Assignment 8

1. **(i)** Suppose f is a function defined on an interval I. If $f'(x)$ is positive for all x in I, what can you conclude about the behaviour of f on I?

   Answer. f is strictly increasing on I.

   (ii) Let f be defined on an interval I with an interior point c. If f has a local maximum at c, which of the following statements are true? (a) $f'(c) > 0$; (b) $f'(c) < 0$; (c) $f'(c) = 0$; (d) None of the above; (e) c is a critical point.

   If f has a local maximum at c and $f'(c)$ exists then what can you conclude about $f'(c)$?

   State whether the following is true or false. If c is a critical point of f, then f must have a local maximum or local minimum at c. Justify your answer.

   Answer. (d), (e). We choose (d) because f may not be differentiable at c, thus (a) (b) (c) are incorrect. (e) is correct by definition.

   We conclude that $f'(c) = 0$, since c is assumed to be an interior point.

   False. Counterexample: take $f(x) = x^3$ and look at $x = 0$.

   (iii) If $f''(x) > 0$ on an interval, what can you conclude about $f'(x)$ in that interval?

   Answer. We conclude that $f'(x)$ is strictly increasing, and f is concave up on that interval.

2. Consider the function

$$f(x) = \frac{x^3 + x^2 - 2x - 3}{x^2 - 3}.$$

Its first and second derivatives are given by

$$f'(x) = \frac{(x^2-1)(x^2-6)}{(x^2-3)^2}, \quad f''(x) = \frac{2x(x^2+9)}{(x^2-3)}.$$

(a) Find all x where $f(x)$ is defined and such that $f'(x) = 0$ or $f'(x)$ does not exist.

Solution. $f'(x) = 0$ precisely when $x = \pm 1$ or $x = \pm\sqrt{6}$. $f'(x)$ does not exist when $x = \pm\sqrt{3}$. However, at these two points, $f(x)$ is not defined. Therefore we shall **not** consider these two points.

(b) Find all x where $f(x)$ is defined and such that $f''(x) = 0$ or $f''(x)$ does not exist.

Solution. $f''(x) = 0$ precisely when $x = 0$. (Notice that $x^2 + 9$ is always positive.) $f''(x)$ does not exist when $x = \pm\sqrt{3}$. At these two points, $f(x)$ is not defined. Therefore again we shall **not** consider these two points.

(c) On which intervals is $f(x)$ increasing? On which intervals is $f(x)$ decreasing?

Solution. For the first part, it suffices to identify the intervals on which $f'(x)$ is non-negative. This is easy; these intervals are: $(-\infty, -\sqrt{6}\,]$, $[-1, 1]$, $[\sqrt{6}, \infty)$. So on *each* of these intervals, f is increasing.

For the second part, one needs to identify the intervals on which $f'(x)$ is non-positive. These intervals are: $[-\sqrt{6}, -\sqrt{3}\,)$, $(-\sqrt{3}, -1]$, $[1, \sqrt{3}\,)$, $(\sqrt{3}, \sqrt{6}\,]$. Therefore on *each* of these intervals, f is decreasing.

Warning. It is **incorrect** to say the following:

$$f \text{ is decreasing on } [-\sqrt{6}, -\sqrt{3}\,) \cup (-\sqrt{3}, -1] \cup [1, \sqrt{3}\,) \cup (\sqrt{3}, \sqrt{6}\,].$$

To see this, just notice that, for example $f(1.5) < f(2)$, and both 1.5 and 2 lie in the above union of intervals. This can also be seen by looking at the graph of f (see the last page).

It is indeed true that f is increasing on $(-\infty, -\sqrt{6}] \cup [-1, 1] \cup [\sqrt{6}, \infty)$. However, this is purely by coincidence.

Therefore, in exams, after you find the intervals, you should only say: f is increasing/decreasing on each of the intervals. **Never** say that f is increasing/decreasing on the *union* of the intervals.

(d) On which intervals is $f(x)$ concave up? On which intervals is $f(x)$ concave down?

Solution. $f(x)$ concaves up when $f''(x) > 0$, and $f''(x) > 0$ on $[0, \sqrt{3}\,)$, and $(\sqrt{3}, \infty)$. So on *each* of these two intervals, $f(x)$ is concave up.

