Error term bounds

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These notes explain how to find the error in linear approximation. Assume that we have a function $f(x)$, which we are approximating, centred at a, and that we have approximated a particular value $f(x_0)$ by $L(x_0)$, where $L(x_0)$ is the linear approximating polynomial and x_0 is close to the center a. Let I be the open interval with endpoints x_0 and a. Here are the steps which help you determine how good the linear approximation is:

Step 1: Compute $f''(X)$

- If $f''(x) > 0$ for all x in I, then f is concave up on I, so $L(x_0) < f(x_0)$, so your approximation is an underestimate.
- If $f''(x) < 0$ for all x in I, then f is concave down on I, so $L(x_0) < f(x_0)$, so your approximation is an overestimate.

Step 2: Error term

The error term is given by

$$E(x) = f(x) - L(x) = \frac{f''(x)}{2}(x-a)^2.$$

Step 3: Bounding

Bound for the Error in the linear approximation to f centred at a is given by

$$|f(x) - L(x)| \leq \frac{1}{2}M|x-a|^2,$$

where M is a constant such that $|f''(c)| \leq M$ for all c between a and x. Most often, $f''(x)$ will be either be always increasing or always decreasing on I, or will involve simple trigonometric functions, whose bounds are known. Remember that you can check if a function is increasing or decreasing by looking at its first derivative.

- If $|f''(x)|$ is increasing, then we may take $M = f''(b)$ where b is the right endpoint of I.

- If $f''(x)|$ is decreasing, then we may take $M = f''(d)$ where d is the left endpoint of I.
- If you get a nice number in the above steps, then you may use this for M; if not you should search past the endpoints to get a cleaner upper bound.
- Once you have a nice upper bound, then you can conclude that $|E(x_0)|$ is also less than this upper bound. This is your estimate of the size of the error. The smaller it is, the better is your approximation.

Step 4: Finding an interval that contains the exact value $f(x_0)$

Suppose you are approximating $f(x_0)$ using linear approximation centred at a, so that x_0 is close to a. We want to find the smallest interval which we can guarantee contains the true value of $f(x_0)$. To do this, let M be the upper bound for $|E(x_0)|$. Then

$$\begin{array}{ccccc} -M & < & E(x_0) & < & M \\ -M & < & f(x_0) - L(x_0) & < & M \\ L(x_0) - M & < & f(x_0) & < & L(x_0) + M. \end{array}$$

- Therefore $f(x_0)$ is contained in the open interval $(L(x_0) - M, L(x_0) + M)$. If your linear approximation was an over-estimate, the replace the right endpoint $L(x_0) + M$ by $L(x_0)$ for your interval.
- If your linear approximation was an under-estimate, then replace the left endpoint $L(x_0) - M$ by $L(x_0)$.
- The true value $f(x_0)$ is then contained in this interval.