

# Moderate Optimization Example

**Problem:** An open-top cylindrical tank with a volume of ten cubic feet is to be made from a sheet of steel. Find the dimensions of the tank that will require as little material used in the tank as possible.

**Solution:**

**1 - Preparation** Re-reading the problem statement and writing down what is given and what is to be found.

*“An open-top cylindrical tank with a volume of ten cubic feet is to be made from a sheet of steel. Find the dimensions of the tank that will require as little material used in the tank as possible.”*

- Rewrite the given info

*A cylindrical tank open at the top.*

*Made from as little sheet material as possible.*

*Volume is 10 ft<sup>3</sup>.*

- **To Find** Dimension of the tank that minimizes the sheet material needed.

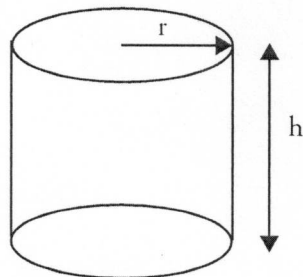
**2 - Translation** to an optimization problem:

“As little material as possible” translates to “as little outside surface area of the tank as possible.” Or better: “minimum outside surface area.”

- **Drawings:**

$V = \text{Volume} = 10 \text{ ft}^3$

$A = \text{Area}$   
 $= \pi r^2 + 2 \pi r h$



- **Symbols:** Four definitions are in the drawing above.

Volume =  $V \text{ ft}^3$ .      Surface Area =  $A \text{ ft}^2$

$r = \text{radius of tank (ft)}$      $h = \text{height of tank (ft)}$

**Formulation as an optimization problem**

**Design Variables:**       $r$ , radius (ft)  
    $h$ , height (ft)

The **Criterion Function** in this example is the surface area,  $A$ , since when it is minimum, the amount of material used will be a minimum.

Criterion Function =  $A = \pi r^2 + 2 \pi r h$

A **Constraint** in this case is the required volume.

Constraint:     $V = \pi r^2 h = 10 \text{ ft}^3$ .

Or:     $h = 10 / \pi r^2$

Now we have the optimization problem in mathematical form. In words, the math problem is the find the values of  $r$  and  $h$  that give a tank of minimum surface area with a volume of  $10 \text{ ft}^3$ .

**3 - Application : Solution by Numerical Search.**

We can almost always proceed by systematic numerical search.

In this example, there is really only a single unknown design variable because the constraint on the volume can be used. The table below shows a solution found by trying a value for  $r$ , computing  $h$  from the volume, and then computing the resulting surface area,  $A$ . This table was produced by using a spreadsheet program which is most convenient for such problems, but not really necessary in such a simple one-variable search.

One can start anywhere, but starting with  $r=1$  seems a reasonable place. After the third trial, it is apparent that the minimum  $A$  is between  $r=1$  and  $r=3$ .

Thus we plan to try  $r=1.5$  and  $r=2.5$ . However, the result of  $r=1.5$  reveals that the minimum  $A$  will be between  $r=1$  and  $r=2$ . We continue with this process as shown.

Trial	$r$	$h = 10/\pi r^2$	$A = \pi r^2 + 20/r$
1	1	3.18	dec 23.14
2	2	0.796	22.56
3	3	0.35	inc 34.93
4	1.5	1.414	20.40
5	1.25	2.04	20.91
6	1.75	1.04	21.04
7	1.4	1.62	20.44
8	1.3	1.88	20.69
9	1.45	1.51	20.39
10	1.46	1.49	20.39
11	1.47	1.47	20.39
12	1.48	1.45	20.39

The optimum appears to be  $20.39 \text{ ft}^2$ , and it is found at a value of  $r$  approximately equal to 1.45 or 1.46 ft. To get more accuracy, we could refine the search.

# Solutions

Math 104 - Fall 2012

Set up the problem: draw a diagram if appropriate, name the variables you are using and use them to express the objective function and all relevant constraints. The idea today is to practice translating word problems into these statements, so stop before you take the derivative (unless you have time at the end).

1. Find the dimensions of the rectangle of maximum area that can be inscribed in a right triangle with sides of length 3 and 4 and hypotenuse of length 5, if two sides of the rectangle lie along the two sides of the triangle. Make sure you justify that your answer is a maximum. In solving this problem, follow these steps:
  - Draw a sketch and assign names to all relevant variables.
  - State what quantities are given.
  - Express in your own words what the problem is asking.
  - Identify the quantity that needs to be optimized.
  - Identify all the constraints given in the problem.
  - Express the quantity that needs to be optimized as a mathematical function. In other words, write down the "objective function".
  - Express in mathematical form all the constraints given in the problem.
  - Simplify the "objective function" as a function of only one variable.
  - Apply appropriate calculus techniques to find maxima/minima of the "objective function".
  - Discuss your results.

