Assignment 6 SOLUTIONS

1. In the year 2000, a student takes out a 20000 dollars loan at an annual interest rate of r (interest compounded continuously). The student makes one loan payment of 10000 dollars in 2015, then one more loan payment of L in 2020 after which the student owes nothing.
(a) If the student's debt is 40000 dollars in 2005, what is r?

Solution: Interest is compounded continuously so we can express debt as a function of time using $A(t) = A_0 e^{rt}$ where t is the time (in years) elapsed since the loan was taken out. From the question, we know that $A_0 = 20000$ and $A(5) = 40000$. To find r, substitute this information into the function to obtain $40000 = 20000e^{5r}$. This solves to

$$r = \frac{\ln 2}{5} \approx 0.1386.$$

(b) At what rate (in dollars per year) is the student's debt increasing in 2010? (use the value of r from part (a)).

Solution: The rate at which the student's debt is increasing after t years have passed is $A'(t) = A_0 r e^{rt}$. In 2010, t=10 so

$$A'(10) = 20000\left(\frac{\ln 2}{5}\right)e^{\frac{10\ln 2}{5}} = 11090.4.$$

The student's debt is increasing a rate of \$11090.40 per year in 2010.

(c) Use the value of r from part (a) to find L.

Solution: The student owes nothing after making a payment of L so L is equal to the student's debt in 2020. To find this, first determine the student's debt in 2015 after 10000 has been paid:

$$A(15) = A_0 e^{rt} = 20000e^{\frac{15 \ln 2}{5}} - 10000 = 150000.$$

We know r remains constant. The student's debt after five additional years is therefore

$$A(20) = A(15)e^{\frac{(20-15)\ln 2}{5}} = 300000.$$

Hence $L = 300000$.

Simplyfing $e^{(t \ln 2)/5}$ to $2^{t/5}$ makes computations easier in this part.

2. Suppose that you invest in a high-interest saving account in which the interest rate is compounding continuously at 3.5%. (a) If you invest 10,000 dollars in 2013, how long will it take for the money to double?

Solution: We want to find t such that $A(t) = 2A_0$ where A_0 is the investment. Since the interest rate is compounded continuously at 3.5%, $r = 0.035$. We obtain $2A_0 = A_0e^{0.035t}$ after substitutions, where t is the doubling time in years. The A_0 cancel, leaving behind $2 = e^{0.035t}$. Hence time required for the money to double is

$$t = \frac{\ln 2}{0.035} \approx 19.8042.$$

It will take approximately 19 years and 10 months for the money to double. If the investment was made in 2013, the money will have doubled by the end of 2033.

(b)You recommend this scheme to your friend who invests 5000 dollars in 2014. When will he have 10,000 dollars in his account?

Solution: The friend will have 10000 dollars in his account when his investment doubles in value. Observe that our solution in the previous part did not require the value of A_0. Doubling time does not depend on the size of an investment so the friend will have 10000 dollars by the end of 2034.

(c)How long should you invest the money so that you have 12,000 dollars in your account?

Solution: The investment A_0 is 10000 and $r = 0.035$. We want to find the time t such that $A(t) = 12000 = 1.2A_0$. Making these substitutions, we have $1.2A_0 = A_0e^{0.035t}$. This solves to

$$t = \frac{\ln 1.2}{0.035} \approx 5.209.$$

An investment of 10000 dollars will grow to 12000 dollars in approximately 5.209 years.

3. Consider the demand equation

$$q = f(p) = \frac{1}{\sqrt{1+p}}$$

(a) Find the elasticity of demand function $E(p)$.

Solution: The derivative of the demand equation $f(p) = (1+p)^{-\frac{1}{2}}$ is

$$f'(p) = -\tfrac{1}{2}(1+p)^{-\frac{3}{2}}$$

Thus

$$E(p) = \frac{-pf'(p)}{f(p)} = \frac{-p(-\frac{1}{2}(1+p)^{-\frac{3}{2}})}{(1+p)^{-\frac{1}{2}}} = \frac{p}{2(1+p)}$$

is the elasticity of demand function.

(b) When $p = 10$, how will an increase in price affect the profit?

Solution: Substituting $p = 10$ into the elasticity of demand function shows that

$$E(10) = \frac{10}{2(1+10)} < 1$$

Demand is inelastic at $p = 10$ so an increase in price will correspond to an increase in profit.

4. In a petri dish, the number of bacteria increase at an exponential rate. After 10 minutes, there are 10,000 bacteria, and after 20 minutes, there are 100,000.

(a) How many bacteria were there initially?

Solution: The number of bacteria increase by a factor of ten every 10 minutes. Hence there were initially 1000 bacteria.

Alternatively: $P(10) = P_0e^{10r} = 10000$ and $P(20) = P_0e^{20r} = 100000$; $e^{10r} = 10$ by dividing $P(20)$ by $P(10)$; $P_0 = 1000$ by $P(10)$.

(b) Write a function which gives the population of the bacteria at time t.

Solution: $r = (\ln 10)/10$ from part b so $P(t) = 1000 \cdot 10^{t/10}$ where t is in minutes.

(c) If penicillin is added at $t = 30$ minutes, causing the population to decay exponentially so that at $t = 60$ minutes the population is 5000, then at what time has the population returned to its initial amount?

Solution: At $t = 30$, the bacterial population is $P(30) = 1000 \cdot 10^{30/10} = 1000000$. After penicillin is added ($t \geq 30$), $P(t) = 1000000e^{r_d(t-30)}$ for some $r_d < 0$. Substituting $P(60) = 5000$ gives $5000 = 1000000e^{30r_d}$. For the desired time t, we have $1000 = 1000000e^{r_d(t-30)}$. It follows that $-\ln 200 = 30r_d$ and $-\ln 1000 = r_d(t-30)$. Dividing the two equations produces

$$\frac{\ln 1000}{\ln 200} = \frac{t-30}{30}.$$

This yields $t \approx 69.1129$ as the solution. The time at which the population has returned to its original level is approximately $t = 69.1129$ minutes.