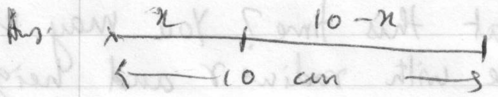
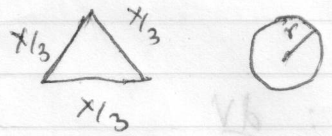
Eg: A piece of wire 10 cm long is cut into two pieces. One piece is bent into a circle while the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimal? Maximal?

Ans:

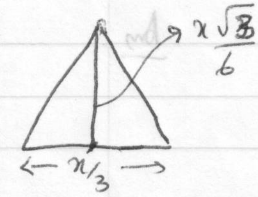


Constraint : $0 \le x \le 10$.



$2\pi r$ = Circumference of Circle

$= 10 - x; \quad r = \dfrac{10-x}{2\pi}$



$$\text{Area of triangle} = \frac{1}{2} \times \frac{x}{3} \times x\sqrt{3}/6$$

$$\text{Area of Circle} = \pi r^2 = \pi \frac{(10-x)^2}{4\pi^2} = \frac{(10-x)^2}{4\pi}$$

$$A(x) = \text{Total area} = \frac{x^2\sqrt{3}}{36} + \frac{(10-x)^2}{4\pi} \quad \Big| \quad A'(x) = \frac{x\sqrt{3}}{18} - \frac{(10-x)}{2\pi}$$

$$\cancel{A'_c(x) = \frac{x\sqrt{3}}{36} - \frac{(10-x)}{2\pi}}$$

$$A'(x) = 0 \Rightarrow \frac{x\sqrt{3}}{18} - \frac{(10-x)}{2\pi} = 0 \Rightarrow \frac{x\sqrt{3}}{9} - \frac{(10-x)}{\pi} = 0$$

$$\Rightarrow 90 - 9x = \pi x\sqrt{3} \Rightarrow 90 = x(9 + \pi\sqrt{3}) \Rightarrow x = \frac{90}{(9+\pi\sqrt{3})} \approx 6.23.$$

$$A(0) = \frac{\sqrt{3} \times 0}{36} + \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.96$$

$$A(10) = \frac{100\sqrt{3}}{36} = \frac{25\sqrt{3}}{9} \approx 4.82.$$

So $x \approx 6.23$ is the point

$$A\left(\frac{90}{9+\pi\sqrt{3}}\right) \approx 3.00$$

that minimises the area to $3\,cm^2$

and $x=0$ is the point that maximises the area to $7.96\,cm^2$.

§ (4) A right circular cone is turned upside down (vertex is at the bottom); the height of cone is 10cm and the diameter at the top is also 10cm. The cone is filled half-way (by depth), and water is draining out. The depth of the water is dropping at a rate of 4cm per second. How fast is the volume of the water changing at this time? You may use the fact that the volume of a cone with radius r and height h is $\frac{\pi r^2 h}{3}$.

(Related rates)

Ans

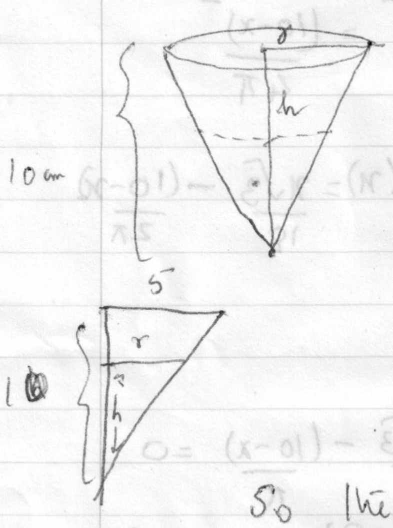
Want : $\frac{dV}{dt}$

$$V = \frac{\pi r^2 h}{3}$$

$$= \frac{\pi h^3}{4 \times 3} = \frac{\pi h^3}{12}$$

Given: $\frac{dh}{dt} = 4\ \text{cm/sec}$, $r = 5\text{cm}$

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}\ h \Rightarrow r = \tfrac{1}{2} h$$



$$\frac{dV}{dt} = \frac{1}{12} \cdot 3\pi \cdot h^2 \times \frac{dh}{dt} = \frac{\pi}{4} h^2 \times \frac{dh}{dt}$$

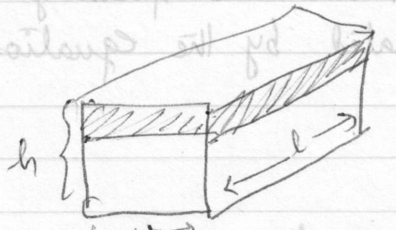
$$\left.\frac{dV}{dt}\right|_{h=10} = \left(\frac{\pi}{4}\right)(100) \times \frac{dh}{dt}, \quad \frac{dh}{dt} = 4$$

$$\frac{dV}{dt} = \frac{\pi}{4} \times 100 \times (-4) = -100\pi \quad (- \text{ as Volume} \downarrow)$$

So the volume is changing at the rate of $-100\pi\ \text{cm}^3/\text{sec}$.

