Review session problems

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December 3, 2013

1 Estimate the value of $\sqrt{26}$ using linear approximation. Is this an underestimate or an overestimate? Use the information to determine an interval which you can be sure contains the exact value $\sqrt{26}$.

(a) : 26 is close to 25, square root of 25 is easy!

Let $f(x) = \sqrt{x}$; we take $a = 25$ to be the centre.

Want to approximate $f(x_0) = f(26)$, so $x_0 = 26$.

$f(a) = f(25) = 5$.

$f'(x) = \frac{1}{2} x^{-1/2}$, $f'(a) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$.

$f''(x) = -\frac{1}{4} x^{-3/2}$.

$L(x) \approx f(x)$ for x close to 25.

$L(x) = 5 + \frac{1}{10}(x - 25)$; $L(26) \approx f(26)$

$L(26) = 5 + \frac{1}{10}(26 - 25) = 5 + \frac{1}{10} = \underline{\underline{5.1}}$

$f''(x) = -\frac{1}{4} x^{-3/2}$, $f''(x) < 0$ for x in $(25, 26)$, so concave down, hence $L(26) = 5.1$ is an overestimate.

2

$E(x) = f(x) - L(x)$

$$|E(x)| \le \frac{f''(x)}{2}(x-a)^2.$$

To bound $E(x)$, we need to find an upper bound for $|f''(x)|$.

$$|f''(x)| = +\frac{1}{4x^{3/2}}; \quad |f''(25)| = \frac{1}{4\times 5^3} = \frac{1}{4\times 125} = \frac{1}{500} = 0.002$$

$f''(x)$ is decreasing for x in $(25, 26)$ as $f'''(x) < 0$ for x in this interval. Hence

$$|f''(x)| \le |f''(25)| = 0.002.$$

$$|E(x)| \le \frac{0.002}{2}(x-25)^2;$$

$$|E(x_0)| = |E(26)| \le 0.001\,(26-25)^2 = 0.001$$

Hence $|E(26)| \le 0.001$.

This means $-0.001 < E(26) < 0.001$.

$E(26) = f(26) - L(26) = f(26) - 5.1$

$-0.001 < E(26) < 0.001 \Rightarrow -0.001 < f(26) - 5.1 < 0.001$

$\Rightarrow 5.1 - 0.001 < f(26) < 5.1 + 0.001$

$\Rightarrow 5.099 < f(26) < 5.106.$

Hence $f(26)$ is contained in the open interval $(5.099, 5.101)$. The linear approximation is an overestimate, hence we know that $f(26) < 5.1$, so $f(26)$ is contained in the smaller interval $(5.099, 5.1)$

2 A steel company ABC steel manufactures nuts and bolts. When x nuts are produced, they can be sold for $-3x+500$ dollars each, When y bolts are produced, they can be sold for $-y+300$ dollars each. Assume that nuts and bolts weight 0.5 kg each. How many nuts and how many bolts must be produced to maximize the revenue from 100kg of steel? Justify your answer.

Let R be the revenue. Then

$R = (-3x+500)x + (-y+300)y$

Constraint: Total weight $= 100$ kg; so $\frac{x}{2} + \frac{y}{2} = 100$

$\Rightarrow x+y = 200 \Rightarrow y = 200-x$; $0 \le x \le 200$.

So $R(x) = (-3x+500)x + (-(200-x)+300)(200-x)$, $0 \le x \le 200$.

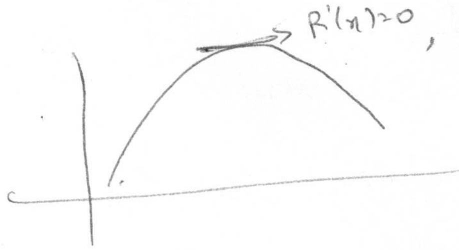
$R(x) = -4x^2 + 600x + 20000$,

$R'(x) = -8x + 600$;

Maximise: $R'(x) = 0 \Rightarrow x = \frac{600}{8} = 75$,

$y = 125$.

$R'(x)=0$, Revenue maximised.



Hence the company should produce 75 nuts and 125 bolts.

Asymptotes: Always take out largest power of x in the denominator to compute $\lim_{x\to\pm\infty}$.

$$f(x) = \frac{p(x)}{q(x)} \; ; \; p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 , \quad a_m \neq 0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 , \quad b_n \neq 0 .$$

$m < n$: $\lim_{x\to\pm\infty} f(x) = 0$ Horizontal asymptote : $y = 0$

$m = n$: $\lim_{x\to\pm\infty} f(x) = \frac{a_m}{b_n}$ Horizontal asymptote : $y = \frac{a_m}{b_n}$.

