Problems in Finding Derivatives and Tangent Lines

SOLUTION

1. Use the definition of the derivative to compute \( f'(1) \) for \( f(x) = \frac{13}{x + 7} \). NO CREDIT will be given for any other method.

Solution:
\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{13}{1 + h + 7} - \frac{13}{8}}{h} = \lim_{h \to 0} \frac{13 \cdot 8 - 13(8 + h)}{8h(8 + h)} = \lim_{h \to 0} \frac{-13h}{8h(8 + h)} = \lim_{h \to 0} \frac{-13}{8(8 + h)} = -\frac{13}{64}
\]

2. Compute the derivatives of the following functions. DO NOT SIMPLIFY.
(a) \( f(x) = (\sin x + x^2 + 1)(2x^3 + x)^2 \)
(b) \( g(x) = \frac{x^2 + 12x + e^3}{x + e^x} \)
(c) \( h(t) = e^{3t}(t^2 + x^2) \)

Solution:
(a) \( f'(x) = (\cos x + 2x)(2x^3 + x)^2 + 2(\sin x + x^2 + 1)(2x^3 + x)(6x^2 + 1) \)
(b) \( g'(x) = \frac{(2x + 12)(x + e^x) - (x^2 + 12x + e^3)(1 + e^x)}{(x + e^x)^2} \)
(c) \( h'(t) = 3e^{3t}(t^2 + x^2) + e^{3t}(2t) \)
3. Find $h'(1)$ where $h(x) = \frac{xg(x) + 7}{f(x)}$, $f'(1) = 4$, $g'(1) = -2$, and $f(1) = 1$, $g(1) = 1$. EXPREE YOUR ANSWER AS AN INTEGER.

Solution:

$$h'(x) = \frac{f(x)(xg(x) + 7)' - (xg(x) + 7)f'(x)}{[f(x)]^2}$$

$$= \frac{f(x)g(x) + xg'(x) - [xg(x) + 7]f'(x)}{[f(x)]^2}$$

∴ $h'(1) = \frac{f(1)[g(1) + g'(1)] - [g(1) + 7]f'(1)}{[f(1)]^2}$

$$= \frac{1[1 + (-2)] - [1 + 7]4}{1^2}$$

$$= -33$$

4. Find the equation of the tangent line to $y = f(x) = \frac{x + 3}{2x + 1}$ at the point corresponding to $x = 0$.

Solution:

The slope of the tangent line at $x = 0$ is $f'(0)$.

$$f'(x) = \frac{(2x + 1) - 2(x + 3)}{(2x + 1)^2}.$$ 

∴ $f'(0) = \frac{1 - 6}{1} = -5$.

Therefore, the equation of the tangent line is: $y = -5x + b$.

At $x = 0$, $y = f(0) = \frac{3}{1} = 3$. Therefore $3 = 0 + b$, i.e. $b = 3$.

The equation of the tangent line is: $y = -5x + 3$. 

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5. Find the $y$-intercept of the tangent line to the curve $y = x^3 + 1$ at the point $(2, 9)$.

Solution:

$y'(x) = 3x^2$.

At the point $(2, 9)$, $y'(2) = 3 \cdot 2^2 = 12$.

Therefore the equation of the tangent line is: $y = 12x + b$.

Hence, the $y$-intercept $b = y - 12x = 9 - 12 \cdot 2 = -15$.

6. Find the $x$ and $y$ coordinates of the point on the graph of $y = \frac{1}{4}(2x + 1)^2$ where the tangent line is parallel to the line $y - 3x = 1$.

Solution:

The slope of the tangent line of $y = \frac{1}{4}(2x + 1)^2$ is:

$y'(x) = \frac{1}{4} \cdot 2(2x + 1) \cdot 2 = 2x + 1$.

Since the tangent line is parallel to the line $y - 3x = 1$, the tangent line has a slope equal to 3.

Therefore $y'(x) = 2x + 1 = 3 \implies x = 1$.

$y = \frac{1}{4}(2 \cdot 1 + 1)^2 = \frac{9}{4}$.

The $x, y$ coordinates are $x = 1, y = \frac{9}{4}$. 

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