(5) A shoebox, complete with overlapping lid, has a base with length $5/3$ times its width. The rim of the lid overlaps the side of the main box by $1/5$ of the height of the box. Assuming the thickness of the box is negligible and of uniform density, and the total volume of the box is $3000\ \text{cm}^3$. What dimensions will result in the minimum weight of the empty box?

Ans



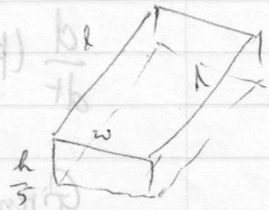
Total Volume $= lhw = 3000 \text{ cm}^3$

Given: $l = \frac{5}{3} w$ (Constraints)

$$V = \frac{5}{3} \times w \times h \times w = \frac{5}{3} w^2 h = 3000 \text{ cm}^3$$

$$h = \frac{1800}{w^2} = \left(\frac{3000 \times 3}{5 w^2}\right)$$

Area of Cardboard is proportional to the weight.



Area of bottom of box $= wl + 2lh + 2wh$

— " — lid $= wl + 2l\frac{h}{5} + 2w\frac{h}{5}$

$$A = \text{Total area} = 2wl + 2lh\left(\frac{6}{5}\right) + 2wh\left(\frac{6}{5}\right)$$

$$= 2wl + (2l + 2w)\frac{6}{5}h.$$

$h = \frac{1800}{w^2}$ $\quad l = \frac{5}{3}w$

$$\Rightarrow A(w) = 2w\left(\frac{5}{3}w\right) + \left(2\left(\frac{5}{3}w\right) + 2w\right) \times \frac{6}{5} \times \frac{1800}{w^2}$$

$$= \frac{10}{3}w^2 + \frac{16}{3}w \times \frac{6}{5} \times \frac{1800}{w^2}$$

$$= \frac{10}{3}w^2 + \frac{2160}{w}.$$

Goal: Minimise $A(w)$

~~Wish~~

$$A'(w) = 0 \Rightarrow \frac{20}{3}w - \frac{11520}{w^2} = 0$$

$$\Rightarrow w^3 - 1728 = 0 \Rightarrow w = 12. \text{ When } w = 12,\ l = 20,\ h = 12.5.$$

Depth of lid $= \frac{1}{5}(12.5) = 2.5$.

Local minimum? $A''(w) = \frac{20}{3} + \frac{2(11520)}{w^3}$ $\Big|$ $w = 12$ $A''(w) > 0$.

(6) The price p in \$ for a product and the quantity q of demand for a product are related by the equation

$$p^2+2q^2=1100.$$

If the price is increasing at a rate of \$2 per month when the price is \$30, find the rate of change of the revenue R in dollars per month.

Ans: $p^2+2q^2=1100$ ——— (1)

$$\frac{d}{dt}(p^2+2q^2) = 2p\frac{dp}{dt} + 4q\frac{dq}{dt} = 0 \quad \text{——— (2)}$$

Given $\frac{dp}{dt}=2$; and $p=30$. Substituting in (1), we get

$(30)^2+2q^2=1100$; $\quad q^2=100 \quad \Rightarrow q=10.$

Substitute in (2), put $p=30$, $\frac{dp}{dt}=2$. Get

$$60\times 2 + 40\times\frac{dq}{dt}=0 \quad \Rightarrow 40\frac{dq}{dt}=-120 \Rightarrow \frac{dq}{dt}=-3.$$

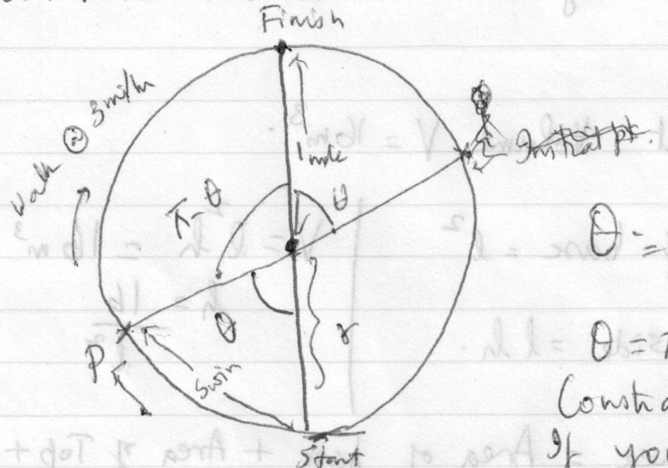
Want: $\frac{dR}{dt}$: $\quad R=pq$; $\frac{dR}{dt} = p\frac{dq}{dt} + q\frac{dp}{dt}$

$$= 30(-3)+10(2) = -90+20 = -70$$

So the rate of change of revenue is \$70 per month & the revenue is decreasing.

Eg: Suppose you are standing on the shore of a circular pond with radius 1 mi and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 mi/hr from your current position to another point P on the shore and then walk at 3 mi/hr along the shore to the terminal point. How should you choose P to minimize the total time for the trip?

