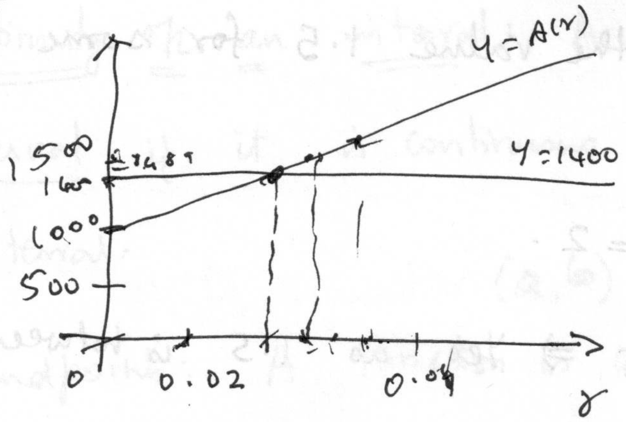
②



Graphs of $A(r)$ and $y=1400$ intersect for an r between $r=0$ and $r=0.08$.

Solving $A(r) = 1400 \rightsquigarrow r \approx 0.0675$.

<u>Derivative</u> :

Problems on limits:

Find the following limits:

a) $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4}$: $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+2)}$

$= \lim_{x\to 2} \frac{(x-1)}{(x+2)} = \frac{1}{4}$

b) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$; $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{((\sqrt{x+1})-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}$

$= \lim_{x\to 0} \frac{((\sqrt{x+1})^2-1)}{x(\sqrt{x+1}+1)}$

$= \lim_{x\to 0} \left(\frac{x+1-1}{x(\sqrt{x+1}+1)}\right) = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$

$= \lim_{x\to 0} \frac{1}{(\sqrt{x+1}+1)} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

c) Given the function

$$f(x) = \begin{cases} 7-4x & x<1 \\ x^2+2 & x \geq 1 \end{cases}$$

Evaluate the following limits, if they exist.

i) $\lim_{x\to -6} f(x)$: Ans $= \lim_{x\to -6} f(x) = \lim_{x\to -6} 7-4x$ (as $-6<1$)

$= 7-4(-6) = 31$

ii) $\lim_{x\to 1} f(x)$: Ans: $LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 7-4x = 3$

$$RHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3+2) \quad (\text{as } x > 1)$$

$$= (1)^3 + 2 = 3.$$

$RHL = LHL = 3$, hence limit exists and equals 3.

Eg. $\lim_{x \to 5} (10 + |x-5|)$

Absolute value function $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x > 0 \end{cases}$

$$\lim_{x \to 5^-} (10 + |x-5|) = \lim_{x \to 5^-} (10 + (+5-x)) \quad \left(\begin{array}{l} x \to 5^- \text{ means} \\ |x-5| = -x+5 \end{array}\right)$$

$$= 10 + (5-5) = 10.$$

$$\lim_{x \to 5^+} (10 + |x-5|)$$

$$= \lim_{x \to 5^+} (10 + (x-5)) \quad (\text{as } x \to 5^+ \text{ means } x > 5, \text{ hence } x-5 > 0$$

$$= 10$$

So $LHL = RHL = 10$

Continuity and Intermediate Value Theorem.

(1) Determine if the following function is continuous at $x=1$.

$$f(x) = \begin{cases} 3x-5 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$$

Ans: The function $f(x)$ is defined at $x=1$ and $f(1)=2$.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} (3x-5) = 3(1)-5 = -2.$$

Thus $\lim_{x\to 1} f(x) = -2$ and does not equal $f(1)=2$. Therefore $f(x)$ is not continuous at $x=1$.

(2) Let $f(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that $f(x)$ is continuous for all values of x.
Show that f is differentiable for all values of x, but that the derivative f' is not continuous at $x=0$. Assume $f'(x) = \sin(1/x) + 2x\cos(1/x)$ for $x \neq 0$.

Ans: $f(x) = k(x) \cdot h(g(x))$; where $k(x) = x^2$, $g(x) = 1/x$ and $h(x) = \cos x$. Thus $h(g(x)) = \cos(1/x)$ is continuous and $k(x)$ is continuous; $h(g(x))$ is continuous as it is the composite of two continuous functions. The function $f(x)$ is therefore a product of two continuous functions and is continuous, when $x \neq 0$.

