Problems in Continuity and Differentiability; Derivatives as rate of change; Deriving functions

SOLUTION

1. Find the number $c$ that makes

$$ f(x) = \begin{cases} 
\frac{x-c}{c+1}, & \text{if } x \leq 0 \\
 x^2 + c, & \text{if } x > 0 
\end{cases} $$

continuous for every $x$.

Solution:
Note that $f(x)$ is continuous for every $x \neq 0$.

$$ f(0) = \frac{0 - c}{c + 1} = \frac{-c}{c + 1}. $$

$$ \lim_{x \to 0^+} f(x) = 0^2 + c = c. $$

$$ \lim_{x \to 0^-} f(x) = \frac{-c}{c + 1}. $$

Since $f(x)$ is continuous for every $x$, hence continuous for $x = 0$.

$$ \Rightarrow f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x). $$

$$ \Rightarrow \frac{-c}{c + 1} = c. $$

$$ \Rightarrow c = 0 \text{ or } c = -2. $$

2. Find the values of $a$ and $b$ so that

$$ f(x) = \begin{cases} 
a x + b, & \text{if } x < 0 \\
2 \sin(x) + 3 \cos(x) & \text{if } x \geq 0 
\end{cases} $$

is differentiable at $x = 0$.

Solution:
First of all, $f(x)$ must be continuous at $x = 0$. Hence

$$ \lim_{x \to 0^-} f(x) = f(0). $$

$$ \Rightarrow b = 2 \sin 0 + 3 \cos 0 = 3. $$

Second, find $f'(x)$:

$$ f'(x) = \begin{cases} 
a, & \text{if } x < 0 \\
2 \cos(x) - 3 \sin(x) & \text{if } x \geq 0 
\end{cases} $$

Since $f(x)$ is differentiable at $x = 0$, $\lim_{x \to 0^-} f'(x) = f'(0)$.

$$ \Rightarrow a = 2 \cos 0 - 3 \sin 0 = 2. $$

Therefore $a = 2$, $b = 3$. 

1
3. The price $p$ (in dollars) and the demand $q$ (in thousands of units) of a commodity satisfy the demand equation $6p + q + qp = 94$. Find the rate at which demand is changing when $p = 9, q = 4$, and the price is rising at the rate of $2$ per week.

Solution:
From demand equation, we can get: $q = \frac{94 - 6p}{1 + p}$.

To find the rate of change of demand, we need to find the derivative: $q'(p) = \frac{-100}{(p + 1)^2}$.

At $p = 9$, $q'(9) = \frac{-100}{100} = -1$.

Since: $\Delta p / \Delta q = q'(p)$.

The Rate of Change of Demand at $(p = 9, q = 4) = q'(9) \cdot \text{The Rate of Change of Quantity}$.
When price is rising at the rate of 2, the demand should be changing at the rate of $-1 \cdot 2 = -2$. Hence demand is decreasing at the rate of 2.

4. When EZ Electronic Company sells surge protectors at $50 a piece, they produce and sell 3000 of them per month. For every $1 increase in price, the number of surge protectors they sell decreases by 15. Find the linear demand function $q = D(p)$, where $p$ is a price of a unit and $q$ is the number of surge protectors made and sold.

Solution:
Notice that the demand function is linear, therefore $q = D(p) = mp + b$.

When price increases by $1$, the demand decreases by 15, so $m = -15$.

Therefore $q = -15p + b$. Since the point $(p, q) = (50, 3000)$ must lie on this line, $3000 = -15 \cdot 50 + b \Rightarrow b = 3750$.

The demand function is: $q = -15p + 3750$. 
5. Find the derivatives of the following functions using differential rules (product rule, quotient rule, etc.). DO NOT SIMPLIFY

(a) \( \frac{\cos 2x}{\sin x} \)

(b) \((\sqrt{x} + x + 2)((x + 1)^3 - 2)\)

(c) \( y = \frac{e^{x^2-1}}{\sin(x^2)} \).

Solution:

(a) \( \left( \frac{\cos 2x}{\sin x} \right)' = \frac{1}{2} \sqrt{\frac{\cos 2x}{\sin x}} = \frac{-\sin 2x \cdot 2 \cdot \sin x - \cos 2x \cdot \cos x}{(\sin x)^2} \)

(b) \([\sqrt{x} + x + 2)((x + 1)^3 - 2)]' = (\frac{1}{2\sqrt{x} + 1})(((x + 1)^3 - 2) + (\sqrt{x} + x + 2) \cdot 3 \cdot (x + 1)^2 \)

(c) \( y' = \frac{e^{x^2-1}(2x)\sin(x^2) - e^{x^2-1}\cos(x^2) \cdot 2x}{\sin^2(x^2)} \).

6. Use the definition of derivatives to find \( f'(x) \) for \( f(x) = \sqrt{x^2 - 1} \). NO CREDIT WILL BE GIVEN FOR USING OTHER METHOD.

Solution:

By the definition of derivatives,

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
 = \lim_{h \to 0} \frac{\sqrt{(x + h)^2 - 1} - \sqrt{x^2 - 1}}{h}
 = \lim_{h \to 0} \frac{\sqrt{(x + h)^2 - 1} - \sqrt{x^2 - 1}}{h} \cdot \frac{\sqrt{(x + h)^2 - 1} + \sqrt{x^2 - 1}}{\sqrt{(x + h)^2 - 1} + \sqrt{x^2 - 1}}
 = \lim_{h \to 0} \frac{(x + h)^2 - 1 - x^2 + 1}{h(\sqrt{(x + h)^2 - 1} + \sqrt{x^2 - 1})}
 = \lim_{h \to 0} \frac{2xh + h^2}{2hx + h}
 = \lim_{h \to 0} \frac{2x + h}{2\sqrt{x^2 - 1} + 1}
 = \frac{2x}{2\sqrt{x^2 - 1}}
\]