Ans



Central angle θ

$\theta = 0$, then you are walking

$\theta = \pi$: You are swimming the length.

Constraint : $0 \le \theta \le \pi$.

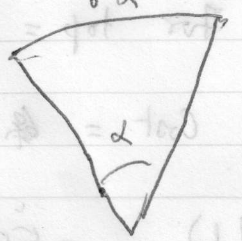
If you swim at an angle θ, then the total time varies with θ.

~~An~~ chord length = length swam = $2r\sin(\theta/2)$

Time for swimming leg, $r = 1$, speed = 2 mi/hr

$$= \frac{\text{distance}}{\text{rate}} = 2\,\frac{\sin(\theta/2)}{2} = \sin\theta/2.$$

Walking leg: arc length



$$\text{Time} = \frac{1\times(\pi-\theta)}{3}$$

$$T(\theta) = \sin\frac{\theta}{2} + \frac{\pi-\theta}{3}, \quad 0 \le \theta \le \pi.$$

Min Travel time is when $\theta = \pi$, $T(\pi) = 1$ hr. / Entire Trip swimming

Max Travel time $\theta \approx 96°$; $T \approx 1.23$ hr

$$\frac{dT}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{3} = 0 \quad \text{or} \quad \cos\frac{\theta}{2} = 2/3 \ ; \quad \theta \approx 1.68 \text{ rad} \approx 96°$$

$\theta \in [0, 1]$.

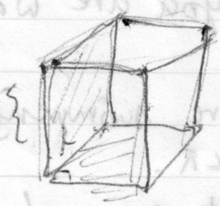
$T(\theta) = T(96°) \approx 1.23$ h ; $T(0) = \pi/3 = 1.05$ hr, $T(\pi) = 1$ hr.

②

Eg. A box-shaped shipping crate with a square base is designed to have a volume of $16\,m^3$. The material used to make the base costs twice as much per square metre as the material in the sides, and the material used to make the top costs half as much per square metre as the material in the side. Suppose the material to make the sides costs \$20 per square metre. What are the dimensions of the crate that minimize the cost of the materials?

Ans

Constraint: Volume $V = 16\,m^3$.



Area of base $= l^2$ | $V = l^2 h = 16\,m^3$

Area of side $= lh$. | $h = \frac{16}{l^2}$

Total area = Area of base + Area of Top + Area of sides

$= l^2 + l^2 + 4lh = 2l^2 + \frac{64}{l}$

Cost of material for sides $= 20 \times 4 \times lh = 80\,lh = 80 \times l \times \frac{16}{l^2} = \frac{1280}{l}$

For base $= 40 \times l^2$

~~For side~~

For top $= 10 \times l^2$.

Total Cost $= \frac{1280}{l} + 40l^2 + 10l^2$

$C(l) = 50l^2 + \frac{1280}{l}$; $C'(l) = 100l - \frac{1280}{l^2}$

$C'(l) = 0 \Rightarrow 100l^3 - 1280 = 0 \Rightarrow 100l^3 = 1280$

$\Rightarrow l^3 = \frac{128}{10} = \frac{64}{5}$

$C''(l) = 20\left(\frac{128}{l^3} + 5\right) > 0$. So $l = (64/5)^{1/3}$, $h = \frac{16}{\left(\frac{64}{5}\right)^{2/3}} = 5^{2/3}$

Minimum: $l = \left(\frac{64}{5}\right)^{1/3}$. Area of side $= lh = \frac{16}{l^2}\,l$

<u>Weeks 11 & 12</u>

<u>Linear approximation</u>: (Quadratic approximation)

Find the quadratic approximating polynomial for $f(x)=\sqrt{x}$, centred at $a=4$ and use it to approximate $\sqrt{3.9}$.

<u>Ans</u> $p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$f(a) = f(4) = \sqrt{4} = 2.$

$f'(a): \; f'(x) = +\frac{1}{2}\cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$; $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f''(\cancel{x}) = \frac{1}{2} \times \frac{-1}{2}\frac{1}{x^{3/2}} = \frac{-1}{4} \times \frac{1}{x^{3/2}}$; $f''(4) = \frac{-1}{4}\left(\frac{1}{8}\right) = \frac{-1}{32}$

$p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$

3.9 is close to 4, so $p_2(x) \approx f(x)$.

$p_2(3.9) = 2 + \frac{1}{4}(3.9-4) - \frac{1}{64}(3.9-4)^2$

$= 2 + \frac{1}{4}(-0.1) - \frac{1}{64}(0.01)$

$= 2 - \frac{0.1}{4} - \frac{0.01}{64} \approx 1.97484375.$

$p_1(3.9) = 2 + \frac{1}{4}(3.9-4) = 1.975.$

Derivatives of inverse trigonometric functions:

Evaluate the derivatives of a) $f(y) = \tan^{-1}(2y^2-4)$;

b) $f(t) = \sin(\sec^{-1}(2t))$.

