PEAK QUASISYMMETRIC FUNCTIONS AND EULERIAN ENUMERATION

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ABSTRACT. Via duality of Hopf algebras, there is a direct association between peak quasisymmetric functions and enumeration of chains in Eulerian posets. We study this association explicitly, showing that the notion of **cd**-index, long studied in the context of convex polytopes and Eulerian posets, arises as the dual basis to a natural basis of peak quasisymmetric functions introduced by Stembridge. Thus Eulerian posets having a nonnegative **cd**-index (for example, face lattices of convex polytopes) correspond to peak quasisymmetric functions having a nonnegative representation in terms of this basis. We diagonalize the operator that associates the basis of descent sets for all quasisymmetric functions to that of peak sets for the algebra of peak functions, and study the g-polynomial for Eulerian posets as an algebra homomorphism.

1. INTRODUCTION

In the enumerative theory of partially ordered sets, one is often interested in enumerative functionals that are nonnegative for a given class of posets. Thus, for example, the *generalized lower bound theorem* for convex polytopes asserts that certain functionals of the flag f-vector, the so-called g-vector, will be nonnegative for all convex polytopes.

In recent years, there have been a number of papers linking the enumerative theory of posets to the study of coalgebras and Hopf algebras, leading to a deeper understanding of one such funtional, the **cd**-index of Eulerian posets. See [1, 5, 14, 15, 19, 20] for a sample of such work and [12] for a relatively recent survey of the state of such enumerative questions.

In the theory of symmetric functions, one is often interested to know when certain symmetric functions can be expressed as nonnegative linear combinations of a preferred basis (the Schur functions, for example). The recent breakthrough of Haiman on the Macdonald positivity conjecture [23] is one such instance.

Setting questions in posets and symmetric functions in the context of Hopf algebras has led to a deep understanding of their relationship. In [21], Gel'fand, *et al.*, show the Hopf algebra of quasisymmetric functions to be dual to the the Hopf algebra $NC = \mathbb{Z}\langle y_1, y_2, \dots\rangle$, which they called *noncommutative symmetric functions*. Billera and Liu [18] considered elements of the algebra $\mathbb{Q}\langle y_1, y_2, \dots\rangle = \mathbb{Q}\otimes NC$ as flag-enumeration functionals on all graded posets, and they defined a quotient $A_{\mathcal{E}}$ of $\mathbb{Q}\langle y_1, y_2, \dots\rangle$, which consists of all such functionals on Eulerian posets. Bergeron, Mykytiuk, Sottile and van Willigenburg [9, 10] showed that the algebra $A_{\mathcal{E}}$ is dual to Stembridge's algebra Π of peak quasisymmetric functions [33]. More precisely, they showed that both of these algebras have natural coproducts that make them

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into Hopf algebras, and that these Hopf algebras are, in fact, dual. This duality links the study of the enumerative properties of Eulerian posets, including associated geometric objects such as convex polytopes and hyperplane arrangements, with that of Stembridge's enriched P-partitions and related questions having to do with peaks and shuffles in permutations.

We will explore some of these links here. In particular, we will show that the natural nonnegativity questions on each side are closely related. The weight enumerators of all enriched P-partitions of chains were shown by Stembridge to be a basis for the peak algebra Π. An immediate consequence of the result of Bergeron, *et al.*, is the fact that the formal quasisymmetric function $F(P)$ of an Eulerian poset P, as defined by Ehrenborg [19], is an element of Π. The coefficients of $F(P)$ in terms of this basis are given by the **cd**-index of P. Thus nonnegative representation for quasisymmetric functions of Eulerian posets is equivalent to their having a nonnegative **cd**-index. More precisely, we show that the linear forms defining the coefficients of the **c**-2**d**-index give a basis for $A_{\mathcal{E}}$ dual to Stembridge's basis for Π. This completely unexpected result shows the **cd**-index to be a natural concept in spite of its initial *ad hoc* definition.

