Assignment 9

1. A small manufacturer wholesales leather jackets to a number of specialty stores. The monthly demand from these stores for the jackets is described by the demand equation $p = 400 - 50q$. Here p is the wholesale price, in dollars per jacket, and q is the monthly demand, in thousands of jackets. Note that the demand equation makes no sense if $q \geq 8$. The manufacturer's marginal cost is given by the equation

$$dC/dq = \frac{800}{q+5}.$$

Determine the number of jackets that must be sold per month to maximize monthly profit. You do not need to justify that your answer provides the maximal profit.

Solution:
Constraint: $0 \leq q \leq 8$
Given: $p = 400 - 50q$; $C(q) =$ Cost function; $\dfrac{dC}{dq} = \dfrac{800}{q+5}$
Goal: Find q so that $P(q)$ is maximal, where P is the profit function.

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= pq - C(q) \\ &= 400q - 50q^2 - C(q) \end{aligned}$$

$$\begin{aligned} P'(q) &= 400 - 100q - C'(q) \\ &= 400 - 100q - \frac{800}{q+5} \end{aligned}$$

Maximize profit by finding q such that $P'(q) = 0$.

$$\begin{aligned} P'(q) = 400 - 100q - \frac{800}{q+5} = 0 &\Rightarrow 400q + 2000 - 100q^2 - 500q - 800 = 0 \\ &\Rightarrow 100q^2 + 100q - 1200 = 0 \\ &\Rightarrow q^2 + q - 12 = 0 \\ &\Rightarrow (q+4)(q-3) = 0 \end{aligned}$$

The constraint is satisfied by $q = 3$. Hence profit is maximized when 3 jackets are sold per month.

2. Find two positive real numbers m and n whose product is 50 and whose sum is as small as possible.

Solution:
Constraint: $m > 0;\ n > 0;\ mn = 50$
Goal: Minimize $m + n$.
Let $M = m + n$.

$$\begin{aligned} M(m) &= m + \frac{50}{m} \\ &= \frac{m^2 + 50}{m} \end{aligned}$$

Minimize $M(m)$ by finding m such that $M'(m) = 0$.

$$\begin{aligned} M'(m) = 1 - \frac{50}{m^2} = 0 &\Rightarrow m^2 - 50 = 0 \\ &\Rightarrow m^2 = 50 \end{aligned}$$

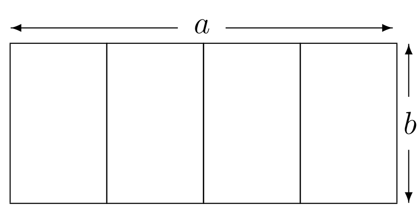
But m is positive so $m = \sqrt{50} = 5\sqrt{2}$. Then

$$n = \frac{50}{5\sqrt{2}} = \frac{10}{\sqrt{2}} = \frac{2 \times 5}{\sqrt{2}} = 5\sqrt{2}.$$

So $m = n = 5\sqrt{2}$.

3. You want to build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

Solution:



Constraint: $2a + 5b = 500$ft. Since there are 5 lengths of b and 500ft of fencing, $0 \leq b \leq 100$; $a = \dfrac{500 - 5b}{2}$ or $0 \leq a \leq 250$.
Area: ab ft^2.
Goal: Maximize area.

$$2a + 5b = 500 \;\Rightarrow\; 5b = 500 - 2a \;\Rightarrow\; b = 100 - \frac{2}{5}a$$

$$\text{Area} = ab = a\left(100 - \frac{2}{5}a\right)$$

$$A(a) = 100a - \frac{2}{5}a^2$$

Maximize area: $A'(a) = 0$.

$$\begin{aligned} A'(a) = 100 - \frac{4}{5}a = 0 &\Rightarrow 100 = \frac{4}{5}a \\ &\Rightarrow 4a = 500 \\ &\Rightarrow a = 125\text{ft} \end{aligned}$$

and so

$$b = 100 - \left(\frac{2}{5}\right)(125) = 100 - 50 = 50\text{ft}.$$

So $a = 125$ft and $b = 50$ft works.
Maximizes area (check): When $a = 125$,

$$A''(a) = -\frac{4}{5} < 0.$$

Hence area is maximized.

4. A container in the shape of a right circular cylinder with no top has a surface area of $3\pi m^2$. What height h and base radius r will maximize the volume of the cylinder?

Solution:
Constraint: Base of box is a circle and only $3\pi\text{m}^2$ of material is available, so $0 \leq r \leq \sqrt{3}$.
Goal: Maximize volume.
Total surface area = Area of base + Area of curved body = $\pi r^2 + 2\pi r h$.

$A = 3\pi$, so

$$3\pi = \pi r^2 + (2\pi r)h \Rightarrow 2\pi rh = 3\pi - \pi r^2$$
$$\Rightarrow h = \frac{3\pi - \pi r^2}{2\pi r}$$

Volume $= V = \pi r^2 h$.

$$V = V(r) = \pi r^2 \left(\frac{3\pi - \pi r^2}{2\pi r}\right) = \frac{3}{2}\pi r - \frac{1}{2}\pi r^3$$

$$V'(r) = \frac{3}{2}\pi - \frac{3}{2}\pi r^2 = \frac{3}{2}\pi(1-r)(1+r)$$

Maximize volume: Letting $V'(r) = 0$, we obtain $r = \pm 1$. Radius cannot be negative so $r = 1$.
It follows that

$$h = \frac{3\pi - \pi r^2}{2\pi r} = 1$$

given $r = 1$.
Second derivative test: $V''(r) = -3\pi r$ so for $r = 1$, $V''(1) = -3\pi < 0$.
Volume is maximized when $r = h = 1$m.

5. Suppose you own a tour bus and you book groups of 20 to 70 people for a day tour. The cost per person is \$30 minus \$0.25 for every ticket sold. If gas and other miscellaneous costs are \$200, how many tickets should you sell to maximize your profit? Treat the number of tickets as a nonnegative real number.

Solution:

Define q as the number of tickets sold. Constraint: $20 \leq q \leq 70$.

$$\begin{aligned}\text{Revenue } R(q) &= q(30 - 0.25q)\\ &= -0.25q^2 + 30q\end{aligned}$$

$$\begin{aligned}\text{Cost } C(q) &= \text{fixed cost}\\ &= 200\end{aligned}$$

$$\begin{aligned}\text{Profit } P(q) &= R(q) - C(q)\\ &= -0.25q^2 + 30q - 200\end{aligned}$$

Maximize $P(q)$: $P'(q) = 0$.

$$P'(q) = -0.5q + 30 = 0 \Rightarrow 30 = 0.5q$$
$$\Rightarrow q = \frac{300}{5} = 60$$

Second derivative test: $P''(q) = -0.5 < 0$ so $q = 60$ is a local maximum.
$q = 60$ is a unique critical point so it is also the absolute maximum.