$m > n$: $\lim_{x\to\pm\infty} f(x) = \infty$ or $-\infty$; No horizontal asymptote.

If $f(x)$ is in reduced form, then vertical asymptotes occur at zeros of $q(x)$.

3 Consider the curve $x^2 + y^3 - 2xy = 0$. Assume that the point $(x, y) = (1, 1)$ lies on the curve, and that nearby points on the curve satisfy $y = f(x)$ for some function of $f(x)$. Find $f(1)$ $f'(1)$ and $f''(1)$. Approximate $f(1.02)$ using the linear approximation of $f(x)$ at $x = 1$. State whether you expect this to be an overestimate or an underestimate of $f(1.02)$.

Ans : $x^2+y^3-2xy=0$; Implicit differentiation:

$$2x + 3y^2y' - 2y - 2xy' = 0 \quad — \ (*)$$

At $x=y=1$, get $\cancel{2} + 3y' - \cancel{2} + 2y' = 0 \Rightarrow y' = 0$.

Hence $f'(1) = 0$; $f(1) = y = 1$.

$f''(1)$: Differentiating $*$ again, we get

$$2 + 6yy'^2 + 3y^2y'' - 2y' - 2xy'' = 0. \quad \text{Put } x=y=1,\ y'=0;$$

get $2 + 3y'' - 2y'' = 0 \Rightarrow y'' + 2 = 0 \Rightarrow y'' = -2 \Rightarrow f''(1) = -2$.

$L(x) = f(1) + f'(1)(x-1) = 1 + 0\,(x-1) = 1$.

Hence $L(1.02) = 1$.

This is expected to be an overestimate as $f''(1) < 0$.

(4) Consider the function $f(x) = \frac{2x^3}{3x^2-9}$. Its first and second derivatives are given by

$$f'(x) = \frac{2x^2(x^2-9)}{3(x^2-3)^2}, \quad f''(x) = \frac{4x(x^2+9)}{(x^2-3)^3}.$$

Find the intervals of increase and decrease. On which intervals is $f(x)$ concave up (resp. concave down)? Find the x-coordinate of all local min, local max, and inflection points. Write down the horizontal and vertical asymptotes. Draw a rough sketch of the graph of $f(x)$.

Ans :

f' :

$+$		$-$		$-$		$-$		$-$		$+$
	-3		$-\sqrt{3}$		0		$\sqrt{3}$		3	

$f'(x)$ and $f''(x)$ defined everywhere on $\mathbb{R}$ except at $\pm\sqrt{3}$.

Intervals of interest : $(-\infty, -3)$, $(-3, -\sqrt{3})$, $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$, $(\sqrt{3}, 3)$

Intervals of increase : $(-\infty, -3)$; $(3, \infty)$ (As $f' > 0$.

Intervals of decrease : $(-3, -\sqrt{3})$, $(-\sqrt{3}, 0)$, $(0, \sqrt{3})$, $(\sqrt{3}, 3)$.

f'' :

$-$		$+$		$-$		$+$
	$-\sqrt{3}$		0		$\sqrt{3}$	

Concave up : $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$ (as $f'' > 0$)

Concave down : $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$ (as $f'' < 0$).

Local max at $x = -3$: (f' changes from $+$ to $-$)

Local min at $x = 3$ (f' changes from $-$ to $+$).

Inflection pt : $x = 0$.

<u>Asymptotes</u> : $f(x) = \dfrac{2x^3}{3x^2 - 9}$ $3 > 2$; No horizontal asymptotes.

$$\lim_{x \to \pm\infty} = \lim_{x \to \pm\infty} \frac{\cancel{x^2}(2x)}{\cancel{x^2}\left(3 - \frac{9}{x^2}\right)} = \pm\infty$$

(7)

Intercepts: x-intercept: where $y=0$; $y=f(x)=\frac{2x^3}{3x^2-9}$; so $x=0$.

y-intercept: where $x=0$; $y=0$.

Vertical Asymptotes: zeroes of $3x^2-9$

$3x^2-9=0 \Rightarrow x^2-3=0 \Rightarrow (x+\sqrt{3})(x-\sqrt{3})=0$

Vertical asymptotes: $x=\pm\sqrt{3}$.

Graph: