

Computing the error term.

(1)

Let $P_n(x)$ be the n^{th} Taylor polynomial for $f(x)$, so that

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \dots + \frac{f^{(n-1)}(a)(x-a)^{(n-1)}}{(n-1)!} + \frac{f^{(n)}(a)(x-a)^{(n)}}{n!},$$

Here a is the centre and x is in an interval containing a , and close to a . Then $P_n(x)$ approximates $f(x)$, denoted

$$P_n(x) \approx f(x).$$

The error term $E_n(x)$ has the form

$$E_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!},$$

for some c that lies between a and x ; $c \in [a, x]$.

To bound the error term therefore, we need to bound $f^{(n+1)}(x)$, i.e. we need to find M such that

$$f^{(n+1)}(c) \leq M \quad \text{for all } c \in [a, x].$$

Thus to bound the error term for linear approximation, we need to find M such that

$$f^{(2)}(c) \leq M \quad \text{for all } c \in [a, x].$$

To bound the error term for Quadratic approximation, we need to find M such that

$$f^{(3)}(c) \leq M \quad \text{for all } c \in [a, x].$$

For linear approximation, we then have

$$|E(x)| = |\text{Error}| \leq \frac{M}{2} (x-a)^2.$$

For Quadratic approximation,

$$|E(x)| = |\text{Error}| \leq \frac{M}{3!} (x-a)^3.$$

Examples: a) Use linear and quadratic approximation to approximate $\ln(0.9)$.

Ans: $f(x) = \ln(x)$; center = $a = 1$.

$$f(a) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x}; \quad f'(a) = \frac{1}{1} = 1.$$

$$L(x) = f(a) + (x-a)f'(a) = 0 + (x-1)1 = (x-1).$$

$$L(0.9) = (0.9-1) = -0.1$$

Quadratic Approximation:

$$f''(x) = -\frac{1}{x^2};$$

$$f''(1) = -1.$$

$$Q(x) = L(x) + \frac{(x-a)^2}{2} f''(a)$$

$$= L(x) + \frac{(x^2)}{2} (-1)$$

$$= (x-1) + \frac{(x^2)}{2}$$

$$Q(0.9) = -0.1 - \frac{(0.1)^2}{2} = -0.1 - \frac{0.01}{2} = -0.11$$

b) ~~Approximate~~ Estimate the error term in both cases

For linear approximation

$$|\text{Error}| \leq \frac{M}{2} (x-a)^2$$

where M is the maximum value of the absolute value of the second derivative on the interval [0.9, 1].

$$f''(x) = -\frac{1}{x^2}$$

This is a decreasing function, so the value is bounded by

$$M = \frac{1}{(0.9)^2} = \frac{100}{81} \quad \text{Hence} \quad |\text{Error}| \leq \frac{\frac{100}{81}}{2} (0.9-1)^2 = \frac{100}{81} \times \frac{0.01}{2} = \frac{100}{81} / 200 = \frac{1}{162}$$

$f(x) = \sin x$
 Center = $a = 0$

$f(a) = \sin 0 = 0$

$f'(x) = \cos x$

$f'(a) = \cos 0 = 1$

$L(x) = f(x) + (x-a)f'(a)$

$= \sin x + x$

$= \sin x + x$

1. Why are you taking this course?
 It is required by my program.
 Other
2. How new is the course material to you so far?
 Completely new
 Somewhat familiar
 Totally familiar
3. The pace of the lectures is:
 Too slow
 Good
 Too fast
4. Reading the lecturer's writing is:
 Easy
 Difficult
5. How many hours a week do you spend studying and doing homework for this course?
 1 hour
 2-3 hours
 4-6 hours
 7-9 hours
6. Do you practice problems from the book on your own?
 Yes
 No
7. How do you complete your assignments for this course?
 Normally on my own
 Communicate with friends/classmates
 Seek help of the instructor if stuck
8. When do you usually start doing a homework assignment?
 As soon as it is posted
 I postpone it to the last day/night
 Something in between
 I never do assignments
9. Estimate your level of understanding of the course material so far.
 I understand almost everything
 I understand most of the material
 I still don't understand many things that we have covered
10. What do you like or dislike about the course?
 Like - Our lecturer explains most of the problems and concepts clearly.
 Dislike: Doing math problems on computer. I prefer writing on paper or board.
11. What concerns would you like to bring to the instructor's attention?
 No so far.

REMAINDER ESTIMATION

1. Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 0$ up to x^2 and use this to estimate $\sqrt{1.1}$. What is the error?