Ans : a) $f(y) = \tan^{-1}(2y^2-4)$. Put $u = 2y^2-4$; $f(u) = \tan^{-1} u$

$$\frac{df}{dy} = \frac{df}{du} \cdot \frac{du}{dy}$$

$$f'(y) = \frac{1}{1+u^2} \times 4y$$

$$= \frac{1}{1+(2y^2-4)^2} \times 4y \quad //$$

b) $f(t) = \sin(\sec^{-1}(2t))$

$u = \sec^{-1}(2t)$; $f(t) = \sin(u)$

$$\frac{df}{dt} = \frac{df}{du} \times \frac{du}{dt} = \cos u \times \frac{1}{|2t|\sqrt{4t^2-1}} \times 2$$

$$= \cancel{\cos(\sec^{-1}(2t))} \times \cancel{\frac{2}{|\sec^{-1} 2t|\sqrt{(\sec^{-1} 2t)^2-1}}} \quad //$$

$$= \cos(\sec^{-1}(2w)) \times \frac{2}{|2w|\sqrt{4w^2-1}} = \frac{(1/w)}{2|w|\sqrt{4w^2-1}}$$

$$\left(\cos(\sec^{-1} 2w)\right) = \frac{1}{\sec(\sec^{-1} 2w)} = \frac{1}{2w} ; \quad 2 \times \cos(\sec^{-1}(2w)) = \frac{1}{w},$$

a) Find an equation of the tangent line to the curve

$$x^3+xy^2+y^3=13$$

at the point (1,2) on the curve. Please simplify.

(b) Use a suitable linear approximation to estimate the x co-ordinate of the point on the curve whose y coordinate is $\frac{31}{16}$. A calculator-ready answer is enough.

Ans a) $f(x) = 3x^2 + y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$$(2xy+3y^2)\frac{dy}{dx} = -3x^2-y^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2-y^2}{2xy+3y^2}$$

Tangent line at (1,2): Slope $= \frac{-3-4}{4+12} = -\frac{7}{16}$

Equation: $(y-2) = -\frac{7}{16}(x-1)$.

(b) x-coordinate when $y = \frac{31}{16}$: Difference between y-coordinates: $2 - \frac{31}{16} = -\frac{1}{16}$

"dy"; want "dx"; "$dx = \frac{dy}{dx} \cdot dx$".

$$\left(\frac{31}{16}-2\right) = -\frac{7}{16}(x-1) \Rightarrow (31-32) = -7(x-1)$$

$$\Rightarrow -1 = -7(x-1)$$

$$\Rightarrow 1 = 7(x-1) \Rightarrow 1 = 7x-7 \Rightarrow x = \frac{8}{7}.$$

So x is close to 1. Using linear approximation, the ~~Hence since $\frac{8}{7}$ is at~~ tangent line approximates f near $x = \frac{8}{7}$.

Eg: a) Let $f(x)$ be a function with the property that $f'(x) = \frac{x-1}{x+1}$.
If $g(x) = f(e^x)$, compute $g'(x)$.

Ans: $g'(x) = f'(e^x) \cdot e^x$

$$= \left(\frac{e^x - 1}{e^x + 1}\right) e^x.$$

b) Compute the absolute value of the difference between the calculator value of $\sqrt[3]{28}$ and the approximate value of $\sqrt[3]{28}$ computed using the linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 27$.

$f(27) = 3$

Ans: $\sqrt[3]{27} = 3$; so can use linear approximation near 27 as 28 is close to 27.

$f(x) = \sqrt[3]{x}$; $f(x) = x^{1/3}$; $f'(x) = \frac{1}{3} x^{-2/3}$

~~$P_1(x) = f$~~

~~$f'(28) =$~~

$p_1(x) = f(a) + (x-a) f'(a)$, $a = 27$

$p_1(x) = f(27) + (x-27) \times f'(27)$ | $f'(27) = \frac{1}{3}\left(27^{-2/3}\right)$

$= \frac{1}{3}\left(3^{-2}\right) = \frac{1}{3} \times \frac{1}{9} = \frac{1}{27}$

$= 3 + (x-27)\left(\frac{1}{27}\right)$

$p_1(28) = 3 + (28-27)\left(\frac{1}{27}\right)$

$= 3 + \frac{1}{27} = \frac{82}{27} \approx$ ~~3.036~~ ≈ 3.037

Calculator: 3.036588 | "Overestimate".

Eg: Use linear approximation to estimate $\sqrt{0.18}$.

Ans: $f(x) = \sqrt{x}$; we must first find an a close to $\sqrt{0.18}$

such that $f(a)$ is easy to compute. Clearly $a = 0.16$ is good,

$f(x) = \sqrt{0.16} = 0.4$.

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \;;\; f'(a) = \frac{1}{2\sqrt{a}} = \frac{1}{2 \times 0.4} = \frac{1}{0.8}.$$

$$f(x) \approx p_1(x) \;;\; p_1(x) = f(a) + f'(a)(x-a)$$

$$p_1(x) = f(0.16) + f'(0.16)(0.18 - 0.16)$$

$$= 0.4 + \frac{1}{0.8}(0.02) = 0.4 + \frac{1}{40}$$

$$\approx 0.4 + 0.025 = 0.425.$$

$$f(x) \approx p_1(x) \;;\; \sqrt{0.18} \approx 0.425.$$

$\sqrt{0.18} \approx 0.42426$

Overestimate.