SOLUTION:

Imagine the right triangle drawn with the  $x$ -axis and  $y$ -axis forming the two perpendicular sides and the line  $y = -\frac{4}{3}x + 4$  forms the hypotenuse (it intersects the  $x$ -axis at  $(3, 0)$  and intersects the  $y$  axis at  $(0, 4)$ ). Choose a point  $(a, b)$  on the line so that  $b = -\frac{4}{3}a + 4$ . The area of the rectangle is

$$f(a) = a\left(-\frac{4}{3}a + 4\right) = -\frac{4}{3}a^2 + 4a$$

Then  $f'(a) = -\frac{8}{3}a + 4$ . We compute  $f'(a) = 0$  for  $a = \frac{3}{2}$  and so for  $b = 2$ . Does this rectangle of dimensions  $\frac{3}{2} \times 2$  have maximum area? We check that  $f'(a) > 0$  for  $0 < a < \frac{3}{2}$  and  $f'(a) < 0$  for  $\frac{3}{2} < a < 3$ . Thus  $f(a)$  is maximized at  $a = \frac{3}{2}$ .

2. You are in a dune buggy at point P in the desert, 12 km due south of the nearest point A on a straight east-west road. You want to get to a town B on the road 18 km east of A. If your dune buggy can travel at a speed of 15 km/h through the desert, and 39 km/h along the road, toward what point on the road should you head to minimize your travel time from P to B? Apply the problem-solving strategy described above.

SOLUTION:

Again a picture would make this clearer. Let  $C$  be the first point on the east west road that the dune buggy arrives at. Let  $x$  denote the distance from A. Thus the dune buggy will travel  $\sqrt{x^2 + 12^2}$  km in the desert and  $18 - x$  on the road. Thus the travel time in the desert is  $\frac{\sqrt{x^2 + 12^2}}{15}$  and the time on the road is  $\frac{18 - x}{39}$ . Let  $f(x)$  be the time

$$f(x) = \frac{\sqrt{x^2 + 12^2}}{15} + \frac{18 - x}{39}$$

we compute

$$f'(x) = \frac{2x}{30\sqrt{x^2 + 12^2}} - \frac{1}{39}$$

Now  $f'(x) = 0$  for  $\frac{2x}{30\sqrt{x^2+12^2}} = \frac{1}{39}$  and so  $78x = 30\sqrt{x^2+12^2}$  and so  $13x = 5\sqrt{x^2+12^2}$  and so  $13^2x^2 = 5^2(x^2+12^2)$  and so  $12^2x^2 = 5^212^2$  and so  $x = 5$  (we may discard  $x = -5$  which must be worse than  $x = 5$ ). We check the signs of the derivative to ensure that  $x = 5$  yields a minimum for  $f(x)$ , the travel time.

3. A piece of wire 10 cm long is cut into two pieces. One piece is bent into a circle while the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimal? Maximal?

SOLUTION:

Divide the wire into pieces of length  $x$  and  $10 - x$  forming equilateral triangle and circle respectively. The side length of the triangle is  $x/3$  giving area  $(1/2) \cdot x/3 \cdot x\sqrt{3}/6$ . For the circle  $2\pi r = 10 - x$  giving  $\pi r^2 = \pi(10 - x)^2/(2\pi)^2$ . The total enclosed area is the sum  $A(x) = x^2\sqrt{3}/36 + (10 - x)^2/(4\pi)$ . Thus  $A'(x) = x\sqrt{3}/18 - (10 - x)/(2\pi)$ .  $A'(x) = 0 \Rightarrow x = 90/(\sqrt{3}\pi + 9) \approx 6.23$ . We find  $A(0) = 25/\pi \approx 7.96$ ,  $A(10) = 25\sqrt{3}/9 \approx 4.82$ , and  $A(90/(\sqrt{3}\pi + 9)) \approx 3.00$ . The min is  $3.00\text{cm}^2$  and the max is  $7.96\text{cm}^2$ .

4. A right circular cone is turned upside-down (vertex is at the bottom); the height of cone is 10 cm and the diameter of the top is also 10 cm. The cone is filled half-way (by depth), and water is draining out. The depth of the water is dropping at a rate of 4 cm per second. How fast is the volume of the water changing at this time? You may use the fact that the volume of a cone with radius  $r$  and height  $h$  is  $\pi r^2 h/3$ .