We give the basic definitions in the rest of §1. In §1.1 we define the algebra $\mathcal{Q}$ of quasisymmetric functions over $\mathbb{Q}$ and the subalgebra Π of peak functions. In §1.2 we discuss graded and Eulerian posets and the algebras of flag-enumeration functionals on each class. In §1.3 we define the relevant coproducts on these algebras that make them pairs of dual Hopf algebras. Finally, in §1.4, we look at different bases for $\mathcal{Q}$ and corresponding representations.

In §2, we relate the representation of the quasisymmetric function $F(P)$ in terms of Stembridge's basis to the **cd**-index of the poset P, in particular to the **c**-2**d**-index studied in [14]. One consequence is that the quasisymmetric functions corresponding to zonotopes lie in the (half) integral sublattice of Π spanned by the Stembridge basis.

In §3 we consider the map ϑ, defined and studied by Stembridge, associating the weight enumerator of all P-partitions for a fixed labeled poset with that of the corresponding enriched P-partitions for the same data. When applied to a quasisymmetric function coming from a representable geometric lattice, one obtains the quasisymmetric function arising from the corresponding zonotope. We show this map to be diagonalizable on Π, and we give an explicit basis of eigenvectors. The principal eigenvector in any degree is given by the distribution of peak sets in the corresponding symmetric group. In fact, the operator $\frac{1}{2^{n+1}}\vartheta$ can be viewed a giving a random walk on the peak sets of S_{n+1} having this stationary distribution.

Finally, in §4, we extend the usual g-polynomial of Eulerian posets to the algebra Π (in fact to $\mathcal{Q}$), where it defines an algebra homomorphism to the polynomial ring $\mathbb{Q}[x]$. It is hoped that this way of viewing the g-invariant will lead to a better understanding of its properties.

1.1. **Quasisymmetric functions and the peak algebra.** We let $\mathcal{Q}$ denote the algebra of quasisymmetric functions over $\mathbb{Q}$, that is, all bounded degree formal power series F in variables $x_1, x_2, \ldots$ such that for all m, and any $i_1 < i_2 < \cdots < i_m$, the coefficient of $x_{i_1}x_{i_2}\cdots x_{i_m}$ in F is the same as that of $x_1x_2\cdots x_m$. Equivalently, $\mathcal{Q}$ is the linear span of $M_0 = 1$ and all power series M_β, where $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ is a vector of positive integers (a *composition* of $\beta_1+\beta_2+\cdots+\beta_k$)

and

$$M_\beta = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_k}^{\beta_k}. \tag{1.1}$$

We denote by $\mathcal{Q}_{n+1}$ the subspace of $\mathcal{Q}$ consisting of those quasisymmetric functions that are homogeneous of degree $n+1$; equivalently, $\mathcal{Q}_{n+1}$ is the linear span of all M_β, where β is a composition of $n+1$. It is straightforward to see that the 2^n such M_β form a basis for the vector space $\mathcal{Q}_{n+1}$. For integer $k>0$, let $[k] := \{1, 2, \ldots, k\}$ and $[0] = \emptyset$. It will be helpful for us to consider the equivalent indexing of this basis by subsets of $[n]$, where for $S = \{i_1, i_2, \ldots, i_k\} \subset [n]$, $M_S := M_{\beta(S)}$ and $\beta(S) = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n+1-i_k)$. When $n+1$ is not clear from the context, we will write $M_S^{(n+1)}$. For further details about quasisymmetric functions, see [32].

Definition 1.1. *Let $n \geq 0$ and $S \subset [n]$.*

(1) *S is said to be* left sparse *if $1 \notin S$ and $i \in S$ implies $i-1 \notin S$.*
(2) *Similarly, S is* right sparse *if $n \notin S$ and $i \in S$ implies $i+1 \notin S$.*
(3) *For an integer k, let $S + k = \{i + k \mid i \in S\}$.*

We note that [26] uses the terms left and right sparse in the opposite sense than used here.