Eg: Find the linear approximation to $f(x) = \sin x$ at $x = 0$,

and use it to approximate $\sin 2.5°$.

degrees ⟷ radians

$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

Ans: $p_1(x) = f(a) + f'(a)(x-a)$; $f(x) = \sin x$, $a = 0$.

radians → degrees

degrees = radians × 180/π

degrees → radians

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

$f(0) = 0$. $f'(x) = \cos x$; $f'(0) = 1$.

$p_1(x) = (x - 0) = x$.

$$2.5° = 2.5\left(\frac{\pi}{180°}\right) = \frac{\pi}{72} \approx 0.04363 \text{ rad}.$$

$\sin 2.5° \approx 0.04362$!

$\sin(2.5°) \approx p_1(0.04363)$
$= 0.04363$.

Eg. Estimate $f(2\cdot1)$ when $f(x)=12-x^2$, $a=2$ and compute the percent error.

Ans $f(a)=f(2)=12-4=8.$

$f'(x)=-2x;\quad f'(a)=-2(2)=-4.$

$p_1(x)=f(a)+(x-a)f'(a) \quad = 8+(x-2)(-4)$
$= 8-4x+8) = 16-4x.$

$f(2\cdot1) \approx p_1(2\cdot1) = 16-(8.4) = 7\cdot6$

Calculator: $f(2\cdot1)=12-(2\cdot1)^2 = 12-4\cdot4) \approx 7\cdot59$!

% error: $100\times\dfrac{|f(x)-p_1(x)|}{f(x)} \doteq \dfrac{f(2\cdot1)-p_1(2\cdot1)}{7\cdot59}\times100 = \dfrac{|7\cdot59-7\cdot6|}{7\cdot59}\times100$
$\approx 0.13\,\%$.

% error: $100\times\dfrac{|f(x)-p_1(x)|}{|f(x)|}$

$= \dfrac{100\times|(\text{approximation}-\text{exact})|}{|\text{exact}|}.$

Eg: Use the notation of differentials to write the approximate change in $f(x)=3\cos^2x$ given a small change dx.

Ans: $y=f(x)$; $\quad dy=f'(x)\,dx$; $\quad f'(x)=-6\cos x\sin x$

$dy=-6\cos x\sin x\,dx.$

Eg. Estimate $f(2\cdot1)$ when $f(x)=12-x^2$, $a=2$ and compute the percent error.

Ans. $f(a)=f(2)=12-4=8.$

$f'(x)=-2x;\quad f'(a)=-2(2)=-4.$

$p_1(x)=f(a)+(x-a)f'(a) \quad = 8+(x-2)(-4)$
$= 8-4x+8) = 16-4x.$

$f(2\cdot1) \approx p_1(2\cdot1) = 16-(8.4) = 7\cdot6$

Calculator: $f(2\cdot1)=12-(2\cdot1)^2 = 12-4\cdot41 \approx 7\cdot59$!

% error: $100\times\dfrac{|f(x)-p_1(x)|}{f(x)} = \dfrac{f(2\cdot1)-p_1(2\cdot1)}{7\cdot59}\times100 = \dfrac{|7\cdot59-7\cdot6|}{7\cdot59}\times100$
$\approx 0.13\%.$

% error: $100\times\dfrac{|f(x)-p_1(x)|}{|f(x)|}$

$= \dfrac{100\times|(\text{approximation}-\text{exact})|}{|\text{exact}|}.$

Eg. Use the notation of differentials to write the approximate change in $f(x)=3\cos^2 x$ given a small change dx.

Ans: $y=f(x)$; $\quad dy=f'(x)\,dx$; $\quad f'(x)=-6\cos x\sin x$

$dy=-6\cos x\sin x\,dx.$

Eg 3) Approximate the change in the volume of a sphere when its radius changes from $r = 5$ ft to $r = 5.1$ ft $\left(V(r) = \frac{4}{3}\pi r^3\right)$.

Ans: Need $\frac{dV}{dr}$: $\frac{dv}{dr} = \frac{4}{\cancel{3}} \times \cancel{3} \times \pi r^2 = 4\pi r^2$.

ΔV = change in volume ; $\Delta V \approx V'(a)\,\Delta r$; $a = 5$ ft

Δr = change in radius $= 0.1$ ft.

$V'(a) = 4\pi a^2 = 4\pi(25) = 100\pi$

$\Delta V = 100\pi \times 0.1 = 10\pi \approx 31.42\,ft^3$.

Eg 4) Consider the following functions and express the relationship between a small change in x and the corresponding change in y in the form $dy = f'(x)\,dx$.

a) $f(x) = 1/x^3$; $f'(x) = -\frac{3}{x^4}$; so $dy = -\frac{3}{x^4}\,dx$.

(b) $f(x) = \tan x$; $f'(x) = \sec^2 x$, so $dy = \sec^2 x\,dx$.

