Computing the error term.

Let $P_n(x)$ be the n^{th} Taylor polynomial for $f(x)$, so that

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{(n-1)}}{(n-1)!}$$

$$+ \frac{f^{(n)}(a)(x-a)^{(n)}}{n!},$$

Here a is the centre and x is in an interval containing a, and close to a. Then $P_n(x)$ approximates $f(x)$, denoted

$$P_n(x) \approx f(x).$$

The error term $E_n(x)$ has the form

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c that lies between a and x; $c \in [a,x]$.

To bound the error term therefore, we need to bound $f^{(n+1)}(x)$, i.e. we need to find M such that

$$f^{(n+1)}(\cancel{x}) \le M \quad \text{for all } c \in [a,x].$$

Thus to bound the error term for linear approximation, we need to find M such that

$$f^{(2)}(c) \le M \quad \text{for all} \quad c \in [a, x].$$

To bound the error term for Quadratic approximation, we need to find M such that

$$f^{(3)}(c) \le M \quad \text{for all} \quad c \in [a, x].$$

For linear approximation, we then have

$$|E(x)| = |\text{Error}| \le \frac{M}{2}(x-a)^2.$$

For Quadratic approximation,

$$|E(x)| = |\text{Error}| \le \frac{M}{3!}(x-a)^3$$

Examples: a) Use linear and quadratic approximation to approximate $\ln(0.9)$.

Ans: $f(x) = \ln(x)$; center $= a = 1$.

$f(a) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$; $f'(a) = \frac{1}{1} = 1$.

$L(x) = f(a) + (x-a)f'(a) = 0 + (x-1)\,1 = (x-1)$.

$L(0.9) = (0.9 - 1) = -0.1$

Quadratic approximation:

$$f''(x) = -\frac{1}{x^2} \;;$$

$$f''(1) = -1 .$$

$$Q(x) = L(x) + \frac{(x-a)^2 f''(a)}{2}$$

$$= L(x) + \frac{(x^2)(-1)}{2}$$

$$= (x-1) - \frac{(x^2)}{2} .$$

$$Q(0.9) = -0.1 - \frac{(0.1)^2}{2}$$

$$= -0.1 - \frac{(0.01)}{2} = \frac{-0.11}{\cancel{2}} .$$

b) ~~Approximate~~ Estimate the error term in both cases

For linear approximation

$$|\text{Error}| \le \frac{M}{2} (x-a)^2 ,$$

where M is the maximum value of the absolute value of the second derivative on the interval $[0.9, 1]$.

$$f''(x) = -\frac{1}{x^2}$$

This is a decreasing function, so the value is bounded by

$$M = \frac{1}{(0.9)^2} = \frac{100}{81} . \text{ Hence}$$

$$|\text{Error}| \le \frac{\frac{100}{81}}{2} (0.9-1)^2 = \frac{\frac{100}{81} \times 0.01}{2}$$

$$= \frac{100}{81} / 200 = \frac{1}{162}$$

$f(x) = \sin x.$

Center $= a = 0.$

$f(a) = \sin 0 = 0.$

$f'(x) = \cos x$

$f'(a) = \cos 0 = 1.$

$L(x) = f(x) + (x-a) f'(a)$

$= f(x) + x$

$= \sin x + x.$

Remainder estimation

1. **Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 0$ up to x^2 and use this to estimate $\sqrt{1.1}$. What is the error?**

We differentiate:

- $f(x) = (x+1)^{1/2}$ so $f(0) = 1$
- $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(0) = \frac{1}{2}$
- $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$ so $f''(0) = -\frac{1}{4}$
- $f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

Thus the Taylor polynomial of degree 2 around $a = 0$ is:

$$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

We note that we are trying to estimate $\sqrt{1.1}$, which it $f(0.1)$, so we plug in $P_2(0.1)$ to get our estimate:

$$P_2(0.1) = 1 + \frac{1}{20} - \frac{1}{800} = \frac{839}{800}$$

Now we look for the error. When we go up to degree 2, we have a formula for the error:

$E = \frac{f^{(3)}(c)}{3!}(0.1)^3$ for some c between the centre and the point we're estimating at–i.e., $0 < c < 0.1$.

