Pond Scum

If an initial population of surface algae in a pond at midnight (12:00am) is doubling every 10 minutes such that the pond is completely covered at noon, at what time of day was the pond half-covered?

CQ

A. 12:10am
B. 4:00am
C. 6:00am
D. 11:00am
[✓ E. 11:50am] doubles in 10 mins to full at noon.

Pay Raise

Suppose you are negotiating a contract for a new job with a starting salary of $50,000. Which of the following raise schedules would you prefer?

A. Raise of $1000 per year.
B. Raise of 2% per year.
C. It doesn't matter; these are the same.
[✓ D. Need more information.]
E. Confused.

Percent increase will always eventually beat linear increase. Here, it only takes 1 year.

Raise after 1 year: $1000 versus 50 000 . 0.02 = 1000
same at 1 year!

Raise after 2 years: $1000 versus 51,000 . 0.02 = 1020
better!

How is this related to exponential growth? Relative rate of change.

Relative rate is ~~the percent rate~~ (how much change with respect to current amount). For function f(x)

$$\text{Rel. Rate of change} = \frac{f'(x)}{f(x)}.$$

Exponential functions are the (only) functions with a constant relative rate:

$$f(x) = Ce^{kx} \qquad \frac{f'(x)}{f(x)} = \frac{kCe^{kx}}{\cancel{k}Ce^{kx}} = k$$

$$f'(x) = kCe^{kx}$$

Compare with linear function which have a constant rate but changing relative rate.

Exponential Growth and Decay

Math 104 Section 106 - Oct 2012

Have you seen or heard the phrase "growing exponentially" in a news story?

What does it mean when something is "growing exponentially"? The quantity is . . .

CQ

- A. Growing very rapidly.
- B. Growing, maybe quickly, maybe slowly.
- C. Doubling every time period for a certain time period (every year, every day, etc.)
- D. Growing by a fixed % every time period for a certain time period (every year, every day, etc.).
- E. Doing something else not described by A-D.

basically all true.

Explain . . . Depends what slowly vs. quickly means.

And: C and D are equivalent, just over different time scales.

Technically, to say that a function models **exponential growth**, it must have the form:

$$A(t) = Ce^{kt} \quad \text{or} \quad f(x) = Ce^{kx}$$

with $k > 0$ for growth.

(and it models **exponential decay** if $k < 0$). We often write the formula as:

$$A(t) = A(0)e^{kt}$$

$A(0)$ = population at time $t=0$.

t is time.

Are there any other functions that represent exponential growth?

What about $f(x) = 2^x$?

$$2^x = e^{\ln(2^x)}$$

$$2^x = e^{x\ln(2)}$$

so $f(x) = 2^x$ also has form e^{kx} for $k = \ln(2)$.

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Compound Interest

If you start with principal amount P, with a nominal interest rate r compounded n times per period, the formula for computing the amount at time t is:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, interest rates are often quoted at nominal rate meaning annual, though the compounding periods may be annual, monthly, daily, semi-annually, etc.

Loan Shark

Suppose a loan shark offers to loan you $1000 at an annual rate of 100% interest that you will pay back at the end of one year (so if it were not compounded, you would owe a total of $2000). You have the option to either pay $3000 at the end of the year OR let the loan shark choose the compounding rate to apply for the year. Which should you choose?

A. Pay $3000.
[✓ B.] Let the loan shark choose the compounding rate.
C. The results are about the same.
D. There is not enough information to solve the problem.
E. Confused.

If the loan shark chooses to compound n times,

$$A(1) = 1000\underbrace{\left(1 + \frac{1}{n}\right)^n}_{}$$

t=1 year

initial amount

uses r=1 for 100% annual rate

t=1 for one year

An n increases, $\left(1+\frac{1}{n}\right)^n$ increases

BUT $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

so we never owe more than 1000e dollars (about $2700), which is definitely less than $3000 of option A.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Continuously Compounded Interest: nominal rate

$$A(t) = Pe^{rt}$$

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