Eg 5 : Explain why or why not :

(a) The linear approximation to $f(x) = x^2$ at the point $(0,0)$ is $L(x) = 0$

Ans. $f'(x) = 2x$; at $x = 0$, $f'(x) = 0$; $a = 0$; $f(a) = f(0) = 0$

$L(x) = f(a) + f'(a)(x - a)$

$= 0 + 0 \cdot (x - 0) = 0$

True.

(b) If $f(x) = mx + b$, then at any point $x = a$, the linear approximation to f is $L(x) = f(x)$.

True ; since $f(x)$ is itself linear.

use

(5) $f(x) = 1/(x+1)$; $a = 0$; ~~$1/x+1$~~, estimate $\frac{1}{1.1}$

$f'(x) = -\frac{1}{(x+1)^2}$; $f'(a) = -1$; $f(a) = 1$

$p_1(x) \approx f(x)$; $p_1(x) = f(a) + x f'(a)$ | $p_1(x) = f(a) + (x-a) f'(a)$, $a=0$

$p_1(x) = 1 - x$

$p_1\left(\frac{1}{1.1}\right) \approx 1 - \frac{1}{1.1} = \frac{1.1-1}{1.1} = \frac{0.1}{1.1} = \frac{1}{11} = 0.9$.

Percentage error $= \frac{\left|0.9 - \frac{1}{1.1}\right|}{\frac{1}{1.1}} = 1\%$

(c) $f(x) = e^{-x}$; $a = 0$, estimate $e^{-0.03}$

$f'(x) = -e^{-x}$; $f'(a) = -1$; $f(0) = 1$

$p_1(x) = f(a) + (x-0) f'(0) = 1 + x(-1) = 1 - x$. $p_1(x) \approx f(x)$

$p_1(0.03) = 1 - 0.03 = 0.97$

% error: $\frac{\left|0.97 - e^{-0.03}\right|}{e^{-0.03}} \times 100 \approx 0.046\%$.

Eg: $f(x) = x/(x+1)$; use linear approximation to ~~appro~~ estimate $f(1.1)$.

Ans: (1.1) is close to 1; so compute $p_1(x)$ with $a = 1$; $f(1) = \frac{1}{2}$

$f'(x) = \frac{1}{(1+x)^2}$

$p_1(x) \approx f(x)$; $p_1(x) = \frac{1}{2} + (x-1)\left(\frac{1}{4}\right) = \frac{1}{4}(x+1)$.

$f'(1) = \frac{1}{4}$.

$p_1(1.1) \approx 0.525$

$f(1.1) = \frac{1.1}{2.1}$

% age error $= 100 \times \frac{\left|0.525 - \left(\frac{1.1}{2.1}\right)\right|}{\frac{1.1}{2.1}} \approx 0.23\%$.

Errors:

Recall: $p_1(x) = f(a) + f'(a)(x-a)$ is the linear approximation for $f(x)$ centered around a.

Eg $f(x) = \sin x$; x = measured in radians, $a = 0$.

$p_1(x) = x$; $\sin x$; x = measured in radians; $a=0$.

$p_1(x) = x$; so $\sin x \approx x$ near zero when x is expressed in radians.

Eg: Write down the linear and quadratic approximation for $f(x) = \ln(1+x)$ and use it to approximate $\ln(1.05)$.

Ans: $f(x) = \ln(1+x)$; $a = 0$ is the centre! This is because $1.05 = 1 + 0.05$ and 1.05 is close to 1.

$f(0) = \ln(1) = 0$; $f'(x) = \frac{1}{1+x}$; $f'(0) = 1$; $f''(x) = \frac{-1}{(1+x)^2}$; $f''(0) = -1$.

$p_1(x) = x$; $p_2(x) = x - \frac{1}{2}x^2$.

$\ln(1.05) = f(0.05)$; $p_1(0.05) = 0.05$; $p_2(x) = 0.05 - \frac{1}{2}(0.0025)$
$p_2(x) = 0.0525$, $p_1(x) \approx f(x)$; $p_2(x) \approx f(x)$.

Eg: Consider the curve $x^2 + 2y^2 = 1100$. Consider the point $(30, 10)$ on the curve and assume that nearby points are of the form $y = f(x)$ for some function $f(x)$. Use quadratic approximation to find y when $x = 29.5$.

Ans. Given $P = (30, 10)$ is a point on the curve. Notice that 29.5 is close to 30. So $(29.5, f(29.5))$ is a point on the graph and we need to estimate $f(29.5)$ using quadratic approximation, $P_2(x) \approx f(x)$ for x near $a=30$.

$$P_2(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2} f''(a).$$

$f(30) = y = 10$.

$$f'(a) = f'(x)|_{x=a} = \frac{dy}{dx}\Big|_{x=a}$$

$x^2 + 2y^2 = 1100$; $2x + 4y\frac{dy}{dx} = 0$

$$x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}$$

At $x = 30$, $y = 30$, so $\frac{dy}{dx}\Big|_{x=30} = \frac{-30}{20} = -\frac{3}{2}$.