So

$E = \frac{3}{3! \cdot 8(\sqrt{c+1})^5} \frac{1}{1000} = \frac{3}{64000}(\sqrt{c+1})^5$.

And we know c is between 0 and 0.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smallest, so it is largest when $c = 0$, and thus $E < \frac{3}{64000}$.

2. **Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 3$ up to the degree-2 term and use this to estimate $\sqrt{4.1}$. What is the error?**

We differentiate:

- $f(x) = (x+1)^{1/2}$ so $f(3) = 2$
- $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(3) = \frac{1}{4}$
- $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$ so $f''(0) = -\frac{1}{32}$
- $f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

Thus the Taylor polynomial of degree 2 around $a = 0$ is:

$$P_2(x) = 2 + \frac{1}{4}(x-3) - \frac{1}{32}(x-3)^2$$

We note that we are trying to estimate $\sqrt{4.1}$, which it $f(3.1)$, so we plug in $P_2(3.1)$ to get our estimate:

$$P_2(3.1) = 2 + \frac{1}{40} - \frac{1}{3200} = \frac{6479}{3200}$$

Now we look for the error. When we go up to degree 2, we have a formula for the error: $E = \frac{f^{(3)}(c)}{3!}(3.1-3)^3$ for some c between the centre and the point we're estimating at–i.e., $3 < c < 3.1$. So

$E = \frac{3}{3!\cdot 8(\sqrt{c+1})^5}\frac{1}{1000} = \frac{3}{64000}(\sqrt{c+1})^5$.

And we know c is between 3 and 3.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smalllest, so it is largest when $c = 3$, and thus $E < \frac{3}{128000}$.

3. **Let $f(x) = \ln(x+5) - \ln 5$. Approximate this by $\frac{x}{5} - \frac{x^2}{50}$. What is the error in this estimate provided $|x| < 0.1$?**
First, we compute the Taylor series for f around $a = 0$:
$f(x) = \ln(x+5) - \ln(5)$, so $f(0) = 0$.
$f'(x) = \frac{1}{x+5}$, so $f'(0) = \frac{1}{5}$
$f''(x) = -\frac{1}{(x+5)^2}$, so $f''(0) = -\frac{1}{25}$.
$f'''(x) = \frac{2}{(x+5)^3}$.
So we see that the given estimate is the degree-2 Taylor series for f, and therefore by Taylor's theorem, the absolute value of the error for $|x| < 0.1$ is exactly $|\frac{f^{(3)}(c)}{3!}x^3| = \frac{|x|^3}{3|c+5|^3}$. And $|x| < 0.1$ and c between 0 and x, so $-0.1 < x < 0.1$ and $-0.1 < c < 0.1$. So to make the error as big as possible, we make the numerator as big as possible, which happens when $x = 0.1$, and the denominator as small as possible, which happens when $c = -0.1$, so the error is, in absolute value, bounded by $\frac{(0.1)^3}{3(4.9)^3}$.

4. **Let $f(x) = \sin(x)$. Approximate this by $x - \frac{x^3}{6}$. What is the error in this estimate provided $|x| < 0.1$?** Use linear approximation.

$f(x) = \sin x$, center $= a = 0$; $f(a) = \sin 0 = 0$, $f'(x) = \cos x$, $f'(a) = 1$.

$L(x) = f(x) + (x-a) f'(a) = f(x) + x = \sin x + x$.

$|\text{Error}| \leq \frac{M}{2}(x-a)^2$, where M is such that $|f''(x)| \leq M$ for all x in $[0, 0.1]$.

$f''(x) = -\sin x$; $|f''(x)| \leq 1$ for all $x \in [0, 0.1]$. So

$|\text{Error}| \leq \frac{1}{2}(x-a)^2$. //