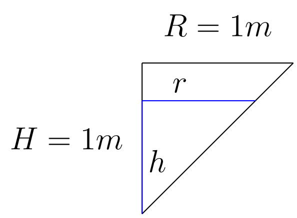
Assignment 7

1. Consider a water tank shaped like an inverted right circular cone of height 1 metre and radius 1 m. Let $h(t)$ denote the height of the water level of the tank in metres with time t in days. Water flows into the tank at a constant rate of 10 litres/day (1 litre=$10^{-3}m^3$.) Water evaporates from the tank at a rate of $0.01A$ in m^3/day, where A is the area in m^2 of the water surface.

   (a) When $h = 0.2m$, how fast is the water level rising?

Solution:



By similar triangle, the radius r of water surface at the water level h is: h. Therefore, the surface area $A = \pi h^2$.

The volume of the water is: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$.

$\therefore \dfrac{dV}{dt} = \pi h^2 \dfrac{dh}{dt}$

Since water flows in in the rate of 10 L/day, and flows out in the rate of $0.01A$ m^3/day,

$$\frac{dV}{dt} = (10 * 10^{-3} - 0.01A) = 0.01 \times (1 - \pi h^2) = \pi h^2 \frac{dh}{dt}.$$

Therefore, at $h = 0.2$, $\dfrac{dh}{dt} = \dfrac{0.01 \times (1 - \pi \times 0.2^2)}{\pi \times 0.2^2} \approx 0.0696.$

Hence the water level is increasing at the rate of 0.0696 m/day.

(b) If the tank is left for a long time, will it overflow?

Solution: From part (a): $\dfrac{dh}{dt} = \dfrac{0.01(1-\pi h^2)}{\pi h^2}$.

$$\frac{dh}{dt} \geq 0 \iff 1-\pi h^2 \geq 0 \iff 0 \leq h \leq 1/\sqrt{\pi}.$$

Therefore, the water will reach its equilibrium point at $h = 1/\sqrt{\pi} < 1$. Therefore, the tank will not overflow.

2. Liquid is being poured into a parabolic bowl at a constant rate of $60\pi\ cm^3/s$. The volume of the bowl is given by $V = \pi x^4/2$, where the equation of the parabola is $y = x^2$, and y is the height of the liquid in the bowl. Find the rate of increase of the height of the liquid in the bowl when the height is 10 centimetres.

Solution: $y = x^2 \implies \dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$.

$V = \pi\dfrac{x^4}{2} \implies \dfrac{dV}{dt} = 2\pi x^3\dfrac{dx}{dt}$

When the height is $10cm$, $10 = x^2$. Hence $x = \sqrt{10}$.

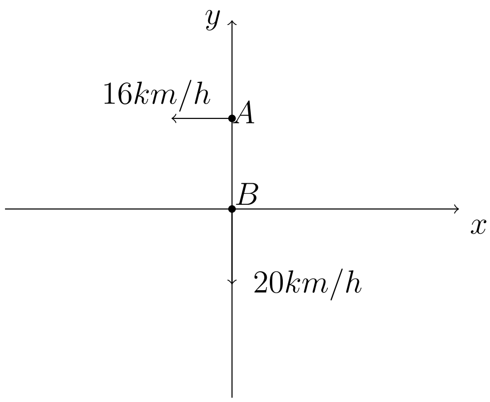
Therefore $\dfrac{dx}{dt} = \dfrac{\frac{dV}{dt}}{2\pi x^3} = \dfrac{60\pi}{2\pi(\sqrt{10})^3} = \dfrac{3\sqrt{10}}{10}$.

Therefore $\dfrac{dy}{dt} = 2 * \sqrt{10} * \dfrac{3\sqrt{10}}{10} = 6$.

Hence, the height of the liquid is increasing in the rate of $6cm/s$.

3. At 1:00 p.m. ship A is 25 km due north of ship B. If ship A is sailing west at a rate of 16km/h and ship B is sailing south at 20km/h, find the rate at which the distance between the two ships is changing at 1:30 p.m. (Be sure to draw a diagram).

Solution:



As shown in the picture, we take the initial point of ship B as the origin. Let t be the time (in hours) after $1:00pm$.

At $1:00pm$: The position for ship $A = (0, 25)$. The position for ship $B = (0, 0)$.

At time t: The position for ship $A = (-16t, 25)$. The position for ship $B = (0, -20t)$.

The distance between ship A and B is:

$D(t) = \sqrt{(-16t)^2 + (25 + 20t)^2} = \sqrt{656t^2 + 1000t + 625}$

$\frac{dD}{dt} = \frac{1312t + 1000}{2\sqrt{656t^2 + 1000t + 625}}$

At $1:30pm$, t=0.5. $\frac{dD}{dt} = \frac{1312 * 0.5 + 1000}{2\sqrt{656 * 0.5^2 + 1000 * 0.5 + 625}} = \frac{828\sqrt{1289}}{1289}$

The rate at which the distance between the two ships is change at $1:30pm$ is $\frac{828\sqrt{1289}}{1289} km/h$

4. Determine all the critical points for the function a) $g(t) = t^{2/3}(2t-1)$, b) $R(x) = \frac{x^2+1}{x^2-x-6}$.

Solution:

(a). $g'(t) = \dfrac{2}{3}t^{-\frac{1}{3}}(2t-1) + 2t^{\frac{2}{3}} = \dfrac{10t-2}{3t^{\frac{1}{3}}}$.

