On Transitive Actions and Fibre Bundles

Toni Annala

1 Introduction

It is elementary to show that for a topological group $G$ and a transitive continuous action $\mu : G \times X \to X$ the function $\pi : G \to X$, defined by sending $g \in G$ to $gx_0$ for some $x_0 \in X$, has homeomorphic fibres, i.e., for any $x, x' \in X$ the spaces $\pi^{-1}\{x\}$ and $\pi^{-1}\{x'\}$ are homeomorphic. The following question then naturally arises: when does $\pi$ give $G$ the structure of a fibre bundle over $X$? We show that the above question is essentially the same as whether or not $\pi$ has local sections at $x_0$. Finally, we show that the existence of local sections is fairly easy to characterize, at least when $G$ is a Lie group smoothly acting on a smooth manifold $X$. This gives a complete answer for our question in the case of smooth actions of Lie groups and smooth vector bundles.

2 Prerequisites

2.1 Local Sections and Fibre Bundles

A local section of a continuous map $f : X \to Y$ is a continuous map $s : U \to X$, where $U$ is an open set of $Y$, such that $f \circ g$ is the identity function of $U$. A local section at $y \in Y$ is a local section $s : U \to X$ where $y \in U$.

Let $f : X \to Y$ be a continuous map of topological spaces and $F$ any nonempty topological space. The map $f$ is called an $F$-fibre bundle, if for every $y \in Y$, we have an open neighbourhood $U$ of $y$ and a homeomorphism $f^{-1}U$ making the diagram (*)

\[
\begin{array}{ccc}
  f^{-1}U & \xrightarrow{\sim} & U \times F \\
  f & \downarrow & p_1 \\
  U & \rightarrow & \\
\end{array}
\]
commute, where the map $p_1$ is just the canonical projection. The map $f$ (or even the space $X$ itself, when the structure morphism $f$ is clear from the context) is called a fibre bundle if $f$ is a $F$-fibre bundle for some topological space $F$.

The following lemma relates the two concepts. Denote by $F$ the fibre of $\pi$ over $x_0$.

**Lemma 2.1.** The map $\pi$ as defined in the introduction makes $G$ an $F$-fibre bundle over $X$ exactly when $\pi$ has a local section at $x_0$.

**Proof.** Assume first that $\pi$ is a fibre bundle. Let $U$ be an open neighbourhood of $x_0$, and $\psi : f^{-1}U \rightarrow U \times F$ a homeomorphism making the diagram (*) commute. Now for $F_0 \in F$, we may define a continuous map $U \rightarrow U \times F$ by $u \mapsto (u, F_0)$ and composing it with $\psi^{-1}$ gives a continuous map $s : U \rightarrow X$. From the fact that the diagram (*) commutes it then follows that $f \circ s$ is the identity function, so $s$ is a local section at $x_0$.

Assume then that there is a local section at $x_0$. The lemma following this one shows that $\pi$ has local sections at every point of $X$. Let $x \in X$ be arbitrary and $s : U \rightarrow G$ a local section at $x$. Define a map $\psi : f^{-1}U \rightarrow U \times F$ by sending $g$ to $(f(g), (s \circ f)(g)^{-1}g)$. The map is well-defined as $(s \circ f)(g)^{-1}g$ is a member of $F$:

$$f((s \circ f)(g)^{-1}g) = s(gx_0)^{-1}gx_0 = x_0$$

where the latter equation follows from the fact that $gx_0 = (f \circ s)(gx_0) = s(gx_0)x_0$. The map $\psi$ is a continuous, and it does make the diagram (*) commute, so the only thing left to show is that $\psi$ is a homeomorphism.

Define a continuous map $\phi : U \times F \rightarrow f^{-1}U$ by sending $(u, g)$ to $s(u)g$. This map is well defined: as $\pi$ is clearly a map of $G$-sets, it follows that $f(s(u)g) = s(u)f(g) = s(u)x_0 = u$. The map $\phi$ is also the inverse of $\psi$ as

$$\phi(\psi(g)) = \phi(f(g), s(f(g))^{-1}g) = s(f(g))s(f(g))^{-1}g = g$$

and

$$\psi(\phi(u, g)) = \psi(s(u)g) = (f(s(u)g), s(f(s(u)g))^{-1}s(u)g)
= (u, s(u)^{-1}s(u)g) = (u, g).$$

Thus $\psi$ is a homeomorphism, which concludes our proof.

The above proof used the following lemma.

