Math 426
Homework 3
Due 3 October 2020

1. Let X be a first-countable topological space. Prove that a subset A of X is closed if it is sequentially closed. Specifically, suppose $x \in \bar{A}$. Produce a sequence of elements $a_n \in A$ such that $(a_n) \to x$.

2. A number of 'universal properties' are asserted in the class notes but not proved. The proofs of these all have a similar flavour, and it is best if you do them yourself. This exercise makes you work through one such proof in detail, and use the property to show the space enjoying the property is determined by it up to homeomorphism. Needless to say, you cannot assert the universal property of the quotient in proving these results—although for other homework exercises, doing so is fine and even encouraged.

Suppose X is a topological space and $\sim$ is an equivalence relation on X. The definition of the topological space $X/\sim$ and the quotient function $q : X \to X/\sim$ was given in the notes.

(a) Suppose that Y is some topological space, $f : X \to Y$ is a continuous function and suppose that $f(x) = f(x')$ whenever $x \sim x'$. Prove that there is a continuous function $\bar{f} : X/\sim \to Y$ such that $f = \bar{f} \circ q$ and $\bar{f}$ is uniquely determined by this equation. This is the universal property of the quotient.

(b) Now suppose that $p : X \to Z$ is some other continuous function satisfying $p(x) = p(x')$ whenever $x \sim x'$ and having the universal property of the quotient: specifically whenever $f : X \to Y$ is a continuous function such that $f(x) = f(x')$ for all $x \sim x'$, then there is a unique continuous function $g : Z \to Y$ such that $f = g \circ p$. Prove that there exists a unique continuous function $\phi : X/\sim \to Z$ such that $\phi \circ q = p$.

(c) Prove that ϕ is a homeomorphism.

3. If X is a Hausdorff topological space, we say that X is *perfectly normal* if, for every closed set A, there exists a continuous function $f_A : X \to [0,1]$ such that $f_A^{-1}(0) = A$. Prove that metric spaces are perfectly normal. Hint: it may help to define a function $X \to [0,\infty)$ by

$$\mathrm{dist}_A(x) = \inf\{d(x,a) \mid a \in A\}$$

and prove that $\mathrm{dist}_A(x)$ is continuous.