Asymptotes

November 7, 2013

Limits at infinity: If a function $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write $\lim_x \to \infty f(x) = L$. In this case the line $y = L$ is a horizontal asymptote of f. The limit at $-\infty$, $\lim_{x\to-\infty} f(x) = M$ is defined analogously, and in this case the horizontal asymptote is $y = M$. The behaviour of f as $x \to \infty$ is called "End behaviour".

Polynomials: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$. Then

- $\lim_{x\to\pm\infty} x^n = \infty$, if n is even
- $\lim_{x\to\infty} x^n = \infty,\ \lim_{x\to-\infty} = -\infty$, if n is odd
- $\lim_{x\to\pm\infty} 1/x^n = 0$
- $\lim_{x\to\pm\infty} p(x) = \infty$ or $-\infty$ depending on the degree of the polynomial and the sign of the leading coefficient a_n.

Rational functions: Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots a_1x + a_0,\ \ q(x) = b_n(x^n) + b_{n-1}x^{n-1} + \cdots + b_1x + b_0,\ \ a_m \neq 0,\ \ b_n \neq 0$. Then

- If $m < n$, then $\lim_{x\to\pm\infty} f(x) = 0$ and $y = 0$ is a horizontal asymptote of f.
- If $m = n$, then $\lim_{x\to\pm\infty} f(x) = a_m/b_n$, and $y = a_m/b_n$ is a horizontal asymptote of f.
- If $m > n$, then $\lim_{x\to\pm\infty} = \infty$ of $-\infty$ and f has no horizontal asymptote.
- Assuming that $f(x)$ is in reduced form (that is, $p(x)$ and $(q(x)$ share no common factors), vertical asymptotes occur at the zeros of q.

For Slant asymptotes, see the earlier notes, this will require carrying out long division.

Infinite limits at infinity: If $f(x)$ becomes arbitrarily large as $x \to \infty$, then we write $\lim_{x\to\infty} f(x) = -\infty$.