The *peak algebra* Π is defined to be the subalgebra of $\mathcal{Q}$ generated by the elements

$$\Theta_S = \sum_{T : S \subset T \cup (T+1)} 2^{|T|+1} M_T, \tag{1.2}$$

where S is a left sparse subset of $[n]$, $n \geq 0$. Here the sum is over $T \subset [n]$ and $M_T = M_T^{(n+1)}$. Defining $\Pi_n = \Pi \cap \mathcal{Q}_n$, we have that $\dim_{\mathbb{Q}}(\Pi_n) = a_n$, the n^{th} Fibonacci number (indexed so that $a_1 = a_2 = 1$) [33, Theorem 3.1].

We consider an equivalent indexing of the basis of Π to that by left sparse subsets in (1.2). Let $\mathbf{c}$ and $\mathbf{d}$ be indeterminates, of degree 1 and 2, respectively. For a **cd**-word $w = \mathbf{c}^{n_1}\mathbf{d}\mathbf{c}^{n_2}\mathbf{d}\cdots\mathbf{c}^{n_k}\mathbf{d}\mathbf{c}^m$ of degree n, define the subset $S_w \subset [n]$ by

$$\begin{aligned} S_w &= \{n_1 + 2, n_1 + n_2 + 4, \ldots, n_1 + n_2 + \cdots + n_k + 2k\} \\ &= \{i_1, i_2, \ldots, i_k\}, \end{aligned}$$

where $i_j = \deg(\mathbf{c}^{n_1}\mathbf{d}\mathbf{c}^{n_2}\mathbf{d}\cdots\mathbf{c}^{n_j}\mathbf{d})$. Note that S_w is always left sparse and every left sparse $S \subset [n]$ is of the form S_w for some **cd**-word w of degree n. Thus, there will be no ambiguity if we relabel this basis to

$$\Theta_w = \Theta_{S_w}, \tag{1.3}$$

where w ranges over all possible **cd**-words. (For $w = \mathbf{1}$, the empty word, we have $\Theta_{\mathbf{1}} = 2M_\emptyset^1$.) Note that $\deg(\Theta_w) = \deg w + 1$, so the ambiguity about the degree in the earlier notation is no longer an issue.

1.2. **Eulerian posets and enumeration algebras.** Recall that a *graded poset* P is one having a unique minimal element $\widehat{0}$ and maximal element $\widehat{1}$ for which every maximal chain has the same number of elements. Thus if $x \in P$ has a maximal chain

$$\widehat{0} = x_0 < x_1 < \cdots < x_k = x,$$

we say that x has *rank* k, denoted $r(x) = k$ (and so $r(\widehat{0}) = 0$). Further, we define the rank of P to be $r(P) := r(\widehat{1})$. For a graded poset P of rank $n+1$ and a subset $S \subset [n]$, we denote by $f_S(P)$ the number of flags (*i.e.*, chains) in P having elements with precisely the ranks in S. Note that the ranks 0 and $n+1$ are not included here. The function $S \mapsto f_S(P)$ is known as the *flag f-vector* of P. Recall that a graded poset is said to be *Eulerian* if its Möbius function μ satisfies $\mu(x,y) = (-1)^{r(y)-r(x)}$ for every pair $x \leq y$. See [29] for general background in this area.

In [18], elements of the free associative algebra $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ were associated to flag numbers of graded posets. If $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ is a composition of $n+1$, let $y_\beta = y_{\beta_1} y_{\beta_2} \cdots y_{\beta_k} \in \mathbb{Q}\langle y_1, y_2, \dots \rangle$. We associate f_S for posets of rank $n+1$ to $y_{\beta(S)}$. For $k \geq 1$, we define

