Mathematics 308, Euclidean Geometry, Sample Final Exam
The University of British Columbia, 2015/16, Winter Session, Term 1.

There are ten (10) questions, they carry equal weight. Answer all questions. Some questions take up two page. The exam should be 14 pages long, and conclude with the symbol ‘.oOo.’. Please check that this is the case before attempting any questions.

Show all your work.

No notes, calculators or textbooks are permitted.

Last Name: First Name:  
Student Number: Signature:  

Rules Governing Examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner or invigilator, and may be subject to disciplinary action:
  (1) speaking or communicating with other candidates, unless otherwise authorized;
  (2) purposely exposing written papers to the view of other candidates or imaging devices;
  (3) purposely viewing the written papers of other candidates;
  (4) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (5) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Examiners’ use only

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(1) A line segment $AB$ is given, along with a point $C$ on it. Construct an isosceles triangle with one side of length $BC$ and two sides of length $AC$ (for this question, you may use a compass that retains its shape even when lifted from the page).
(2) Fix a coordinate system on the plane. Let $s$ and $t$ be the points $(-1, 0)$ and $(2, 0)$.

(a) Give an equation for the set $X$ of all points $r = (x, y)$ such that $\text{dist}(s, r) = 2\text{dist}(t, r)$. Simplify your answer as far as possible.

(b) Give an equation for the line $L$ passing through $(5, 0)$ and $(2, -1)$.  

Continued Overleaf
(c) What, if any, are the coordinates of the point(s) where $L$ and $X$ intersect?
(3) In the following diagram, both circles have centre $A$. The distance $AB$ is 1 and the distance $AD$ is $t < 1$. The point $G$ is the midpoint of $AD$. What is the distance $EF$ (the answer should depend on $t$)?
Two different coordinate systems on the plane are depicted. In coordinate system $C_1$, the points $A$, $B$ and $C$ have coordinates $(13, 0)$, $(18, 12)$ and $(1, 5)$, in that order. The distances $|AD|$ and $|AE|$ are 2 and 1 respectively. The point $D$ is the centre of the circle $e$. In the second coordinate system, $C_2$, the point $A$ is the origin, $B$ lies on the positive $x$ axis and $C$ lies on the positive $y$ axis.

**Figure 1.** A circle and two coordinate systems.

(a) Write down a formula for $T_{2\rightarrow 1}$, the change of coordinates $C_2$ to $C_1$. That is, if $(x, y)_2$ are the $C_2$ coordinates of some point, then $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ should be the $C_1$ coordinates of the same point.
(b) Write down an equation for the circle \( e \) in coordinate system \( C_2 \).

(c) Write down an equation for the circle \( e \) in coordinate system \( C_1 \).
(5) Construct, using a straightedge and compass, the angle $\pi/5$. Indicate what steps you followed, but you do not have to prove your construction is correct.
(6) (a) Suppose $0 < \theta < \pi/2$. Prove the half-angle cosine formula

$$\cos^2(\theta/2) = \frac{1 + \cos \theta}{2}.$$ 

(b) Calculate $\sin(9\pi/8)$. You may assume $\sin(\pi/4) = 1/\sqrt{2}$. 
(7) Suppose a triangle $T$ has sides of length 8, 7 and 4.

(a) Find $\cos \theta$ where $\theta$ is the largest internal angle of $T$

(b) What is the area of $T$? You do not have to simplify your answer.
(8) Let \( \vec{p} \) be the point with coordinates (in vector notation) \[
\begin{bmatrix}
2 \\
-1
\end{bmatrix}.
\]
Give the coordinates of the point:

(a) Obtained by rotating \( \vec{p} \) around the origin by \( \pi/4 \) clockwise.

(b) Obtained by reflecting \( \vec{p} \) across the line passing through the origin and \[
\begin{bmatrix}
\sqrt{3} \\
1
\end{bmatrix}.
\]
(c) Obtained by rotating \( \vec{p} \) around the point \( \left[ \begin{array}{c} 3 \\ -3 \end{array} \right] \) by \( \pi/4 \) clockwise
(9) A regular \textit{rhombicuboctahedron} is a convex polyhedron. It has the property that the boundary of each face is either an equilateral triangle or a square. At each vertex, 3 square faces and 1 triangular face meet.

(a) What is the defect at each vertex?

(b) How many vertices does a regular rhombicuboctahedron have?

(c) There are 8 triangular faces—you may assume this without proof. By placing a charge of +3 at each vertex, or otherwise, count the number of square faces.
(10) Suppose $A, B$ are orthogonal $2 \times 2$ matrices such that for any $\vec{v}$, we have $A\vec{v} \cdot B\vec{v} = 0$. Prove that there exists a number $\lambda \in \mathbb{R}$ such that $\lambda(A + B)$ is an orthogonal matrix. (Hint: remember one can write $\vec{u} \cdot \vec{w} = \vec{u}^T \vec{w}$).