In solving these problems, you may use any proposition in the Elements. Whenever I think you might need a proposition that we have not mentioned in class, I will say so. Please show, in your answers, which propositions from Euclid you are using, if any.

1. Suppose you are given a line segment of length $\ell$. Explain how to use a straightedge & compass to construct a line segment of length $\ell/3$.

2. Let $p, q$ be distinct points in the plane with coordinates $(x_0, y_0)$ and $(x_1, y_1)$. As usual, write $\vec{p}$ for the vector starting at $(0,0)$ and ending at $p$, and similarly for other points. Prove that if $r$ is some point with coordinates $(x, y)$, then

$$\vec{r} = \vec{p} + t(\vec{q} - \vec{p})$$

if and only if

$$(x-x_0)(y_1-y_0) - (y-y_0)(x_1-x_0) = 0.$$

Show that $r = p$ and $r = q$ satisfy these equations. If $L$ is the line passing through $p, q$, we call $L$ the vector equation of $L$.

Added in version 2: a typographic error has been fixed here.

3. Let $v$ be a vector. We call $v$ a unit vector if $\|v\| = 1$. Suppose $u$ and $v$ are unit vectors. Show that $(u + v) \cdot (u - v) = 0$.

4. Suppose $(1,1)$ and $(4,5)$ are the coordinates of two vertices of a square. What are the coordinates of the other two vertices? (give all possible answers)

Added in version 2: a typographic error was fixed in this question. Also, one of the answers requires solving an unpleasant quadratic equation for $x$. We will accept answers that consist of a system of equations such that $(x, y)$ are the coordinates of a vertex if and only if $(x, y)$ is a solution to the system. That is, once you’ve found the equations, you do not need to solve them.

5. Let $ABC$ be a right-angled triangle, with right angle at $A$. As in the proof of Proposition I.47, draw a line $L$ from $A$ meeting $BC$ at right angles. Call the point where $L$ and $BC$ meet $D$. Prove that $ABC$ and $ABD$ are similar triangles. Use the equation

$$\frac{|AB|}{|BD|} = \frac{|BC|}{|AB|}$$

(and the similar equation for $ABC$ and $ACD$) to obtain another proof of Pythagoras’ theorem.

6. Let $ABC$ be a triangle. Let $E, F, G$ be points lying on the line segments $BC, CA$ and $AB$ respectively in such a way that the lines $AE, BF$ and $CG$ meet at a single point, $D$ (we say that the lines are coincident).

(a) Draw a diagram of the situation described above.
(b) Show that
\[
\frac{|AF|}{|FC|} = \frac{\text{area}(ADF)}{\text{area}(FDC)} = \frac{\text{area}(AFB)}{\text{area}(FBC)} = \frac{\text{area}(ADB)}{\text{area}(DBC)}.
\]
(c) Using the above, and the similar results for the other three sides, calculate
\[
\frac{|AF|}{|FC|} \cdot \frac{|CE|}{|EB|} \cdot \frac{|BG|}{|GA|}.
\]
7. Let \(ABCD\) be a square of side-length 1. Let \(P\) be a point on \(AB\). Extend \(BC\) so that the line \(DP\) meets the line \(BC\) at \(E\). Suppose the distance \(PB\) is \(1/5\). What is the distance \(DE\)?