**Lemma 2.2.** If $\pi$ has local section at $x_0$, then it has local sections at every point of $X$. 2
Proof. Denote by \( s \) the local section \( s : U \to X \) at \( x_0 \) whose existence was assumed, and let \( x \in X \) be arbitrary. Because the action of \( G \) on \( X \) is transitive, we can find an element \( g \) of \( G \) which sends \( x_0 \) to \( x \). Define a map \( s' : gU \to X \) with formula \( s'(u) = gs(g^{-1}u) \). The map \( s' \) is clearly continuous. Furthermore, it is a local section of \( \pi \) as

\[
\pi(s'(u)) = \pi(gs(g^{-1}u)) = g\pi(s(g^{-1}u)) = g^{-1}u = u.
\]

This concludes the proof.

\( \square \)

2.2 Some Vector Analysis

It is a well known fact from elementary analysis that a smooth function \( f : U \to V \) between open sets \( U \) and \( V \) of \( \mathbb{R}^n \) has a local inverse at \( x \in U \) if and only if the derivative matrix

\[
Df(x) = \begin{pmatrix}
\partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\
\vdots & \ddots & \vdots \\
\partial_1 f_n(x) & \cdots & \partial_n f_n(x)
\end{pmatrix}
\]

is invertible. This is called the inverse function theorem. As an application of this, we obtain the following lemma:

Lemma 2.3. Let \( U \) and \( V \) be open sets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively, and \( f : U \to V \) a smooth function. Now there exists a smooth local section \( s \) of \( f \) at \( y_0 \in V \) if and only if the rank of \( Df(x_0) \) is maximal for some \( x_0 \) in the fibre of \( y_0 \), i.e., \( Df(x_0) \) defines a surjective linear map.

Proof. If there is a local section \( s \) at \( y_0 \), then as \( f \circ s = \text{id} \), we get that

\[
I = D(f \circ s)(y_0) = Df(s(y_0))Ds(y_0),
\]

and hence \( Df(s(y_0)) \) must have maximal rank.

Assume then that \( Df(x_0) \) has maximal rank. Now from elementary linear algebra it follows that there must be \( m \) columns of \( Df(x_0) \) that are linearly independent. By permuting the coordinates of \( \mathbb{R}^n \) if necessary, we may assume that these are the first \( m \) columns. Define a function \( \tilde{f} : U \to V \times \mathbb{R}^{n-m} \) by sending \( x = (x_1, \ldots, x_n) \) to \((f(x), x_{m+1}, \ldots, x_n)\). The derivative matrix

\[
Df(x) = \begin{pmatrix}
\partial_1 f_1(x) & \cdots & \partial_m f_1(x) & \partial_{m+1} f_1(x) & \cdots & \partial_n f_1(x) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\partial_1 f_m(x) & \cdots & \partial_m f_m(x) & \partial_{m+1} f_m(x) & \cdots & \partial_n f_m(x) \\
0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}
\]
is clearly invertible, so there exists a local smooth inverse function \( h : V' \times W \to U' \) of \( \bar{f} \), where \( U' \) and \( V' \) are open subsets of \( U \) and \( V \) respectively, and \( W \) is an open set of \( \mathbb{R}^{n-m} \). Let \( w \in W \). Now we can define a continuous map \( s : V \to U' \) by sending \( v \) to \( h(v, w) \). This is a local section of \( f \). In order to see this, recall that \( f = p_1 \circ \bar{f} \), and thus

\[
(f \circ s)(v) = p_1(\bar{f}(h(v, w))) = p_1(v, w) = v.
\]

As \( h \) is smooth, it is clear that also \( s \) is.

### 3 G as a Smooth Fibre Bundle

We are ready to prove the main theorem. Assume that \( G \) is a Lie group and that the transitive action on \( X \) is smooth. Defining \( \pi \) as before makes it a smooth map.

**Theorem 3.1.** The map \( \pi \) is a smooth fibre bundle if and only if the induced map on tangent spaces \( T_eG \to T_{x_0}X \) is surjective.

**Proof.** It is easy to show that \( \pi \) is a smooth fibre bundle if and only if there is a smooth local section of \( \pi \) at \( x_0 \). Essentially the same proof holds as in the continuous case, just replace the word "continuous" with the word "smooth" everywhere.

We already know that \( \pi \) has local smooth section at \( x_0 \) if and only if the tangent map \( T_{g_0}G \to T_{x_0}X \) is surjective for some \( g_0 \) in the fibre of \( x_0 \). The problem is that \( g_0 \) is not necessarily \( e \). In any case, if \( T_{g_0}G \to T_{x_0}X \) is surjective for some \( g_0 \), then \( \pi : G \to X \) is a fibre bundle, so finding a local section of \( \pi \) at \( x_0 \) sending \( x_0 \) to \( e \) is trivial. Hence the tangent map \( T_eG \to T_{x_0}X \) is surjective as well, and the theorem follows.

This condition is fairly straightforward to verify, at least in the cases where a closed subgroup of \( GL(n, \mathbb{R}^n) \) acts on a submanifold of \( \mathbb{R}^n \) with its natural action.