We differentiate:

- $f(x) = (x+1)^{1/2}$ so $f(0) = 1$
- $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(0) = \frac{1}{2}$
- $f''(x) = \frac{1}{2}(-\frac{1}{2})(x+1)^{-3/2}$ so $f''(0) = -\frac{1}{4}$
- $f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

Thus the Taylor polynomial of degree 2 around $a = 0$ is:

$$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

We note that we are trying to estimate $\sqrt{1.1}$, which is $f(0.1)$, so we plug in $P_2(0.1)$ to get our estimate:

$$P_2(0.1) = 1 + \frac{1}{20} - \frac{1}{800} = \frac{839}{800}$$

Now we look for the error. When we go up to degree 2, we have a formula for the error:

$$E = \frac{f^{(3)}(c)}{3!}(0.1)^3 \text{ for some } c \text{ between the centre and the point we're estimating at--i.e., } 0 < c < 0.1.$$

So

$$E = \frac{3}{3! \cdot 8(\sqrt{c+1})^5} \frac{1}{1000} = \frac{3}{64000}(\sqrt{c+1})^5.$$

And we know c is between 0 and 0.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smallest, so it is largest when $c = 0$, and thus $E < \frac{3}{64000}$.

2. Let $f(x) = \sqrt{x+1}$. Compute the Taylor series around $a = 3$ up to the degree-2 term and use this to estimate $\sqrt{4.1}$. What is the error?

We differentiate:

- $f(x) = (x+1)^{1/2}$ so $f(3) = 2$
- $f'(x) = \frac{1}{2}(x+1)^{-1/2}$ so $f'(3) = \frac{1}{4}$
- $f''(x) = \frac{1}{2}(-\frac{1}{2})(x+1)^{-3/2}$ so $f''(3) = -\frac{1}{32}$
- $f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(x+1)^{-5/2} = \frac{3}{8(\sqrt{x+1})^5}$

Thus the Taylor polynomial of degree 2 around $a = 3$ is:

$$P_2(x) = 2 + \frac{1}{4}(x-3) - \frac{1}{32}(x-3)^2$$

We note that we are trying to estimate $\sqrt{4.1}$, which is $f(3.1)$, so we plug in $P_2(3.1)$ to get our estimate:

$$P_2(3.1) = 2 + \frac{1}{40} - \frac{1}{3200} = \frac{6479}{3200}$$

Now we look for the error. When we go up to degree 2, we have a formula for the error:

$E = \frac{f^{(3)}(c)}{3!} (3.1 - 3)^3$ for some c between the centre and the point we're estimating at—i.e., $3 < c < 3.1$. So

$$E = \frac{3}{3! \cdot 8(\sqrt{c+1})^5} \frac{1}{1000} = \frac{3}{64000} (\sqrt{c+1})^5.$$

And we know c is between 3 and 3.1, and we want to know how large the error could possibly be. We see that it is largest when the denominator is smallest, so it is largest when $c = 3$, and thus $E < \frac{3}{128000}$.

3. Let $f(x) = \ln(x + 5) - \ln 5$. Approximate this by $\frac{x}{5} - \frac{x^2}{50}$. What is the error in this estimate provided $|x| < 0.1$?

First, we compute the Taylor series for f around $a = 0$:

$$f(x) = \ln(x + 5) - \ln(5), \text{ so } f(0) = 0.$$

$$f'(x) = \frac{1}{x+5}, \text{ so } f'(0) = \frac{1}{5}.$$

$$f''(x) = -\frac{1}{(x+5)^2}, \text{ so } f''(0) = -\frac{1}{25}.$$

$$f'''(x) = \frac{2}{(x+5)^3}.$$

So we see that the given estimate is the degree-2 Taylor series for f , and therefore by Taylor's theorem, the absolute value of the error for $|x| < 0.1$ is exactly $|\frac{f^{(3)}(c)}{3!} x^3| = \frac{|x|^3}{3|c+5|^3}$. And $|x| < 0.1$ and c between 0 and x , so $-0.1 < x < 0.1$ and $-0.1 < c < 0.1$. So to make the error as big as possible, we make the numerator as big as possible, which happens when $x = 0.1$, and the denominator as small as possible, which happens when $c = -0.1$, so the error is, in absolute value, bounded by $\frac{(0.1)^3}{3(4.9)^3}$.

4. Let $f(x) = \sin(x)$. Approximate this by $x - \frac{x^3}{6}$. What is the error in this estimate provided $|x| < 0.1$? Use linear approximation.

$$f(x) = \sin x, \text{ center} = a = 0; \quad f(a) = \sin 0 = 0, \quad f'(x) = \cos x, \quad f'(a) = 1.$$

$$L(x) = f(x) + (x-a) f'(a) = f(x) + x = \sin x + x.$$

$|\text{Error}| \leq \frac{M}{2} (x-a)^2$, where M is such that $|f''(x)| \leq M$ for all x in $[0, 0.1]$.

$$f''(x) = -\sin x; \quad |f''(x)| \leq 1 \text{ for all } x \in [0, 0.1]. \text{ So}$$

$$|\text{Error}| \leq \frac{1}{2} (x-a)^2.$$