$$\chi_k := \sum_{i+j=k} (-1)^i y_i y_j,$$

where the sum is over all $i, j \geq 0$ and we set $y_0 = 1$ for convenience. The element χ_k corresponds to the Euler relation for rank k posets. Let $I_{\mathcal{E}}$ be the two-sided ideal in $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ generated by the χ_k, $k \geq 1$, and define the *algebra of forms on Eulerian posets* to be $A_{\mathcal{E}} = \mathbb{Q}\langle y_1, y_2, \dots \rangle / I_{\mathcal{E}}$. Letting $\deg(y_i) = i$, $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ is a graded algebra and, since $I_{\mathcal{E}}$ is a homogeneous ideal, so is $A_{\mathcal{E}}$. It is shown in [18] that

$$A_{\mathcal{E}} \cong \mathbb{Q}\langle y_1, y_3, y_5, \dots, y_{2k+1}, \dots \rangle \tag{1.4}$$

as graded $\mathbb{Q}$ algebras. As a result, we have that for $n \geq 1$, the dimension of $(A_{\mathcal{E}})_n$ is again a_n, the n-th Fibonacci number.

1.3. **Coproducts and graded Hopf duality.** Noting the equality of the dimensions of Π_n and $(A_{\mathcal{E}})_n$, Bergeron, *et al.*, studied the relationship between them. To do so, they described coproducts on Π and $A_{\mathcal{E}}$, respectively, that make each a Hopf algebra [9]. The coproduct on the subalgebra Π is inherited from the usual coproduct on $\mathcal{Q}$, defined by $\Delta(M_0) = M_0 \otimes M_0$ and

$$\Delta(M_\alpha) = \sum_{\alpha = \alpha_1 \cdot \alpha_2} M_{\alpha_1} \otimes M_{\alpha_2},$$

where $\alpha_1 \cdot \alpha_2$ is the concatenation of compositions α_1 and α_2, and either α_1 or α_2 may be the empty composition of 0. It was shown in [10, Theorem 2.2] that Π is closed under this coproduct.

There is a coproduct on $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ defined by

$$\Delta(y_k) = \sum_{i+j=k} y_i \otimes y_j, \tag{1.5}$$

where the sum is over all $i, j \geq 0$, which extends to all of $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ by virtue of its being an algebra map. In [9], it is shown that this coproduct is well-defined on the quotient $A_{\mathcal{E}}$.

With the augmentation map that is zero in positive degree and the identity in degree 0, both $\mathcal{Q}$ and $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ are bialgebras. The existence of an antipode on each of these bialgebras, making them Hopf algebras, follows from the fact that they are graded (see, *e.g.*,[19, Lemma 2.1]). More precisely, if X has degree n, then

$$\Delta(X) = X \otimes 1 + \sum_{i=0}^{n-1} Y_i \otimes Z_{n-i},$$

where Y_j and Z_j have degree j, and the antipode is defined recursively by $s(1) = 1$ and

$$s(X) = -\sum_{i=0}^{n-1} s(Y_i) Z_{n-i}. \tag{1.6}$$

We can compute the antipode explicitly for Π and $A_{\mathcal{E}}$. If we denote by w^* the reverse of the **cd**-word w, *e.g.*, $(\mathbf{ccd})^* = \mathbf{dcc}$, then we have the following. We delay the proof until §2.1.

Proposition 1.1. *In terms of the basis $\{\Theta_w\}$, the antipode of Π is given by*

$$s(\Theta_w) = (-1)^{\deg w + 1} \Theta_{w^*}.$$

Recall that if $\beta = (\beta_1, \ldots, \beta_k)$ is a composition of $n+1$, then we denote $y_\beta = y_{\beta_1} \cdots y_{\beta_k} \in \mathbb{Q}\langle y_1, y_2, \ldots \rangle$. If $\beta^* = (\beta_k, \ldots, \beta_1)$ is the reverse composition, then we have

Proposition 1.2. *In terms of the basis $\{y_\beta\}$, the antipode of $A_{\mathcal{E}}$ is given by*

$$s(y_\beta) = (-1)^{n+1} y_{\beta^*},$$

where β is a composition of $n+1$.

