MATH 308 QUESTIONS

These are some questions which are about the level of a medium to hard final exam question. I may add to this list. Solutions to selected questions will be made available eventually.

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1. CONSTRUCTIONS & PROOFS

Make sure you know how to carry out the following constructions:

(1) Given a line segment, \(AB\), construct an equilateral triangle with base \(AB\).
(2) Bisect a given line segment
(3) Bisect a given angle
(4) Construct a triangle with given side lengths—also you should know when this is not possible.
(5) Construct a triangle with two given side lengths and a given angle between them
(6) Given a line and a point, construct a parallel line through that point
(7) Given a line, \(L\), and a point, construct a line through the point perpendicular to \(L\).
(8) Construct a square on a given line segment.
(9) Given three points \(A, B, C\), construct a circle passing through \(A, B, C\). This is the \textit{circumcircle} of the triangle \(ABC\).
(10) Given a line segment \(AB\), construct an angle of \(2\pi/5\) at \(A\)
(11) Given a circle with centre \(O\) and a point \(B\) on the circle, construct a regular pentagon having vertices on the circle, one of which is \(B\).
(12) Given a line segment \(AB\), construct a regular pentagon having \(AB\) as one of its sides.

And at least one proof of
(13) Pythagoras’ Theorem.

SOME PROBLEMS

Use any method you like to solve these problems.

(1) Suppose \(ABC\) are three points lying on a circle \(\Gamma\) with centre \(O\), in the order \(A, B, C\). Let \(D\) be a fourth point on \(\Gamma\), lying between \(B\) and \(C\). By considering the triangles \(ABO, BDO, DCO\) and \(CAO\) or otherwise, show that \(\angle BAC + \angle BDC = \pi\).
(2) Given a square $ABCD$ of side length $|AB| = 1$, construct a point $Y$ on the segment $AB$ such that

$$\frac{|AY|}{|AB|} = \frac{|YB|}{|AY|}.$$ 

Hint: Start by bisecting $AD$.

(3) Consider the lines $L$ and $M$ given by $x = 3$ and $y = -1$ in some coordinate system. Consider also the points $p = (-3, 0)$ and $q = (0, 1)$. Suppose $r$ is a point with the following properties

- The distance from $r$ to $L$ is the same as the distance from $r$ to $p$. 

**Figure 1.** A square of side length 1
• The distance from \( r \) to \( M \) is the same as the distance from \( r \) to \( q \).

What are the coordinates of \( r \)? There may be more than one answer.

(4) In Figure 2, the axes of two coordinate systems are represented. One in black, \( C_1 \), and one in blue, \( C_2 \). All coordinates in the diagram are given with respect to \( C_1 \). The point \( A \) is the origin for \( C_2 \), the points \( B \) and \( C \) lie on the \( x \) and \( y \) axes of \( C_2 \) respectively, in both cases in the positive direction from the origin.

The circle has centre (16,0) and radius 4 (in \( C_1 \)). What are the \( C_2 \) coordinates of the centre of the circle? What is the equation of the circle in coordinate system \( C_2 \)?

If a line \( L \) passes through (2, 0) and (0, –4) in the system \( C_1 \), write down an equation for \( L \) in the system \( C_2 \).

Figure 2. A line and a circle and two coordinate systems.
(5) Write down a formula for \( \cos(3\theta) \) in terms of \( \sin(\theta) \) and \( \cos(\theta) \). Suppose \( \theta \) is an angle \( 0 < \theta < \pi/2 \) such that \( 2\cos(3\theta) = 1 \). Write down a cubic equation \( ax^3 + bx + cx + d = 0 \) (with \( a \neq 0 \)) satisfied by \( x \). Hence prove that \( 0.9 < \cos(\pi/9) < 1 \). It may be helpful to know \( (0.9)^2 = 0.729 \).

(6) We know that
\[
\sin \frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{8}}, \quad \cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}.
\]
Write down \( \sin \frac{3\pi}{5} \) and \( \cos \frac{3\pi}{5} \).

Calculate \( \sin \frac{\pi}{5} \) using
\[
\sin(A - B) = \sin(A) \cos(B) - \sin(B) \cos(A).
\]

A regular pentagon is given in Figure 3. If the distance \( AB \) is 1 unit, what is the distance \( FG \)?

(7) In a convex polyhedron, the edges of some faces are equilateral triangles and the edges of the others are regular pentagons. Suppose that, at each vertex, three triangular faces and one pentagonal face meet. Write down the angle defect at each vertex. Using:
\[
V - E + F = 2 \quad \text{Euler's formula for a convex polyhedron.}
\]
\[
D = 2\pi(V - E + F) \quad \text{Descartes' formula}
\]
deduce the total number of vertices. By placing a charge of +1 at each vertex, or otherwise, deduce the number of pentagonal faces. How many edges does this polyhedron have? (Hint: count the number of triangular faces first)

(8) A regular heptagon is given, inscribed in a circle, in Figure 4. Suppose \( |AB| = 1 \). Calculate the angle \( \angle AHB \). Express the distance \( |AH| \) in terms of \( \sin(\pi/7) \). (Note: \( \sin(\pi/7) \) does not have a simple algebraic description without using complex numbers). You may refer to previous questions in your answer.

(9) Suppose \( T(\vec{x}) = A\vec{x} + \vec{c} \) is an orientation-reversing isometry (i.e. the orthogonal matrix \( A \) satisfies \( \det(A) = -1 \)).

(a) Show the eigenvalues of \( A \) are \( \pm 1 \).

(b) It is possible to write any \( \vec{v} \in \mathbb{R}^2 \) in a unique way as \( \vec{v} = \vec{v}_+ + \vec{v}_- \), where \( \vec{v}_+ \) satisfies \( A\vec{v}_+ = \vec{v}_+ \) and \( \vec{v}_- \) satisfies \( A\vec{v}_- = -\vec{v}_- \) (this is equivalent to diagonalising the matrix). Express \( T(T(\vec{0})) \) in terms of \( \vec{c}_+ \) and \( \vec{c}_- \).

(c) Show that \( T(T(\vec{x})) = \vec{x} \) for all \( \vec{x} \in \mathbb{R}^2 \) if and only if \( \vec{c}_+ = \vec{0} \).

(10) A coordinate system is fixed. Give equations for each of the following:

(a) A line passing through \((5, 3)\) parallel to the line passing through \((2, -4)\) and \((-1, 8)\).

(b) A circle passing through \((5, 7), (9, 13)\) and \((8, 12)\)
Figure 3. A regular pentagon.
Figure 4. A regular heptagon of side-length 1