SOLUTIONS: ONE-SIDED AND TWO-SIDED LIMIT PROBLEMS

1. Evaluate the one-sided limits below.

a) i) \( \lim_{x \to 2^-} |x - 2| \) \hspace{1cm} ii) \( \lim_{x \to 2^+} |x - 2| \)

i) As \( x \) approaches 2 from the left, it must be true that \( x < 2 \). We further obtain \( x - 2 < 0 \) by subtracting 2 from both sides of the inequality. The absolute value \( |x - 2| \) is therefore equal to \(-(x - 2)\) for \( x < 2 \). Evaluating the limit yields
\[
\lim_{x \to 2^-} |x - 2| = \lim_{x \to 2^-} (2 - x) = 2 - 2 = 0
\]

ii) Approaching \( x \) from the right of 2, we know that \( x > 2 \). This gives \( x - 2 > 0 \) so \( |x - 2| = x - 2 \). Hence
\[
\lim_{x \to 2^+} |x - 2| = \lim_{x \to 2^+} (x - 2) = 2 - 2 = 0
\]

b) i) \( \lim_{x \to -1^-} \sqrt{x^2 - 1} \) \hspace{1cm} ii) Why do we not evaluate \( \lim_{x \to -1^+} \sqrt{x^2 - 1} \)?

i) Check: if \( x < -1 \), then \( x^2 > 1 \). It follows that \( x^2 - 1 > 0 \) and so the square root is defined. Thus
\[
\lim_{x \to -1^-} \sqrt{x^2 - 1} = \sqrt{(-1)^2 - 1} = 0
\]

ii) When \( x > -1 \) and \( x \) is sufficiently close to \(-1\) (e.g. \( x < 1 \)), \( x + 1 > 0 \) and \( x - 1 < 0 \). It follows that \( x^2 - 1 = (x + 1)(x - 1) < 0 \). Taking this limit would require us to consider the square root of negative numbers, an operation that we have not defined. (Search “sqrt(x^-2-1)” in Google to generate a graph of the function.)

c) i) \( \lim_{x \to 1^-} \sqrt[3]{\frac{x^3 - 4x^2 + 3x}{x^2 - 2x + 2}} \) \hspace{1cm} ii) \( \lim_{x \to 1^+} \sqrt[3]{\frac{x^3 - 4x^2 + 3x}{x^2 - 2x + 2}} \)

The cube root of any real number is defined, so there is no need to consider whether or not the term inside the root is negative. Thus
\[
\lim_{x \to 1^-} \sqrt[3]{\frac{x^3 - 4x^2 + 3x}{x^2 - 2x + 2}} = \lim_{x \to 1^+} \sqrt[3]{\frac{x^3 - 4x^2 + 3x}{x^2 - 2x + 2}} = \sqrt[3]{\frac{1^3 - 4 \cdot 1^2 + 3 \cdot 1}{1^2 - 2 \cdot 1 + 2}} = 0.
\]
2. Compute the following limits:

a) \( \lim_{x \to 2} (|x - 2| + x)^5 \)

\[ = (0 + 2)^5 \]

\[ = 32 \quad (\text{since } \lim_{x \to 2} |x - 2| = 0 \text{ from 1a}) \]

b) \( \lim_{x \to 3} \frac{x^2 - 9}{x^2 - 8x + 15} \)

\[ = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{(x - 5)(x - 3)} \]

\[ = \lim_{x \to 3} \frac{x + 3}{x - 5} \]

\[ = \frac{3 + 3}{3 - 5} \]

\[ = -3 \]

c) \( \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \)

\[ = \lim_{x \to 0} \frac{(\sqrt{x + 1} - 1)(\sqrt{x + 1} + 1)}{x(\sqrt{x + 1} + 1)} \]

\[ = \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)} \]

\[ = \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} \]

\[ = \frac{1}{\sqrt{0 + 1} + 1} \]

\[ = \frac{1}{2} \]

d) \( \lim_{x \to 1} \frac{x - 1}{\sqrt{2x - 1} - 1} \)

\[ = \lim_{x \to 1} \frac{(x - 1)(\sqrt{2x - 1} + 1)}{(\sqrt{2x - 1} - 1)(\sqrt{2x - 1} + 1)} \]

\[ = \lim_{x \to 1} \frac{(x - 1)(\sqrt{2x - 1} + 1)}{2x - 1 - 1} \]
\[
\lim_{x \to 1} \frac{\sqrt{2x - 1} + 1}{2} = \lim_{x \to 1} \frac{\sqrt{2} \cdot 1 - 1 + 1}{2} = 1
\]

3. Find the values of the parameters \(a\) and \(b\) such that the function

\[
f(x) = \begin{cases} 
(2x + a)^3, & \text{if } x < 0 \\
5bx + 8, & \text{if } 0 \leq x < 1 \\
x^2 + 12, & \text{if } x \geq 1 
\end{cases}
\]

is continuous at all the points in its domain.

We know polynomial functions are continuous. Hence, the only points at which discontinuities can occur are the points where pieces of \(f(x)\) join. That is, the pieces of \(f(x)\) must connect at \(x = 0\) and \(x = 1\). This requires the left-hand and right-hand limits of \(f(x)\) to be equal. We begin by evaluating these limits:

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + a)^3 = (2 \cdot 0 + a)^3 = a^3
\]
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5bx + 8) = 5b \cdot 0 + 8 = 8
\]
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5bx + 8) = 5b \cdot 1 + 8 = 5b + 8
\]
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 12) = 1^2 + 12 = 13
\]

Letting \(\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)\) and \(\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)\) gives \(a^3 = 8\) and \(5b + 8 = 13\), which solves to \(a = 2\) and \(b = 1\).