Proof. We show first that $s(y_n) = (-1)^n y_n$. By (1.5) and (1.6) we have $s(y_1) = -y_1$. By induction,

$$s(y_n) = -\left(\sum_{i=0}^{n-1} (-1)^i y_i y_{n-i} \right) = -\left(\chi_n - (-1)^n y_n \right).$$

The assertion follows since χ_n vanishes in $A_{\mathcal{E}}$.

The proposition now follows from the fact that the antipode is an algebra antihomomorphism. $\square$

A key result for our purposes is that $A_{\mathcal{E}}$ and Π are dual as graded Hopf algebras [9, Theorem 5.4]. By dual we will always mean *graded dual*; that is, if a graded algebra is of the form $V = V_0 \oplus V_1 \oplus \cdots$ as a graded vector space, then its graded dual is, as a vector space, $V^* = V_0^* \oplus V_1^* \oplus \cdots$, where V_i^* is the usual dual space to the finite dimensional space V_i.

Thus, we have that elements of Π are characterized by having coefficients that satisfy the generalized Dehn-Sommerville equations for Eulerian posets [3].

Proposition 1.3. *If $F = \sum_{S \subset [n]} f_S M_S \in \mathcal{Q}_{n+1}$, then $F \in \Pi$ if and only if $\sum_{S \subset [n]} a_S f_S = 0$ whenever $\sum_{S \subset [n]} a_S y_{\beta(S)} \in I_{\mathcal{E}}$.*

If P is any graded poset of rank $n+1$, then following [19] we define the formal quasisymmetric function associated to P by

$$F(P) = \sum_{S \subset [n]} f_S(P) M_S \in \mathcal{Q}_{n+1}. \tag{1.7}$$

Then it follows from Proposition 1.3 that the quasisymmetric functions of Eulerian posets are elements of Π. However, the converse does not hold; it is possible for a graded poset P not to be Eulerian, yet still satisfy $F(P) \in \Pi$. The smallest such example has $f_\emptyset = 1$, $f_1 = f_2 = 3$ and $f_{12} = 6$.

1.4. **Bases and interval representations.** It will be helpful to consider two other bases for $\mathcal{Q}$ and the corresponding representations of arbitrary $F \in \mathcal{Q}$. For $S \subset [n]$, we define

$$F_S = \sum_{T \supset S} M_T \tag{1.8}$$

and

$$K_S = \sum_{T \supset S} F_T = \sum_{T \supset S} 2^{|T|-|S|} M_T. \tag{1.9}$$

Again all sums are over $T \subset [n]$ and $M_T = M_T^{(n+1)}$; when the context does not make it clear we will write $F_S^{(n+1)}$ and $K_S^{(n+1)}$. It is easy to check that the F_S and K_S are again bases for $\mathcal{Q}_{n+1}$ and that

$$M_S = \sum_{T \supset S} (-1)^{|T|-|S|} F_T \tag{1.10}$$

and

$$F_S = \sum_{T \supset S} (-1)^{|T|-|S|} K_T. \tag{1.11}$$

Define the *flag h-vector* and *flag k-vector* by the relations $f_S = \sum_{T \subset S} h_T$ and $h_S = \sum_{T \subset S} k_T$. The following is immediate from the definitions.

Proposition 1.4. *For $F \in \mathcal{Q}_{n+1}$, if $F = \sum_{S \subset [n]} f_S M_S$ then*

$$F = \sum_{S \subset [n]} h_S F_S = \sum_{S \subset [n]} k_S K_S.$$

Note that Proposition 1.4 holds, more specifically, for a graded poset P of rank $n+1$: if $F(P) = \sum_{S \subset [n]} f_S(P) M_S \in \mathcal{Q}_{n+1}$ then $F(P) = \sum_{S \subset [n]} h_S(P) F_S = \sum_{S \subset [n]} k_S(P) K_S$, where $f_S(P) = \sum_{T \subset S} h_T(P)$ and $h_S(P) = \sum_{T \subset S} k_T(P)$.

