1. Recall that a Hausdorff space $X$ is perfectly normal if, for every closed set $A$, there exists a continuous function $f_A : X \to [0, 1]$ such that $f_A^{-1}(0) = A$. In such a space, for any two disjoint closed sets $A, B$, there exists a continuous function

$$f : X \to [0, 1], \quad f = \frac{f_A}{f_A + f_B}$$

for which $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

Suppose $X$ is perfectly normal, that $C$ is a closed subset of $X$ and $g : C \to [-1, 1]$ is a continuous function.

(a) By considering $g^{-1}([-1, -1/3])$ and $g^{-1}([1/3, 1])$, construct a continuous function $h_1 : X \to [-1/3, 1/3]$ such that $|h_1(c) - g(c)| \leq 2/3$ for all $c \in C$.

(b) Produce a sequence $(h_n)$ of continuous functions $h_n : X \to [-1 + (2/3)^n, 1 - (2/3)^n]$ such that

i. $|h_n(c) - g(c)| \leq (2/3)^n$ for all $c \in C$.

ii. $|h_n(x) - h_{n-1}(x)| \leq (1/3)(2/3)^{n-1}$ for all $x \in X$.

(a) The sets $A = g^{-1}([-1, -1/3])$ and $B = g^{-1}([1/3, 1])$ are closed and disjoint, and so, since $X$ is perfectly normal, we can produce a continuous $h_1 : X \to [-1/3, 1/3]$ that takes the value $-1/3$ on $A$, the value $1/3$ on $B$ and values in-between elsewhere. We can estimate $|h_1(c) - g(c)|$ by dividing into three distinct cases. If $c \in A$, then $h_1(c) = -1/3$ and $g(c) \in [-1, -1/3]$, so $|h_1(c) - g(c)| \leq 2/3$. If $c \in B$, then $h_1(c) = 1/3$ and $g(c) \in [1/3, 1]$, so $|h_1(c) - g(c)| \leq 2/3$. Finally if $c \in C - (A \cup B)$ then $g(c) \in (-1/3, 1/3)$ and $h_1(c) \in (-1/3, 1/3)$ so $|h_1(c) - g(c)| \leq 2/3$, as required.

(b) We proceed by induction. The previous part of the question established the case of $n = 1$. Let us suppose $h_{n-1}$ has been constructed meeting the conditions for $n - 1$. Then let us consider $g_n = g - h_{n-1}$. The conditions on $h_{n-1}$ ensure that $g_n : C \to [-2(2/3)^{n-1}, (2/3)^n - 1]$. By the same argument used in the previous part, we can construct a continuous function $f_n : X \to [-2(2/3)^{n-1}(1/3), (2/3)^n - 1(1/3)]$ such that $|f_n(c) - g_n(c)| \leq (2/3)^n$ for all $c \in C$. Now set

$$h_n = h_{n-1} + f_n.$$

Finally can carry out some estimates:

- First,

$$|h_n(x)| \leq |h_{n-1}(x)| + |f_n(x)| \leq (1 - (2/3)^{n-1}) + (2/3)^n - 1(1/3) = 1 - (2/3)^n.$$

This implies that $h_n : X \to [-1 + (2/3)^n, 1 - (2/3)^n]$, as required.
• Second, if \( c \in C \), then

\[
|h_n(c) - g(c)| = |h_{n-1}(c) + f_n(c) - g_n(c) - h_{n-1}(c)| = |f_n(c) - g_n(c)| \leq (2/3)^n
\]

as required.

• Third, \( |h_n(x) - h_{n-1}(x)| = |f_n(x)| \leq (1/3)(2/3)^{n-1} \), as required.

\[\square\]

2. Suppose \((f_n : X \to \mathbb{R})\) is a sequence of functions, where \(X\) is a topological space. We say that \((f_n)\) converges to a function \(f : X \to \mathbb{R}\) uniformly if, for all \(x \in X\) and all \(\epsilon > 0\), there exists some \(N_\epsilon \in \mathbb{N}\) such that \(|f_n(x) - f(x)| < \epsilon\) for all \(n > N_\epsilon\). Suppose you are given a sequence of continuous functions \(f_n : X \to \mathbb{R}\) converging uniformly to \(f : X \to \mathbb{R}\). Prove that \(f\) is continuous.

