1. Suppose $X$, $Y$ and $X \times Y$ are non-compact locally compact Hausdorff spaces$^1$. Let $X \cup \{\infty_X\}$, $Y \cup \{\infty_Y\}$ and $(X \times Y) \cup \{\infty\}$ denote the one-point compactifications of $X$, $Y$ and $X \times Y$ respectively. To keep the notation simple, we write $\hat{X}$, $\hat{Y}$ and $\hat{X} \times \hat{Y}$ for these compactifications.

(a) Prove that the inclusion $i : X \times Y \to \hat{X} \times \hat{Y}$ is a compactification.

(b) Prove that the function $f : \hat{X} \times \hat{Y} \to \hat{X} \times \hat{Y}$ given by

$$f(x, y) = (x, y) \quad \forall (x, y) \in X \times Y$$

and

$$f(\infty_X, y) = f(x, \infty_Y) = f(\infty_X, \infty_Y) = \infty \quad \forall x \in X, \forall y \in Y$$

is continuous.

(c) Prove that the spaces

$$\hat{X} \times \hat{Y}$$

and

$$\hat{X} \wedge \hat{Y} = \frac{\hat{X} \times \hat{Y}}{(\hat{X} \times \{\infty_Y\}) \cup \{(\infty_X) \times \hat{Y}\}}$$

are homeomorphic.

2. Recall $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x||_2 = 1\}$, given the subspace topology. Whenever a basepoint of $S^n$ is required in this question, use $(1, 0, \ldots, 0)$.

You may assume that there is a continuous function $f : \mathbb{R}^n \to S^n$ given by

$$f(x) = f(x_1, \ldots, x_n) = \left(\frac{||x||_2^2 - 1}{||x||_2^2 + 1}, \frac{2x_1}{||x||_2^2 + 1}, \frac{2x_2}{||x||_2^2 + 1}, \ldots, \frac{2x_n}{||x||_2^2 + 1}\right).$$

Prove this is a one-point compactification of $\mathbb{R}^n$. Hint: first construct an inverse map $g : f(\mathbb{R}^n) \to \mathbb{R}^n$. You can rely on standard facts about continuous functions and should not have to devote effort to proving $g$ is continuous. Deduce that $S^n \wedge S^m \approx S^{n+m}$ for all $n, m \in \mathbb{N}$.

3. Let $X$ and $Y$ be topological spaces and suppose $Y$ is compact. Let $f : X \times Y \to \mathbb{R}$ be a continuous function. Prove that

$$g(x) = \min_{y \in Y} \{f(x, y)\}$$

defines a continuous function. It may be helpful to begin by fixing an $\epsilon > 0$ and for any $(x, y)$, producing an open $U_{x,y,\epsilon} \ni (x, y)$ such that $f(U_{x,y,\epsilon}) \subseteq (g(x) - \epsilon, \infty)$.

Produce a continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(x) = \inf_{y \in \mathbb{R}} \{f(x, y)\}$ is a well-defined function $g : \mathbb{R} \to \mathbb{R}$, but is not continuous.

$^1$It is sufficient to assume $X$ and $Y$ have these properties.