If $\mathcal{I}$ is a family of subsets of $[n]$, the we denote by $b[\mathcal{I}]$ the *blocking family* of $\mathcal{I}$, defined by

$$b[\mathcal{I}] = \{S \subset [n] \mid S \cap I \neq \emptyset \text{ for all } I \in \mathcal{I}\}.$$

We note that if $\mathcal{I}$ is an antichain in the Boolean lattice $2^{[n]}$, then we can recover $\mathcal{I}$ as the set of minimal elements, under inclusion, in $b[b[\mathcal{I}]]$.

We are particularly interested in the case in which the family $\mathcal{I}$ consists of *intervals* in $[n]$, *i.e.*, subsets of the form $\{i, i+1, \ldots, i+k\}$. If $\mathcal{I}$ is such an interval family, we denote by $F_{\mathcal{I}}$ the element of $\mathcal{Q}_{n+1}$ defined by

$$F_{\mathcal{I}} = \sum_{\substack{S \subset [n] \\ S \in b[\mathcal{I}]}} M_S. \tag{1.12}$$

We call the $F_{\mathcal{I}}$ *interval quasisymmetric functions.* For $S \subset [n]$, if we set $\mathcal{I} = \mathcal{I}(S) = \{\ \{i\} \mid i \in S\}$ then $F_{\mathcal{I}(S)} = F_S$. We will see in the next section that the basis for Π can be represented in a similar manner.

In [16], antichains of intervals were used to describe the extreme rays of the closed convex cone generated by all flag f-vectors of graded posets. Equivalently, the same description can be used to describe the closed convex cone in $\mathcal{Q}$ generated by all $F(P)$ arising from graded posets. The following is essentially [16, Theorem 2.1].

Proposition 1.5. *The extreme rays of the closed convex cone in $\mathcal{Q}_{n+1}$ generated by all elements $F(P)$, where P is a graded poset of rank $n+1$, are precisely the interval quasisymmetric functions $F_{\mathcal{I}}$ corresponding to interval antichains in $[n]$.*

Finally, we note that one can interpret the chain decompositions of [16, 17] as giving multiplication formulae for the $F_{\mathcal{I}}$. In particular, the proof of [16, Proposition 2.8] yields the expression

$$F(P) = \sum_{C} \sum_{S\in b[\mathcal{I}(C)]} M_S = \sum_{C} F_{\mathcal{I}(C)}, \tag{1.13}$$

where the first sum is over all maximal chains C in P and $\mathcal{I}(C)$ is the interval antichain defined in [16, p. 86]. As in [16, Corollary 2.6], we have $F_{\mathcal{I}}$ is the limit, as $N\to\infty$, of elements of the form $\frac{1}{f_{[n]}(P_N)}F(P_N)$, and one can use (1.13) to compute the representation of $F_{\mathcal{I}_1}\cdot F_{\mathcal{I}_2}$ in terms of the $F_{\mathcal{I}}$.

2. The **cd**-index and the peak algebra

Now suppose P is an Eulerian poset of rank $n+1$ and $F(P)=\sum_{S\subset[n]} f_S\, M_S$. We wish to express $F(P)$ in terms of the basis $\{\Theta_w\}$ for Π_{n+1}. An unexpected outcome is that such a representation is provided by the **cd**-index of P.

2.1. **Blocking representations of Θ_w.** We begin by giving a representation of the basis Θ_w in terms of interval families associated with sparse subsets.