Consider an open set \(U \subset \mathbb{R}\). We show that \(f^{-1}(U)\) is open in \(X\). To do this, suppose \(x \in f^{-1}(U)\). We produce an open set \(V \ni x\) such that \(f(V) \subset U\).

Choose \(\epsilon > 0\) sufficiently small so that \((f(x) - 3\epsilon, f(x) + 3\epsilon) \subset U\). Choose some \(n\) sufficiently large that \(|f(y) - f_n(y)| < \epsilon\) for all \(y\). Let \(V\) be the open set \(f_n^{-1}((f(x) - \epsilon, f(x) + \epsilon))\). We show \(f(V) \subset U\). For any \(y \in V\), we have

\[
|f(y) - f(x)| = |f(y) - f_n(y) + f_n(y) - f_n(x) + f_n(x) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| < \epsilon + \epsilon + \epsilon = 3\epsilon
\]

so that \(f(y) \in (f(x) - 3\epsilon, f(x) + 3\epsilon) \subset U\), as required. This proves that \(f\) is continuous. \[\square\]

3. You may use the results of the previous two problems in answering this one.

Suppose \(X\) is a perfectly normal topological space and \(C \subset X\) is a closed subset. Suppose \(g : C \to [-1, 1]\) is a continuous function. Construct a continuous function \(h_\infty : X \to [-1, 1]\) such that \(h_\infty(c) = g(c)\) for all \(c \in C\). Give an example, with proof, of a perfectly normal (e.g., metric) space \(X\), a subset \(A \subset X\), and a continuous function \(g : A \to [-1, 1]\) such that there does not exist a continuous \(h : X \to [-1, 1]\) for which \(h(a) = g(a)\) for all \(a\).

Using the results of Question 1, we know that there exists a sequence of continuous functions \(h_n : X \to [-1, 1]\) satisfying certain conditions defined there. Define a function \(h_\infty : X \to [-1, 1]\) by \(h_\infty(x) = \lim_{n \to \infty} h_n(x)\). This exists because property (ii) of Question 1 ensures \((h_n(x))\) is a Cauchy sequence:

\[
|h_n(x) - h_{n+d}(x)| \leq \sum_{i=n}^{n+d-1} \frac{2^i}{3^i} = \frac{2^{n+1} - 2^n}{3^n} \leq \frac{2}{3^n}
\]

and the upper bound can be made arbitrarily small.

2
Observe that \( h_\infty(c) = g(c) \) if \( c \in C \), since \( |h_n(c) - g(c)| \to 0 \) as \( n \to \infty \).

We claim that \( h_n \to h \) uniformly. Using the result of Question 2, this will prove that \( h \) is continuous. This is similar to the proof of the Cauchy property. For all \( x \in X \), for all \( n \in \mathbb{N} \) and all \( d \in \mathbb{N} \):

\[
|h_\infty(x) - h_n(x)| \leq |h_\infty(x) - h_{n+d}(x)| + |h_n(x) - h_{n+d}(x)| < |h_\infty(x) - h_{n+d}(x)| + \frac{2^n}{3^n}
\]

using an estimate we established earlier.

Taking the limit as \( d \to \infty \) gives us:

\[
|h_\infty(x) - h_n(x)| \leq \frac{2^n}{3^n}.
\]

The bound is independent of \( x \) and tends to 0 as \( n \to \infty \), so we see that convergence of \( h_n \to h_\infty \) is indeed uniform.

To see that the closure hypothesis is necessary, consider the function \( g : (-\infty, 0) \cup (0, \infty) \to [-1, 1] \) given by \( g(x) = -1 \) if \( x < 0 \) and \( g(x) = 1 \) if \( x > 0 \). This function is continuous, but it is not possible to define \( h(0) \) in such a way as to make \( h : \mathbb{R} \to [-1, 1] \) continuous, since \( \lim_{x \to 0^-} h(x) = -1 \) and \( \lim_{x \to 0^+} h(x) = 1 \).

The result proved in this question is called the “Tietze Extension Theorem”. In fact, one does not need the space \( X \) to be perfectly normal, merely normal. If the space is assumed to be normal, one may use a result called Urysohn’s lemma to produce the functions required. Since metric spaces are perfectly normal, however, assuming perfect normality is good enough for almost all purposes.

\( \square \)