$f''(x) = \frac{d^2y}{dx^2}$: Differentiating $2x + 4y\frac{dy}{dx} = 0$ or $x + 2y\frac{dy}{dx} = 0$,

we get $1 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$.

At $x = 30$, $y = 10$, $\frac{dy}{dx} = -\frac{3}{2}$; hence $1 + 2\left(-\frac{3}{2}\right)^2 + 20\left(\frac{d^2y}{dx^2}\right)_{x=30} = 0$.

$$\Rightarrow 1 + 2\left(\frac{9}{4}\right) + 20f''(a) = 0 \Rightarrow 1 + \frac{9}{2} + 20f''(a) = 0$$

$$\Rightarrow 20f''(a) = -\frac{11}{2} \Rightarrow f''(a) = \frac{-11}{40}.$$

$$P_2(x) = f(30) + (x-30)\left(-\frac{3}{2}\right) + \frac{(x-30)^2}{2}\left(\frac{-11}{40}\right)$$

At $x = 29.5$, get

$$P_2(29.5) = 10 + (-0.5)\left(-\frac{3}{2}\right) + \frac{(-0.5)^2}{2}\left(\frac{-11}{40}\right)$$

$$= 10 + \frac{1.5}{2} + \left(\frac{0.25}{2}\right)\left(\frac{-11}{40}\right).$$

Errors:

Recall: $p_1(x) = f(a) + f'(a)(x-a)$ is the linear approximation for $f(x)$ centered around a.

Eg $f(x) = \sin x$; x measured in radians, $a = 0$.

$p_1(x) = x$; $\sin x$; x measured in radians; $a=0$.

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$p_1(x) = x$; $p_2(x) = x - \frac{1}{2}x^2$.

$\ln(1.05) = f(0.05)$; $p_1(0.05) = 0.05$; $p_2(x) = 0.05 - \frac{1}{2}(0.0025)$

$p_2(x) = 0.0525$; $p_1(x) \approx f(x)$; $p_2(x) \approx f(x)$.

Eg: Consider the curve $x^2 + 2y^2 = 1100$. Consider the point $(30, 10)$ on the curve and assume that nearby points are of the form $y = f(x)$ for some function $f(x)$. Use quadratic approximation to find y when $x = 29.5$.

Ans. Given $P = (30, 10)$ is a point on the curve. Notice that 29.5 is close to 30. So $(29.5, f(29.5))$ is a point on the graph

__Errors__:

Recall: $p_1(x) = f(a) + f'(a)(x-a)$ is the linear approximation for $f(x)$ centered around a.

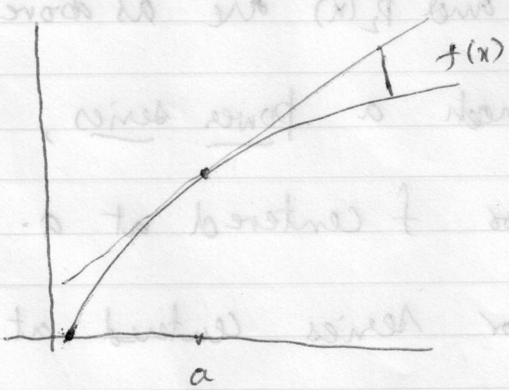
Eg: $f(x) = \sin x$; x measured in radians, $a = 0$.

$p_1(x) = x$; so $\sin x \approx x$ near zero, when x is expressed in radians.

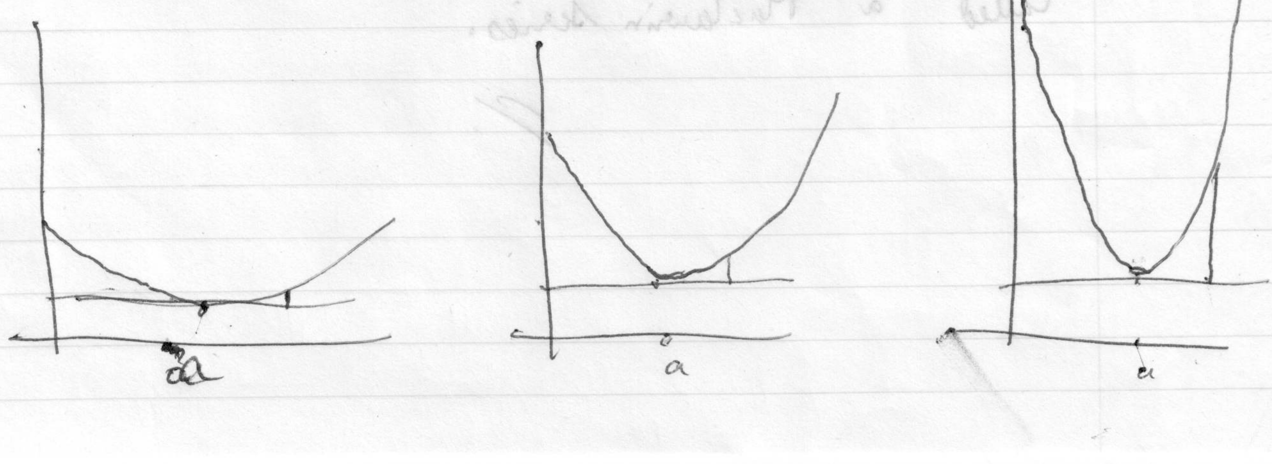
Defn: The error in the linear approximation is M, where

$$|R(x)| = |f(x) - p_1(x)| \le M.$$

In other words, we want to find an upper bound for the difference between the actual value and the approximated value $p_1(x)$.