SOLUTION:

The height of the cone is 10 and the radius is 5. By similar triangles (use a picture) we have that  $r = \frac{5}{10}h = h/2$ . Using  $V = \frac{1}{3}\pi(r^2h)$  we have  $V = (\pi/12)h^3$  so  $dV/dt = (\pi/4)h^2(dh/dt) = (\pi/4)(100)(-4) = -100\pi$ , i.e. the volume of water is decreasing at a rate of  $100\pi$  cubic cm per second.

5. A shoebox, complete with overlapping lid, has a base with length  $5/3$  times its width. The rim of the lid overlaps the side of the main box by  $1/5$  of the height of the box. Assuming the thickness of the box is negligible and of uniform density, and the total volume of the box is 3000 cubic centimetres, what dimensions will result in the minimum weight of the empty box? Make sure you follow the steps outlined in the problem-solving strategy above. Draw a picture of the box and remember that if your computations lead to large numbers you can always express such numbers as a product of their factors: this will help you to carry out algebraic calculations by hand.

SOLUTION:

The problem is more about accounting carefully for the weight. Let  $l$  denote the length, let  $w$  denote the width, and let  $h$  denote the height. We have been told the length is  $5/3$  times the width, namely

$$l = \frac{5}{3}w$$

Also the volume is 3000 and so  $wlh = 3000$  and so we can substitute for  $l$  and so

$$\frac{5}{3}w^2h = 3000$$

From this we can express  $h$  in terms of  $w$  and deduce that

$$h = \frac{1800}{w^2}$$

The amount or area of cardboard is proportional to the weight (from the uniform density hypothesis). The area of the bottom of the box is

$$wl + 2lh + 2wh$$

the area of the lid is

$$wl + 2l\frac{h}{5} + 2w\frac{h}{5}$$

and so the total area is

$$2wl + (2l + 2w)\frac{6}{5}h$$

If we substitute for  $l, h$  we get that total area is

$$A(w) = \frac{10}{3}w^2 + \left(\frac{16}{3}w\right)\frac{6}{5}\frac{1800}{w^2} = \frac{10}{3}w^2 + \frac{2160}{w}$$

We wish to minimize the area and so we differentiate to get

$$A'(w) = \frac{20}{3}w - \frac{11520}{w^2}$$

Thus  $A'(w) = 0$  for  $w^3 = 11520 \times \frac{3}{20} = 1728 = (12)^3$  and so for  $w = 12$ . Thus we have  $A'(w) = 0$  for  $w = 12$  and so  $l = 20$  and  $h = 12.5$ . The lid will have depth  $\frac{1}{5}12.5 = 2.5$

Now we might not be sure that this is the minimum area but can verify that our solution is a local minimum. The possible values for  $w$  are  $0 < w$ . For  $w \rightarrow 0$  we have  $l \rightarrow 0$  and so  $h \rightarrow \infty$  but in order to have a volume of 3000, we can show that  $A(w) \rightarrow \infty$ . For  $w \rightarrow \infty$  we have  $l \rightarrow \infty$  and  $h \rightarrow 0$  and so as  $w \rightarrow \infty$  we have  $A(w) \rightarrow \infty$  just based on the top and bottom surfaces.

6. The price  $p$  in \$ for a product and the quantity  $q$  of demand (that is, the quantity of units sold) for a product are related by the equation  $p^2 + 2q^2 = 1100$ . If the price is increasing at a rate of \$2 per month when the price is \$30, find the rate of change of the revenue  $R$  in dollars per month.

SOLUTION:

The price  $p$  and demand  $q$  are related by the equation

$$p^2 + 2q^2 = 1100$$

Thus

$$\frac{d}{dt}(p^2 + 2q^2) = 2p\frac{dp}{dt} + 4q\frac{dq}{dt} = \frac{d}{dt}(1100) = 0$$

We are given that  $\frac{dp}{dt} = 2$  and that currently  $p = 30$ . Substituting into our first equation we have

$$(30)^2 + 2q^2 = 1100 \text{ and so } q^2 = 100 \text{ and so } q = 10$$

where we discard the negative root. Substituting in the above equation  $p = 30$ ,  $q = 10$  and  $\frac{dp}{dt} = 2$ , we have  $2 \times 30 \times 2 + 4 \times 10 \times \frac{dq}{dt} = 0$  from which we solve to obtain  $\frac{dq}{dt} = -\frac{12}{4} = -3$  units per month.