Similarly $f(x)$ concaves down on each of the following intervals: $(-\infty, -\sqrt{3}\,)$, $(\sqrt{3}, 0]$.

The warning for part (c) still stands here. You *cannot* that f is concave up on $[0, \sqrt{3}\,) \cup (\sqrt{3}, \infty)$.

(e) Find the coordinates of all local extrema and the inflection points. Be sure to indicate which is which.

Solution. Local maximum attained at: $x = -\sqrt{6}$, $x = 1$.

Local minimum attained at: $x = -1$, $x = \sqrt{6}$.

Inflection point (where $f''(x)$ changes sign): $x = 0$.

(f) Find any asymptotes of the function $f(x)$ and write their equations.

Solution. Two vertical asymptotes are easily observed: $x = \pm\sqrt{3}$.

Next we find the slanted asymptotes. Take limit:

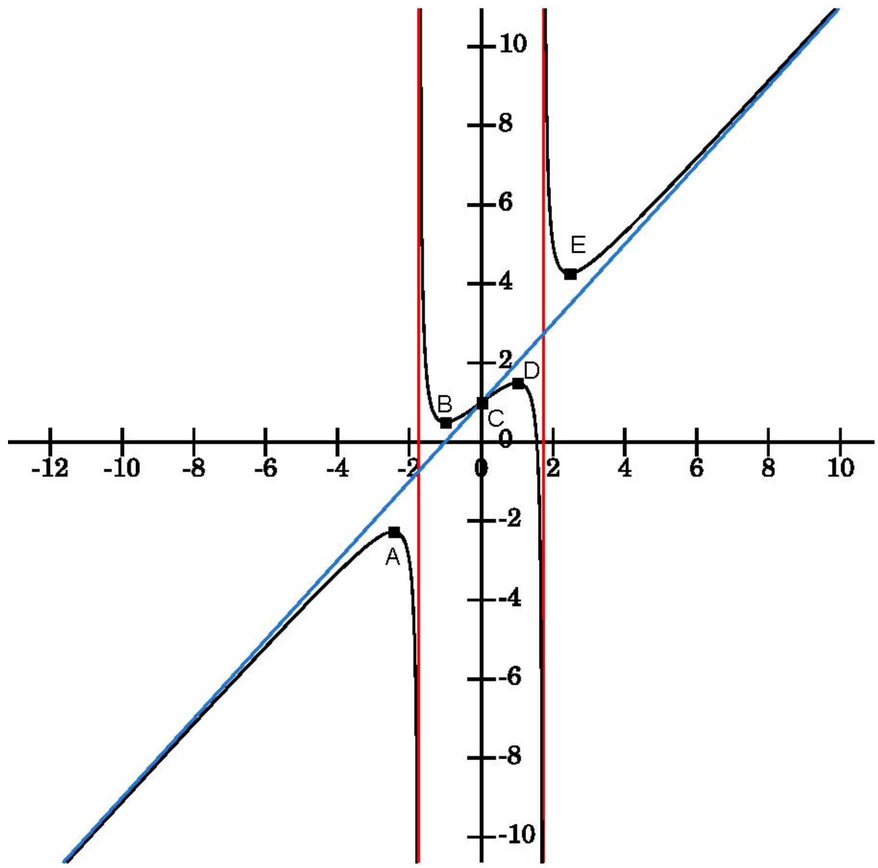
$$\lim_{x \to \pm\infty} f'(x) = 1.$$

This would be the slope of the "possible" asymptotes. Compute

$$f(x) - x = 1 + \frac{x}{x^2 - 3}.$$

Take limit: $\lim_{x \to \pm\infty} (f(x) - x) = 1$. This would be the y intercept of the asymptote. Conclusion: the slanted asymptote of f is $y = x + 1$.

(g) Draw a rough sketch of the graph of $f(x)$. Accurately place all critical points and inflection points, indicate all asymptotes, and make sure your graph shows where $f(x)$ is increasing and decreasing and correctly shows its concavity.



Black curve: $y = f(x)$.

Red lines: two vertical asymptotes, $x = \pm\sqrt{3}$.

Blue line: slanted asymptote, $y = x + 1$.

Local max points: A, D.

Local mins: B, E.

Inflection point: C.