MATH 308 MIDTERM

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There are 5 questions, they carry equal weight. Answer all questions. This exam should be 8 pages long, and conclude with the symbol ‘.oOo.’ Please check that this is the case.

Show all your work.

No notes, calculators or textbooks are permitted.

Name (print):

Student ID number:

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Total
(1) Three points $A, B, C$ are given. Using a straightedge and compass, construct a triangle $ABD$ such that $|AD| = |AC|$ and such that the angle $\angle DAB$ is a right angle. No points will be given if it is not clear what steps you followed.

The following diagram should make a construction clear. There are other constructions, but this is the easiest. The large circles have centres $E$ and $F$. The small circle has centre $A$. 
(2) In the given diagram, 6 lines and a circle are given. The circle \( c \) has radius \( \frac{1}{2} \) and passes through \( B = (0, 0) \) and \( A = (0, 1) \). The point \( D \) has coordinates \((t,1)\), where \( t > 0 \). The lines \( AD \) and \( EF \) are both parallel to the \( x \)-axis, and \( DF \) is parallel to the \( y \)-axis.

(a) Write down an equation for the line \( L \) passing through \( B, D \) and \( E \).

\[
t y - x = 0
\]

(b) Write down an equation for the circle \( c \) passing through \( A \) and \( B \) with radius \( \frac{1}{2} \).

\[
x^2 + (y - 1/2)^2 = 1/4, \text{ or, expanding out, } x^2 + y^2 - y = 0.
\]
(c) Give the coordinates of the point $F$. The answer will depend on $t$.

First we find $E$ the point of intersection of the line $L$ and the circle. Since this point lies on $L$, its coordinates satisfy $ty = x$. Then, since it lies on the circle, its coordinates satisfy $(ty)^2 + y^2 - y = 0$. We are interested in the point with nonzero $y$ coordinate, so we may divide by $y$, obtaining $(t^2 + 1)y = 1$, or $t = \frac{1}{t^2 + 1}$. The $y$ coordinates of the points $E$ and $F$ are equal. The $x$ coordinates of the points $F$ and $(t, 1)$ are equal. Therefore

$$F = \left(t, \frac{1}{t^2 + 1}\right)$$
(3) Suppose two coordinate systems are given: $C_1$ and $C_2$.

Let $p_2$ denote the origin in $C_2$, and let $L_2, M_2$ denote the $x$-axis and $y$-axis for $C_2$ respectively. We give the coordinates

$$p_2 = \begin{bmatrix} 13 \\ 26 \end{bmatrix}$$

Moreover, $L_2$ and $M_2$ are described in $C_1$ by

$$\begin{bmatrix} x \\ y \end{bmatrix} = p_2 + t \begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = p_2 + t \begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

respectively.

(a) Describe all points on $L_2$ and $M_2$ having distance 1 from $p_2$.

Points on $L$ having distance 1 from $p_2$ are points of the form $p_2 + t \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ where

$$\left\| \begin{bmatrix} 5 \\ 12 \end{bmatrix} \right\| = 1.$$ A simple calculation shows that this is the case when $t = \pm 1/13$. The points are therefore

$$\begin{bmatrix} 13 \\ 26 \end{bmatrix} + \frac{1}{13} \begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 26 \end{bmatrix} - \frac{1}{13} \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$
Note: This much suffices to do the rest of the question, you do not have to simplify further. Similarly, the points on $M$ are the points
\[
\begin{pmatrix} 13 \\ 26 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 12 \\ -5 \end{pmatrix}, \quad \begin{pmatrix} 13 \\ 26 \end{pmatrix} - \frac{1}{13} \begin{pmatrix} 12 \\ -5 \end{pmatrix}.
\]

(b) How many coordinate systems $C_2$ are there that meet the conditions outlined above?

There are 2 choices for the point $(1,0)$ in $C_2$ and 2 choices for the point $(0,1)$. Therefore there are 4 choices in all. Therefore there are 4 such coordinate systems.

(c) Choose one particular coordinate system $C_2$ such that all the conditions in the question are satisfied. Give an explicit (matrix-and-vector) description of the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ that converts $C_2$-coordinates to $C_1$-coordinates.

We choose the vectors $\begin{pmatrix} 13 \\ 26 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 13 \\ 26 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 5 \\ 12 \end{pmatrix}$ to be $(1,0)$ and $(0,1)$ in $C_2$, respectively. The other choices lead to similar, but different, answers.

With these choices, we can write down
\[
T \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 13 \\ 26 \end{pmatrix}.
\]

Note: At this point, it is wise to verify that $T$ translates $(0,0)$ in $C_2$ coordinates into $(13,26)$ in $C_1$ coordinates (so the point called $(0,0)$ in $C_2$ is called $(13,26)$ in $C_1$).
(d) Give an explicit description of the function $S : \mathbb{R}^2 \to \mathbb{R}^2$ that converts $C_1$-coordinates to $C_2$-coordinates

The inverse of $T$ can be calculated explicitly (as was done in the class notes). If $T(\vec{x}) = A\vec{x} + \vec{c}$, then $S(\vec{x}) = A^T\vec{x} - A\vec{c}$ (exploiting the fact that $A^T = A^{-1}$).

$$S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \begin{pmatrix} 13 \\ 26 \end{pmatrix} = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 29 \\ 2 \end{pmatrix}.$$  

**Note:** Again, it is wise to check some points. For example, $S$ sends $(13, 26)$ (the origin of $C_2$ in $C_1$-coordinates) to $(0, 0)$. 
(4) In the following diagram, the distances $|AB|$ and $|AD|$ are both 1 unit. The line $AB$ is perpendicular to $AD$, and a line parallel to $AB$ is drawn through $D$. Using only a straightedge and compass, construct a line segment of length $\sqrt{6}$. Explain your construction.

The following diagram should explain one construction. There are many others.
(5) A diagram is given consisting of an isosceles triangle \( ABC \) and some marked points and line-segments. The angle \( \angle AEC \) is a right angle. Suppose the distance \(|AB| = |AC|\) is \( b \) and the distances \(|CD|, |DB|\) are \( f \) and \( g \) respectively. Express the distance \( e = |AD| \) in terms of \( b, f \) and \( g \).

First observe that \( f + |ED| = g - |ED| \), so

\[
|ED| = \frac{g - f}{2}.
\]

Second \( 2|EB| = |CB| = g + f \), so

\[
|EB| = \frac{g + f}{2}.
\]

Now apply Pythagoras’ theorem to \( AEB \) to get

\[
|AE|^2 = b^2 - \left( \frac{g + f}{2} \right)^2
\]

and apply Pythagoras’ theorem to \( AED \) to get

\[
e^2 = |AE|^2 + \left( \frac{g - f}{2} \right)^2 = b^2 - \left( \frac{g + f}{2} \right)^2 + \left( \frac{g - f}{2} \right)^2 = b^2 - \frac{1}{4}(g^2 + 2fg + f^2) + \frac{1}{4}(g^2 - 2fg + f^2) = b^2 - fg.
\]

.oOo.