1. Fix a coordinate system for the Euclidean plane. Any isometric transformation of the plane may be written as
\[ T(\vec{x}) = A\vec{x} + \vec{c} \]
where \( A \) is an orthogonal matrix and \( \vec{c} \) is a fixed vector. We say \( T \) is orientation preserving if any of the following conditions are satisfied:

- \( \det(A) = 1 \),
- \( A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \) for some angle \( \theta \),
- \( A \in SO_2 \).

We say \( T \) is a translation if \( A = I_2 \) (this agrees with a definition given in class).

(a) Suppose \( T \) is orientation preserving but not a translation. Show that 1 is not an eigenvalue of the matrix \( A \).

(b) Suppose \( T \) is orientation preserving but not a translation. Show there is a unique \( \vec{p} \in \mathbb{R}^2 \) such that \( T(\vec{p}) = \vec{p} \) (such an \( \vec{p} \) is called a fixed vector).

(c) Suppose \( T \) is orientation preserving but not a translation. Let \( \vec{p} \) denote the fixed vector from the previous part. Show that \( T(\vec{x}) = A(\vec{x} - \vec{p}) + \vec{p} \). Describe the transformation \( T \) geometrically.

(d) Suppose \( T \) is not orientation preserving. Show that \( T \) has either infinitely many fixed vectors, or none. Hint: find the eigenvalues of \( A \).

2. By constructing the angle \( 2\pi/5 \) first, construct the angle \( 2\pi/15 \) using only a straightedge and compass.

3. Let \( A, B, C \) be the vertices of a triangle, and let \( \Gamma \) be the unique circle passing through \( A, B, C \). We have seen \( \Gamma \) several times in this course, it is called the circumcircle of the triangle. Let \( O \) denote the centre of \( \Gamma \) (the circumcentre of the triangle). Draw a diameter of the circle from \( C \) through \( O \) to a point \( D \) also on \( \Gamma \). Use the Law of Sines to show that
\[ |CD| = \frac{|BC|}{\sin|\angle BDC|}. \]

Hence, calculate \( |CD| \), the diameter of the circumcircle, in terms of \( |BC| \) and an angle of the triangle \( ABC \).
4. A regular pentagon is given inscribed in a circle. The circle has centre $O$. The distances $|AO|, |BO|, |CO|$ are all 1.

(a) We know

$$\sin \frac{2\pi}{5} = \sqrt{\frac{5 + \sqrt{5}}{8}}, \quad \cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}. \quad (1)$$

Calculate $\sin(3\pi/5)$ and $\cos(3\pi/5)$, giving your answers in the same kind of exact form as in (1).

(b) Calculate the side length $|AB|$ (again, give your answer in an exact form, do not use decimals).

(c) Calculate the area of the triangle $|ABC|$. Give your answer in an exact form and also give a decimal approximation.