We are asked for

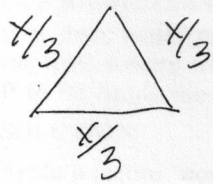
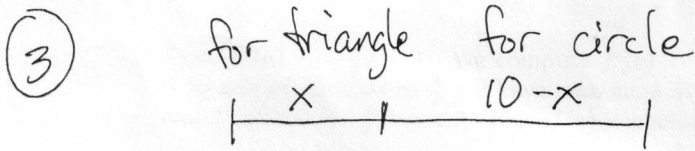
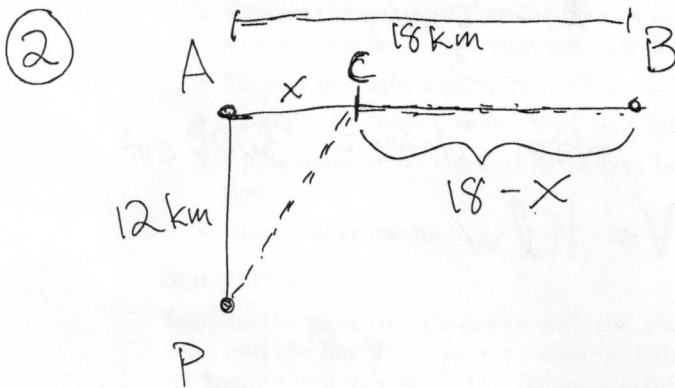
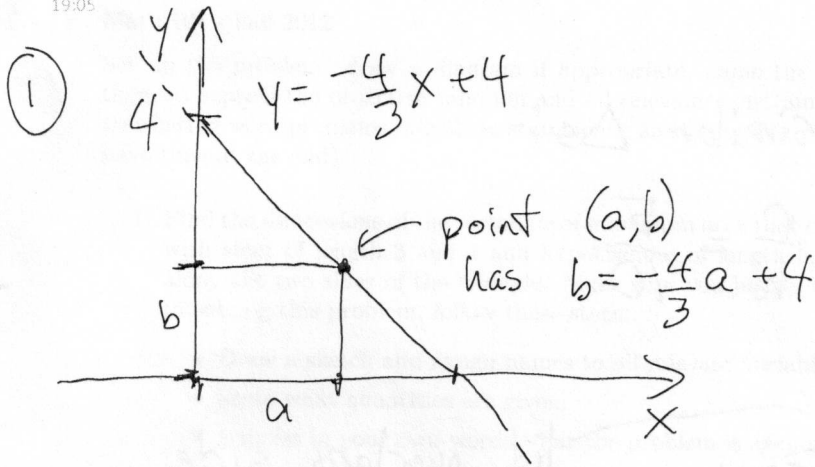
$$\begin{aligned} \frac{d}{dt}R &= \frac{d}{dt}pq = \frac{dp}{dt}q + p\frac{dq}{dt} \\ &= 2 \times 10 + 30 \times (-3) = -70 \end{aligned}$$

a rate of change of \$ -70 per month. Thus revenue is decreasing with respect to time. (As price increases, demand goes down but the overall revenue is decreasing.)



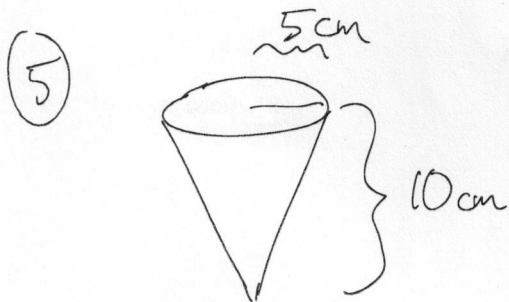
# Word Problem Diagrams

Friday, June 03, 2011  
19:05

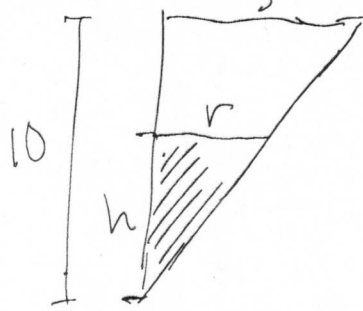


perimeter

$$2\pi r = 10 - x$$



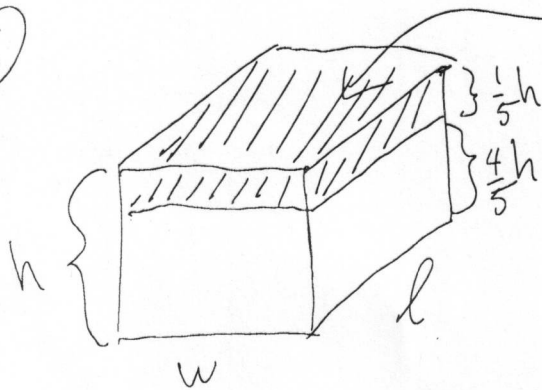
from the side:



similar  $\Delta s$ :

$$\frac{r}{h} = \frac{5}{10}$$

⑥



lid overlaps side

$$l = \frac{5w}{3}$$

$$\text{total volume} = 3000 \text{ cm}^3$$

$$V = hlw$$