<u>At $x=0$</u>: f is defined since $f(0)=0$.

The limit $\lim_{x\to 0} \cos(1/x)$ does not exist since the values of $\cos(1/x)$ oscillates between -1 and $+1$ as $x \to 0$.

However, for $x \neq 0$, $-1 \le \cos(1/x) \le +1$; and hence
$-x^2 \le x^2 \cos(1/x) \le x^2$.

Since $\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$, it follows from the Squeeze theorem that $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \cos(1/x) = 0$.

Since $\lim_{x \to 0} f(x) = 0 = f(0)$, f is continuous at $x = 0$ and hence $f(x)$ is continuous for all values of x.

Given $f'(x) = \sin 1/x + 2x \cos(1/x)$, when $x \neq 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cos(1/h) - 0}{h}$$

$$= \lim_{h \to 0} h \cos(1/h) = 0. \text{ (use squeeze theorem)}$$

Hence f is differentiable for all values of x. Since $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left\{ \sin(1/x) + 2x \cos\left(\frac{1}{x}\right) \right\}$ does not exist as x approaches zero, the derivative $f'(x)$ ~~does~~ is not continuous at $x = 0$. This limit does not exist because the values of $\sin(1/x)$ oscillates between -1 and $+1$ as x approaches zero.

Note that the continuity of the function f for all values of x also follows from the fact that f is differentiable for all values of x.

B

Intermediate Value Theorem: ① Use IVT to show that the function $f(x) = xe^{x-1} - \frac{1}{2}$ has a zero in the interval $[0,1]$.

Ans. $f(x) = xe^{x-1} - \frac{1}{2}$ is continuous on the interval $[0,1]$ (in fact, it is continuous everywhere; check this!). We have

$$f(0) = 0 \cdot e^{0-1} - \tfrac{1}{2} = -\tfrac{1}{2}\ ;\ f(1) = 1 \cdot e^{1-1} - \tfrac{1}{2} = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Thus $f(0) = -\frac{1}{2} < 0$, $f(1) = \frac{1}{2} > 0$.

As 0 lies between $-\frac{1}{2}$ and $+\frac{1}{2}$, the IVT tells us that $f(c) = 0$ for some c in the interval $[0,1]$.

② Use IVT in order to show that the equation $x^5 - x + 1 = 0$ has at least one real solution.

Ans: $f(x) = x^5 - x + 1$; it is continuous everywhere since x is a polynomial. We must find an interval $[a,b]$ such that $f(a)$ and $f(b)$ have opposite signs. Choosing $a = -2$ and $b = -1$, we get $f(-2) = (-2)^5 - (-2) + 1 = -29$, $f(-1) = (-1)^5 - (-1) + 1 = 1$. Thus $f(-2) < 0$ and $f(-1) > 0$.

By IVT, there is a c in the interval $[-2,-1]$ such that $f(c) = 0$.

Q: Determine all values of the constants A and B so that the following function is continuous for all values of x.

$$f(x) = \begin{cases} Ax - B & \text{if } x \le -1 \\ 2x^2 + 3Ax + B & \text{if } -1 < x \le 1 \\ 4 & \text{if } x > 1 . \end{cases}$$

Ans: For $x \le -1$, $f(x) = Ax - B$ and it is continuous for any values of A and B since it is a polynomial. The same is true for $x \in (-1, 1]$. For $x > 1$, $f(x)$ is a constant and is continuous.

$$f(-1) = A(-1) - B = -A - B.$$

LHL $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (Ax - B) = A(-1) - B = -A - B.$

RHL $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x^2 + 3Ax + B) = 2(-1)^2 + 3A(-1) + B$
$= 2 - 3A + B.$

For the limit to exist, we must have LHL = RHL.
So $-A - B = 2 - 3A + B$,
or $2A - 2B = 2 \Rightarrow A - B = 1$

Next consider continuity at $x = 1$; the function must be defined at $x = 1$, so $f(1) = 2(1)^2 + 3A + B = 2 - 3A + B$.

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4 = 4.$

LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x^2 + 3Ax + B) = 2 + 3A + B.$

RHL = LHL gives $2 + 3A + B = 4$ or $3A + B = 2$.

Solving $3A + B = 2$ and $A - B = 1$, we get $B = -1/4$ and $A = 3/4$; these are the values for which f is continuous at $x = 1$ and $x = -1$.