Definition 2.1. *Let $S=\{i_1,\dots,i_k\}\subset[n]$ and w a **cd**-word of degree n. Then*

(1) *if S is right sparse, let*
$$\mathcal{I}_S=\{\{i_1,i_1+1\},\{i_2,i_2+1\},\dots,\{i_k,i_k+1\}\},$$
(2) *if S is left sparse, let*
$$\mathcal{I}^S=\{\{i_1-1,i_1\},\{i_2-1,i_2\},\dots,\{i_k-1,i_k\}\}\ \textit{and}$$
(3) $\mathcal{I}^w=\mathcal{I}^{S_w}$

When defined, both $\mathcal{I}_S$ and $\mathcal{I}^S$ are antichains of disjoint two-element intervals in $[n]$. The interval antichains $\mathcal{I}_S$ and $\mathcal{I}^S$ are among what Bayer and Hetyei refer to as *even interval systems* and so give rise (after their doubling operation) to limits of flag f-vectors of Eulerian posets [6, Proposition 2.6]. We show that in this way $\mathcal{I}^w$ will give rise to Θ_w.

It is straightforward to see that for a degree n **cd**-word w and subset $S\subset[n]$, $S_w\subset S\cup(S+1)$ if and only if $S\in b[\mathcal{I}^w]$. Thus it follows from (1.2) and (1.3) that

$$\Theta_w=\sum_{S\in b[\mathcal{I}^w]} 2^{|S|+1}M_S. \tag{2.14}$$

If we define the map $D:\mathcal{Q}_{n+1}\longrightarrow\mathcal{Q}_{n+1}$ by $D(M_S)=2^{|S|+1}M_S$, then (2.14) is equivalent to

$$\Theta_w=D\left(F_{\mathcal{I}^w}\right),$$

where $F_{\mathcal{I}^w}$ is defined by (1.12).

It follows from [16, Corollary 2.6] and the remark following [6, Definition 4] that $\frac{1}{2}\Theta_w$ is the quasisymmetric function corresponding to what Bayer and Hetyei call the doubled limit poset $DP(n,\mathcal{I}^w)$. From [6, Theorem 4.2] we obtain

Proposition 2.1. *The Θ_w are among the extreme rays of the closed convex cone in $\mathcal{Q}$ generated by all $F(P)$ arising from Eulerian posets.*

It will be helpful in what follows to have a representation of the Θ_w in terms of the basis $\{F_T\}$ of $\mathcal{Q}$. We let $|w|_{\mathbf{d}}$ denote the $\mathbf{d}$-degree of the word w, *i.e.*, the number of $\mathbf{d}$'s in w. The following is essentially [33, Proposition 3.5].

Proposition 2.2. *For any* $\mathbf{cd}$*-word w of degree n,*

$$\Theta_w = 2^{|w|_{\mathbf{d}}+1} \sum_{T,\overline{T} \in b[\mathcal{I}^w]} F_T,$$

where the sum is over all $T \subset [n]$, and $\overline{T} = [n] \setminus T$.

Proof. By Proposition 1.4 and (2.14),

$$\Theta_w = \sum h_T F_T, \tag{2.15}$$

where the h_T are defined uniquely by

$$\sum_{T \subset S} h_T = f_S = \begin{cases} 2^{|S|+1} & S \in b[\mathcal{I}^w] \\ 0 & \text{otherwise.} \end{cases}$$

Since $|w|_{\mathbf{d}} = |S_w|$, we need to show that

$$h_T = \begin{cases} 2^{|S_w|+1} & T, \overline{T} \in b[\mathcal{I}^w] \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Assuming (2.16), we compute

$$\sum_{T \subset S} h_T = 2^{|S_w|+1} \cdot n_S^w, \tag{2.17}$$

where

$$n_S^w = \#\{T \subset S \mid T, \overline{T} \in b[\mathcal{I}^w]\}.$$

If $S \notin b[\mathcal{I}^w]$, then $n_S^w = 0$. If $S \in b[\mathcal{I}^w]$, let

$$\begin{aligned} T_1 &= \{i \in S_w \mid i \in S, i-1 \notin S\}, \\ T_2 &= \{i \in S_w \mid i-1 \in S, i \notin S\}, \\ T_3 &= \{i \in S_w \mid \{i-1, i\} \subset S\}, \text{ and} \\ S' &= S \setminus (T_1 \cup (T_2 - 1) \cup T_3 \cup (T_3 - 1)). \end{aligned}$$

