Problems in Continuity and Differentiability; Derivatives as rate of change; Deriving functions SOLUTION

1. Find the number c that makes

$$f(x) = \begin{cases} \dfrac{x-c}{c+1}, & \text{if } x \leq 0 \\ x^2 + c, & \text{if } x > 0 \end{cases}$$

continuous for every x.

Solution:

Note that $f(x)$ is continuous for every $x \neq 0$.

$f(0) = \dfrac{0-c}{c+1} = \dfrac{-c}{c+1}$.

$\lim_{x \to 0^+} f(x) = 0^2 + c = c$.

$\lim_{x \to 0^-} f(x) = \dfrac{-c}{c+1}$.

Since $f(x)$ is continuous for every x, hence continuous for $x = 0$.

$\Rightarrow f(0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

$\Rightarrow \dfrac{-c}{c+1} = c$.

$\Rightarrow c = 0$ or $c = -2$.

2. Find the values of a and b so that

$$f(x) = \begin{cases} ax + b, & \text{if } x < 0 \\ 2\sin(x) + 3\cos(x) & \text{if } x \geq 0 \end{cases}$$

is differentiable at $x = 0$.

Solution:

First of all, $f(x)$ must be continuous at $x = 0$. Hence $\lim_{x \to 0^-} f(x) = f(0)$.

$\Rightarrow b = 2\sin 0 + 3\cos 0 = 3$.

Second, find $f'(x)$:

$f'(x) = \begin{cases} a, & \text{if } x < 0 \\ 2\cos(x) - 3\sin(x) & \text{if } x \geq 0 \end{cases}$.

Since $f(x)$ is differentiable at $x = 0$. $\lim_{x \to 0^-} f'(x) = f'(0)$.

$\Rightarrow a = 2\cos 0 - 3\sin 0 = 2$.

Therefore $a = 2$, $b = 3$.

3. The price p (in dollars) and the demand q (in thousands of units) of a commodity satisfy the demand equation $6p + q + qp = 94$. Find the rate at which demand is changing when $p = 9, q = 4$, and the price is rising at the rate of \$2 per week.

Solution:

From demand equation, we can get: $q = \dfrac{94 - 6p}{1 + p}$.

To find the rate of change of demand, we need to find the derivative: $q'(p) = \dfrac{-100}{(p+1)^2}$.

At $p = 9$, $q'(9) = \dfrac{-100}{100} = -1$.

Since: $\dfrac{\Delta p}{\Delta q} = q'(p)$.

The Rate of Change of Demand at $(p = 9, q = 4) = q'(9)*$ The Rate of Change of Quantity.

When price is rising at the rate of 2, the demand should be changing at the rate of $-1 \cdot 2 = -2$. Hence demand is decreasing at the rate of 2.

4. When EZ Electronic Company sells surge protectors at \$50 a piece, they produce and sell 3000 of them per month. For every \$1 increase in price, the number of surge protectors they sell decreases by 15. Find the linear demand function $q = D(p)$, where p is a price of a unit and q is the number of surge protectors made and sold.

Solution:

Notice that the demand function is linear, therefore $q = D(p) = mp + b$.

When price increases by \$1, the demand decreases by 15, so $m = -15$.

Therefore $q = -15p + b$. Since the point $(p, q) = (50, 3000)$ must lie on this line, $3000 = -15 \cdot 50 + b$. $\Rightarrow b = 3750$.

The demand function is: $q = -15p + 3750$.

5. Find the derivatives of the following functions using differential rules(product rule, quotient rule, etc.). DO NOT SIMPLIFY

(a) $\sqrt{\dfrac{\cos 2x}{\sin x}}$

(b)$(\sqrt{x}+x+2)((x+1)^3-2)$

(c)$y=\dfrac{e^{x^2-1}}{\sin(x^2)}$.

Solution:

(a) $(\sqrt{\dfrac{\cos 2x}{\sin x}})' = \dfrac{1}{2\sqrt{(\dfrac{\cos 2x}{\sin x})}} \dfrac{-\sin 2x \cdot 2 \cdot \sin x - \cos 2x \cdot \cos x}{(\sin x)^2}$

(b) $[(\sqrt{x}+x+2)((x+1)^3-2)]' = (\dfrac{1}{2\sqrt{x}}+1)((x+1)^3-2)+(\sqrt{x}+x+2)\cdot 3\cdot (x+1)^2$

(c) $y' = \dfrac{e^{x^2-1}(2x)\sin(x^2) - e^{x^2-1}\cos(x^2)\cdot 2x}{\sin^2(x^2)}$.

6. Use the definition of derivatives to find $f'(x)$ for $f(x)=\sqrt{x^2-1}$. NO CREDIT WILL BE GIVEN FOR USING OTHER METHOD.

Solution:

By the definition of derivatives,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{h} \\
&= \lim_{h\to 0} \frac{[\sqrt{(x+h)^2-1}-\sqrt{x^2-1}]\cdot[\sqrt{(x+h)^2-1}+\sqrt{x^2-1}]}{h[\sqrt{(x+h)^2-1}+\sqrt{x^2-1}]} \\
&= \lim_{h\to 0} \frac{(x+h)^2-1-x^2+1}{h[\sqrt{(x+h)^2-1}+\sqrt{x^2-1}]} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h[\sqrt{(x+h)^2-1}+\sqrt{x^2-1}]} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}} \\
&= \frac{2x}{2\sqrt{x^2-1}}
\end{aligned}$$