even though \( \{p, q\} \) is not connected (it is called a quasicomponent of \( X \)).

(a) Write \( A_n \) for \( \{1/n\} \times [-1, 1] \). Observe that \( A_n \) is the image of \( [-1, 1] \) under a continuous map to \( \mathbb{R}^2 \), and each \( A_n \) is therefore connected. Furthermore, consider the intersection of \( (1/n - \epsilon, 1/n + \epsilon) \times [-1, 1] \) with \( X \). For sufficiently small values of \( \epsilon \), this is precisely \( A_n \), so \( A_n \) is open in \( X \). Similarly, considering \( [1/n - \epsilon, 1/n + \epsilon] \times [-1, 1] \cap X \), we see that each \( A_n \) is closed in \( X \). Therefore no proper superset of \( A_n \) can be connected, and \( A_n \) is a connected component.

Since connected components form a partition of the space, it remains to determine the decomposition of \( \{p, q\} \) into connected components, but this set is discrete and therefore disconnected. The components are \( \{p\}, \{q\} \).

(b) Any such continuous function must be constant on connected components. Suppose without loss of generality that \( f(p) = 1 \). Take the sequence \( (1/n, 1) \) in \( X \). This sequence converges to \( p \) and so \( f((1/n, 1)) \to f(p) = 1 \). Therefore for some tail of this sequence \( f((1/n, 1)) \equiv 1 \). Since \( f \) is constant on the \( A_n \), it follows that \( f((1/n, -1)) \equiv 1 \), and so \( f(q) = \lim_{n \to \infty} f((1/n, -1)) = 1 \) as well.

\[\Box\]

3. Let \( f : X \to Y \) be a function between two topological spaces. We may define the graph of \( f \), as the subset of \( X \times Y \) consisting of pairs of the form \((x, f(x))\). Denote the graph by \( \Gamma \) and write \( g : X \to \Gamma \) for the map \( g(x) = (x, f(x)) \).

(a) Let \( U \) be an open interval in \( \mathbb{R} \) and let \( f : U \to \mathbb{R} \) be a function. Prove that \( f \) is continuous if and only if the graph \( \Gamma \) is path connected. This validates the phrase “a function is continuous if you can draw its graph without taking your pencil off the paper”.

(b) Give an example, without proof, of a discontinuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) with path-connected graph.

(a) One direction is easy. If \( f \) is continuous then \( \Gamma \) is the image of \( U \) in \( \mathbb{R}^2 \) under the continuous function \( x \mapsto (x, f(x)) \), and the graph is therefore path connected, since \( U \) is.

Conversely, suppose the graph \( \Gamma \) is path connected. Let \( y \in U \) be a point. It suffices to show that \( f \) is continuous at \( y \).
There exist points $a < y$ and $b > y$ in $U$. There is a path $\gamma : I \to \Gamma$ from $(a, f(a))$ to $(b, f(b))$, since $\Gamma$ is path connected. The image $\text{im}(\gamma)$ is a compact connected subset of $\Gamma \subset \mathbb{R}^2$. The image of $\text{im}(\gamma)$ under $\text{proj}_1 : U \times \mathbb{R} \to U$ is a compact interval $[a', b']$ containing $[a, b]$. Since $\text{proj}_1$ is continuous and $\text{proj}_1|_{\Gamma}$ is injective, we see that $\text{proj}_1|_{\text{im}(\gamma)} : \text{im}(\gamma) \to [a', b']$ is a continuous bijection with compact source and Hausdorff target. It is therefore a homeomorphism. Let $q$ denote the inverse, which is continuous, and now consider $\text{proj}_2 \circ q : [a', b'] \to \mathbb{R}$. This is a composite of two continuous functions, and is therefore continuous. Explicitly, $q(x) = (x, f(x))$ and $\text{proj}_2 \circ q(x) = f(x)$. In particular, $f$ is continuous at any interior point of $[a', b']$, including $y$.

(b) There are many examples. Consider

$$f(x, y) = \begin{cases} 
2x^2 & \text{if } (x, y) \neq (0, 0) \\
\frac{2x^2}{x^2 + y^2} & \text{if } (x, y) = (0, 0)
\end{cases}$$

This is discontinuous, since the limit at $(0, 0)$ along the line $x = 0$ is 0 but along the line $y = 0$ it is 2. The graph is path-connected, however. This is elementary away from $(0, 0)$ because the graphs of $g_k(x) = f(x, k)$ and $h_k(x) = f(k, x)$ are path-connected for all $k \neq 0$, so we can join any point $(x, y, z)$ on the graph, other than $(0, 0, 0)$, to the point $(1, 1, 1)$. But $(0, 0, 0)$ on the graph of $f$ also lies on the line $(0, y, 0)$ on the graph, which is plainly path-connected.