Assignment 10

November 29, 2013

1. Use linear approximation and quadratic approximation to approximate $63^{1/3}$.

   Answer. Notice that $64^{1/3} = 4$. We use the function $f(x) = x^{1/3}$ and choose the center $a = 64$. Then by linear approximation,

   $$63^{1/3} = f(63) \approx f(64) + f'(64)(63 - 64) = 4 - \frac{1}{48} = 2.97917...$$

   By quadratic approximation,

   $$63^{1/3} = f(63) \approx f(64) + f'(64)(-1) + f''(64)(-1)^2/2 = 2.979058...$$

   Remark. The true value of $63^{1/3}$ is 3.979057...

2. Find the linear approximation to $y = \sin x$ centered at $x = 0$.

   Answer. We have $(\sin x)' = \cos x$; at the center, $\cos 0 = 1$. Hence the linear approximation of $\sin x$ centered at 0 is

   $$\sin x \approx \sin 0 + 1 \cdot (x - 0) = x.$$

3. Find $\sqrt{9.02}$ approximately using linear approximation.

   Answer. Use the function $f(x) = \sqrt{x}$, and choose the center $a = 9$. By linear approximation, we have

   $$\sqrt{9.02} = f(9.02) \approx f(9) + f'(9)(9.02 - 9) = 3 + (1/6) \times 0.02 = 3 + \frac{1}{300}.$$

4. For a function $f(x)$ we know that $f(3) = 2$ and $f'(3) = -3$. Give an estimate for $f(2.91)$.

Answer. We use linear approximation to get

$$f(2.91) \approx f(3) + f'(3)(2.91 - 3) = 2 + 3 \times 0.09 = 2.27.$$

5. The function $f(x)$ has the following properties: $f(5) = 2$, $f'(5) = 0.6$, $f''(5) = -0.4$.
   - (a) Find the tangent line to $y = f(x)$ at the point $(5, 2)$.
   - (b) Use (a) to estimate $f(5.2)$.
   - (c) If f is known to be concave down, could your estimate in (b) be greater than the actual value of $f(5.2)$? Justify your answer.

Answer. (a) The slope of the tangent line is $f'(5) = 0.6$. The tangent line is given by $y = 0.6(x - 5) + 2$, or equivalently, $y = 0.6x - 1$.

(b) We have $f(5.2) \approx 0.6 \times 5.2 - 1 = 2.12$.

(c) The error term is given by $R_1(x) = f''(\xi)(x-a)^2/2$. $f(x)$ being concave down means that $f'' < 0$, thus $R_1(x) < 0$. So the approximated value is greater than the actual value.

6. What is the maximum error in approximating $\ln(1 - x)$ centered at 0 by the quadratic polynomial $p_2(x)$ in the interval $[-1/2, 1/2]$.

Answer. The absolute error is given by $|R_2(x)| = |f'''(\xi)x^3|/6$. The third derivative of $\ln(1 - x)$ is $2(x - 1)^{-3}$, so $|R_2(x)|$ becomes

$$|R_2(x)| = \frac{1}{3} \cdot \frac{1}{|\xi - 1|^3} |x|^3.$$

Since $x \in [-1/2, 1/2]$, we have $|x|^3 \leq 1/8$. Now ξ lies between the center 0 and x, but in anyway, ξ also lies in $[-1/2, 1/2]$, and so it is easy to see that $1/|\xi - 1|^3 \leq 8$. Therefore

$$|R_2(x)| \leq \frac{1}{3} \times 8 \times \frac{1}{8} = \frac{1}{3}.$$

Remark. This upper bound is not sharp. One can improve it to

$$|R_2(x)| \leq \ln(3/2) - 5/8 = 0.21953...$$