Notice the difference between p_1 and f increases as we move farther away from the center a.



Note that the difference between f and p_1 increases as we move farther away from a. Also the greater the concavity, the larger the difference between $p_1(x)$ and $f(x)$.

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§: Taylor polynomials: Suppose a function f has derivatives $f^{(k)}(a)$ of all orders at the point a.

~~Power~~ The Taylor polynomial of degree (or order) n is defined by

$$p_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

where $c_k = \dfrac{f^{(k)}(a)}{k!}$

Ofcourse $p_1(x)$ and $p_2(x)$ are as above. If we let $n \to \infty$, we obtain a <u>power series</u>, called the Taylor series for f centered at a. The special case of a Taylor series centered at 0 is called a Maclaurin series.

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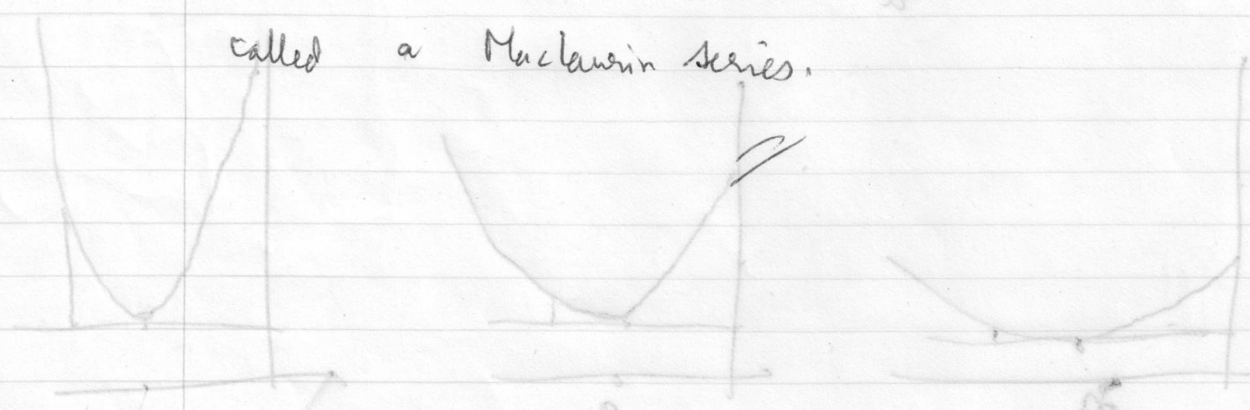

So if f is concave up, then $p_1(x) < f(x)$.

The graph of the function f bends away from the linearly approximating function $p_1(x)$ (which is just the tangent) at a slower pace if the second derivative is smaller.

- The larger the absolute value of the second derivative at a, $|f''(a)|$, the greater the deviation from the tangent line.
- The linear approximation at a is more accurate for f, when the rate of change of f', which is nothing else but f'', is smaller.

Remainder: Write $f(x) = p_n(x) + R_n(x)$, where $n=1$ or 2 ($n=1$ is linear approximation and $n=2$ is Quadratic approximation). To estimate the remainder, suppose that there exists a number M such that $|f^{(n+1)}(c)| \leq M$ for all c between a and x inclusive. Then the remainder satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

where $p_n(x)$ is the linear or quadratic approximation centered around a.

Eg: Find the remainder term R_2 for $f(x) = \sin x$

Centered at $a=0$

Ans: $p_2(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2}x^2$

$f'(x) = \cos x$; $f'(0) = 1$; $f''(x) = -\sin x$, $f''(0) = -0$.

$p_2(x) = x$.

$f^{(3)}(x) = -\cos x$; $|f^{(3)}(x)| \leq 1$.

So can choose $M = 1$

$\therefore |R_2(x)| \leq 1 \cdot \frac{(x-0)^3}{3!} = \frac{x^3}{3!}$

Eg: Suppose $f(x)$ is a function with $f''(x) = \frac{1}{5}\frac{x}{x^2+1}$.

Use linear approximation to estimate $p_1(0.1)$ and estimate the remainder term.

Ans: Center $= a = 0$. Consider the interval $[0, 0.1]$

$$|f''(x)| = \frac{1}{5}\left|\frac{x}{x^2+1}\right|$$

$$\leq \frac{1}{5} \cdot \frac{0.1}{0+1} = \frac{0.1}{5} = 0.02$$

$\therefore$ Can choose $M = 0.02$.

$R_2(x) \leq 0.02 \frac{x^3}{3!}$.