MATH 143 — SLANT ASYMPTOTES

We have seen that a rational function $f(x) = \frac{p(x)}{d(x)}$ will have a horizontal asymptote if the degree of the numerator p is less than or equal to the degree of the denominator d. In particular, if the degree of p is strictly less than that of d, then the x-axis will be the horizontal asymptote—a geometrical condition that can be expressed analytically by saying $f(x) \longrightarrow 0$ as $x \longrightarrow \infty$ and as $x \longrightarrow -\infty$.

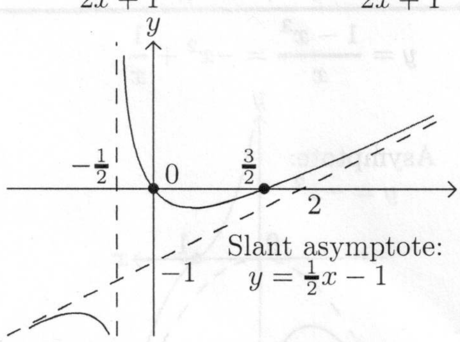
If the degree of p is greater than or equal to the degree of d, then long division can be used to obtain more accurate information about the large scale behavior of the rational function. Recall that $p(x)$ divided by $d(x)$ gives a quotient $q(x)$ and a remainder $r(x)$, provided that $p(x) = d(x) \cdot q(x) + r(x)$ and provided that the degree of r is strictly less than the degree of d. In terms of rational functions, we have

$$f(x) = \frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

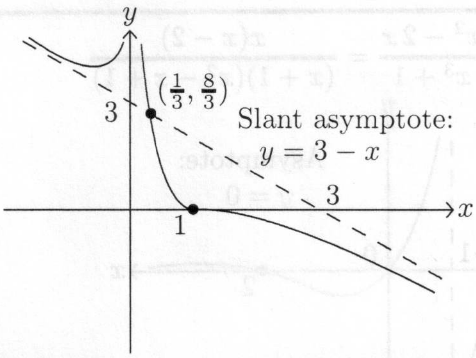
Because of the degree condition on r, it is clear that $\frac{r(x)}{d(x)}$ approaches zero as $x \longrightarrow \pm\infty$, so that $f(x)$ and $q(x)$ are close to each other when $|x|$ is large. Thus the graph of the rational function $f(x)$ is **asymptotic** to the graph of the polynomial $q(x)$ as $x \longrightarrow \pm\infty$. In other words, the two graphs are close to each other as $x \longrightarrow \infty$ and as $x \longrightarrow -\infty$. In the special case where the degree of p is one more than the degree of d, the quotient is a linear function, whose graph is a nonhorizontal line in the plane. That line is called an **oblique** or **slant** asymptote to the graph of the particular rational function.

Examples:

$$y = \frac{x^2 - \frac{3}{2}x}{2x+1} = \left(\tfrac{1}{2}x - 1\right) + \frac{1}{2x+1}$$



$$y = \frac{(1-x)^3}{x^2} = (3 - x) + \frac{1 - 3x}{x^2}$$



Problems: Determine all intercepts and asymptotes for the graphs of the following rational functions and use that information to help you sketch the graphs of the functions.

(a) $f(x) = \dfrac{2x^2}{1-x}$

(b) $f(x) = \dfrac{x^3 - 3x^2}{x^2 - 1}$

(c) $f(x) = \dfrac{(2+x)(2-3x)}{(2x+3)^2}$

(d) $f(x) = \dfrac{x^3 - 1}{x^2 - x - 2}$

(e) $f(x) = \dfrac{x^2 - 2x}{x^3 + 1}$

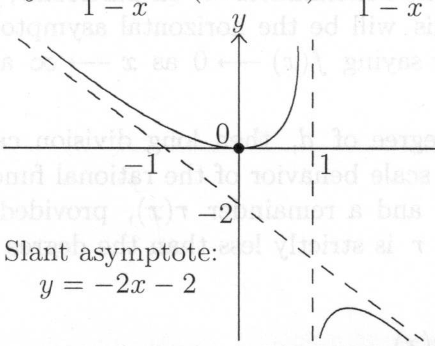
(f) $f(x) = \dfrac{1 - x^3}{x}$

(g) $f(x) = \dfrac{x^3 - 1}{2(x^2 - 1)}$

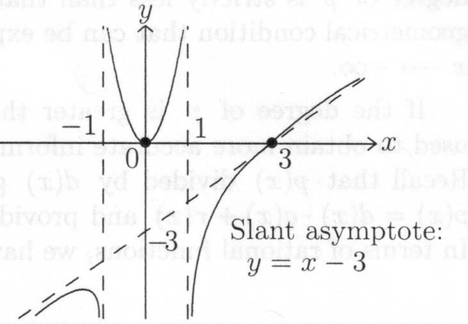
(h) $f(x) = \dfrac{x^4 - 2x^3 + 1}{x^2}$