$g'(t) = 0 \Rightarrow 10t - 2 = 0 \Rightarrow t = \frac{1}{5}$

$g(\frac{1}{5}) = -\dfrac{3}{10\sqrt[3]{2}}$

$g'(t)$ Does Not Exist $\Rightarrow t = 0$.

$g(0) = 0$

The critical points of $g(t)$ are: $(\dfrac{1}{5}, -\dfrac{3}{10\sqrt[3]{2}})$ and $(0, 0)$

(b). $R'(x) = \dfrac{(2x)(x^2-x-6) - (x^2+1)(2x-1)}{(x^2-x-6)^2} = \dfrac{-x^2-14x+1}{[(x+2)(x-3)]^2}$

$R'(x) = 0 \Rightarrow -x^2 - 14x + 1 = 0 \Rightarrow x = 5\sqrt{2} - 7$ or $x = -7 - 5\sqrt{2}$.

$R(5\sqrt{2}-7) = \dfrac{2}{5}(1-\sqrt{2})$; $R(-7-5\sqrt{2}) = \dfrac{2}{5}(1+\sqrt{2})$;

$R'(x)$ Does Not Exist $\Rightarrow (x+2)(x-3) = 0 \Rightarrow R(x)$ Does Not Exist.

The critical points of $R(x)$ are: $(5\sqrt{2}-7, \dfrac{2}{5}(1-\sqrt{2}))$ and $(-7-5\sqrt{2}, \dfrac{2}{5}(1+\sqrt{2}))$

5. Suppose f is a function defined on an interval I . If $f'(x)$ is positive for all x in I what can you conclude about the behaviour of f on I?

Solution: f is increasing in the interval I.

Let f be defined on an interval I with an interior point c. If f has a local maximum at c, which of the following statements are true? (a) $f'(c) > 0$, (b) $f'(c) < 0$. (c) $f'(c) = 0$. (d) None of the above. (e) c is a critical point.

Solution: (e) c is a critical point.

If f has a local maximum at c and $f'(c)$ exists then what can you conclude about $f'(c)$?.

Solution: $f'(c) = 0$.

State whether the following is true or false. If c is a critical point of f, then f must have a local maximum or local minimum at c. Justify your answer.

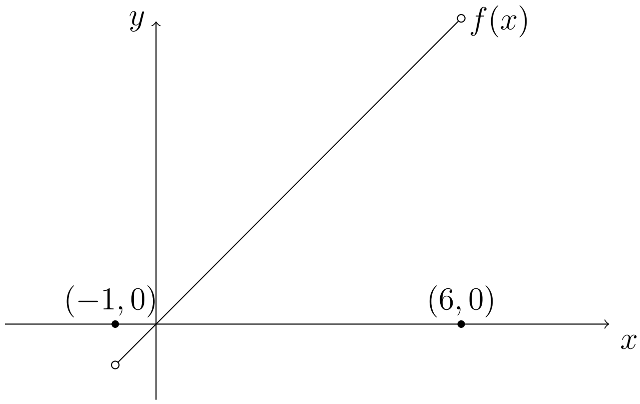
Solution: False. Consider the function $f(x) = x^3$. $f'(x) = 2x^2$. *Rightarrow* f has a critical point at $c = 0$. However $(0, 0)$ is neither a local maximum nor a local minimum. Notice that, $f'(x) > 0$ for both $x < 0$ and $x > 0$. Therefore $f(x)$ is increasing on both intervals $(-\infty, 0)$ and $(0, \infty)$. Hence $f(0)$ is not a local maximum or a local minimum.

If $f''(x) > 0$ on an interval, what can you conclude about $f'(x)$ in that interval?

Solution: If $f''(x) > 0$ on an interval, $f'(x)$ is increasing in the interval.

Can you draw the graph of a function $f(x)$ defined on $[-1, 6]$ that has no absolute minimum (if so, sketch it below). What does the extreme value theorem have to say about this situation.

Solution:

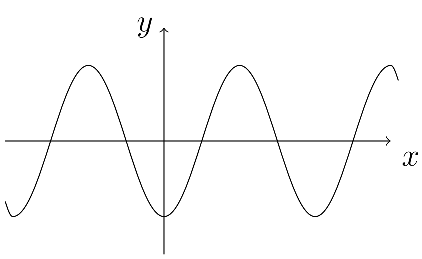


$$f(x) = \begin{cases} x & x \in (-1,6) \\ 0 & x = -1, 6 \end{cases}$$

$f(x)$ doesn't not have an extreme max/min on the interval $[-1,6]$. Notice that the function is **not continuous** on the interval $[-1,6]$. So it does not satisfy the condition of the extreme value theorem.

Draw the graph of a function $f(x)$ defined for all real values of x that is continuous and has both an absolute minimum and an absolute maximum.

Solution:



Let $f(x) = \cos x$ as shown in the graph. $f(x)$ is defined for all real values of x and is continuous. $f(x)$ has an absolute maximum 1 and an absolute minimum -1.