ANSWERS TO SLANT ASYMPTOTE PROBLEMS

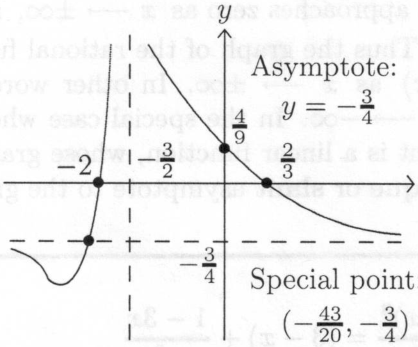
(a) $y = \dfrac{2\,x^2}{1-x} = (-2x-2) + \dfrac{2}{1-x}$



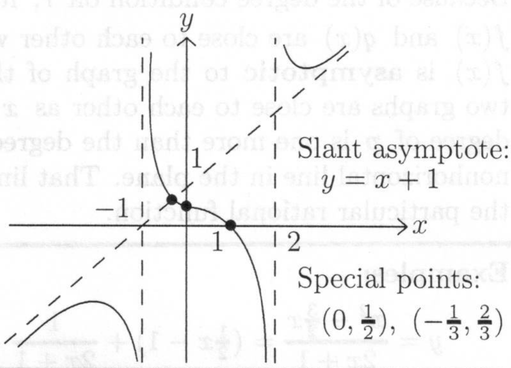
(b) $y = \dfrac{x^3 - 3x^2}{x^2 - 1} = (x-3) + \dfrac{x-3}{x^2-1}$



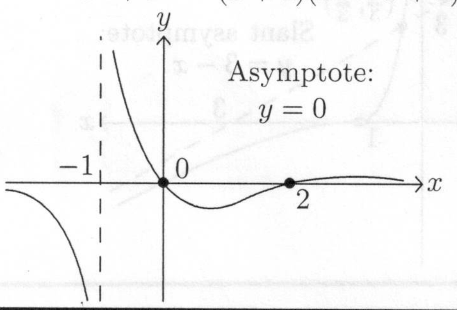
(c) $y = \dfrac{(2+x)(2-3x)}{(2x+3)^2} = (-\frac{3}{4}) + \dfrac{5x + \frac{43}{4}}{(2x+3)^2}$



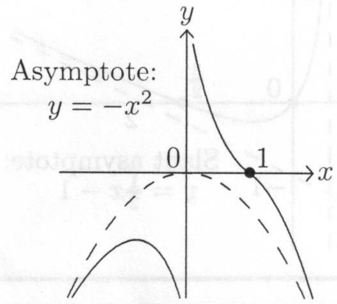
(d) $y = \dfrac{x^3 - 1}{x^2 - x - 2} = (x+1) + \dfrac{3x+1}{(x-2)(x+1)}$



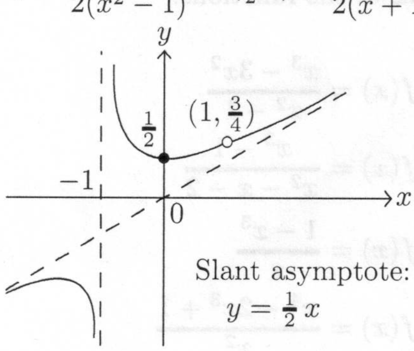
(e) $y = \dfrac{x^2 - 2\,x}{x^3 + 1} = \dfrac{x(x-2)}{(x+1)(x^2 - x + 1)}$



(f) $y = \dfrac{1 - x^3}{x} = -x^2 + \dfrac{1}{x}$



(g) $y = \dfrac{x^3 - 1}{2(x^2 - 1)} = (\frac{1}{2}\,x) + \dfrac{1}{2(x+1)},\ x \neq 1$



(h) $y = \dfrac{x^4 - 2x^3 + 1}{x^2} = (x^2 - 2x) + \dfrac{1}{x^2}$