We have $|T_1| + |T_2| + |T_3| = |S_w|$ and $|S'| = |S| - |T_1| - |T_2| - 2|T_3|$. For a subset $T \subset S$, both T and $\overline{T}$ are in $b[\mathcal{I}^w]$ if and only if

$$T = [T_1 \cup (T_2 - 1)] \cup R_3 \cup R_4,$$

where R_3 consists of one element from each pair $\{i-1, i\}$, $i \in T_3$ (these pairs are disjoint), and R_4 is *any* subset of S'. Thus

$$n_S^w = 2^{|T_3|} \cdot 2^{|S|-|T_1|-|T_2|-2|T_3|} = 2^{|S|-|S_w|},$$

and by (2.17)

$$\sum_{T \subset S} h_T = 2^{|S_w|+1} \cdot n_S^w = f_S,$$

verifying (2.16). ∎

We can now verify the form of the antipode of Π.

Proof of Proposition 1.1. It follows from [19, Proposition 7.2] that if s is the antipode on $\mathcal{Q}$, and P is Eulerian, then

$$s\,(F(P)) = (-1)^{r(P)} F(P^*),$$

where P^* is the dual or opposite or polar poset to P. Thus the antipode of Π is simply the antipode s restricted to Π. Recall from [25, Corollary 2.3] that on $\mathcal{Q}$, s is given on the F basis by

$$s(F_T) = (-1)^{n+1} F_{\overline{T}^\vee}$$

for $F_T = F_T^{(n+1)} \in \mathcal{Q}_{n+1}$, where, for $S \subset [n]$, $S^\vee = \{n+1-i \mid i \in S\}$. Therefore

$$\begin{aligned} s(\Theta_w) &= s\left(2^{|w|_{\mathbf{d}}+1} \sum_{T,\overline{T} \in b[\mathcal{I}^w]} F_T\right) \\ &= (-1)^{n+1} 2^{|w|_{\mathbf{d}}+1} \sum_{T,\overline{T} \in b[\mathcal{I}^w]} F_{T^\vee} \\ &= (-1)^{\deg w+1}\left(2^{|w|_{\mathbf{d}}+1} \sum_{T,\overline{T} \in b[(\mathcal{I}^w)^\vee]} F_T\right) = (-1)^{\deg w+1}\Theta_{w^*}, \end{aligned}$$

where $(\mathcal{I}^w)^\vee = \{I^\vee \mid I \in \mathcal{I}^w\} = \mathcal{I}^{w^*}$. □

2.2. Ψ_w **and the cd-index.** For any Eulerian poset P of rank $n+1$, there is a polynomial of degree n, $\psi_P \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$, called the **cd**-*index* [4]. (Here we assume $\deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$.) We denote by $[w]$ or $[w]_P$ the coefficient of w in ψ_P. The coefficient $[w]_P$ can be expressed linearly in terms of the *sparse flag k-vector*, that is, in terms of the numbers $k_S(P)$ for right sparse $S \subset [n]$. See [13, Proposition 7.1] for this expression. Of interest here is the inversion of this relation [15, Definition 6.5], which we write as follows.

Proposition 2.3. *For right sparse $S \subset [n]$,*

$$k_S = \sum_{\substack{S_w \in b[\mathcal{I}_S] \\ |w|_{\mathbf{d}} = |S|}} [w].$$

Proof. The expression in [15, Definition 6.5] sums over all w of degree n that *cover* S and have $|S|$ $\mathbf{d}$'s. Noting that in [15], the indexing is by dimension, not by rank as in this paper (and in [13]), it follows that w covers S if and only if $S \subset S_w \cup (S_w - 1)$. Since $|S| = |S_w|$, we can conclude w covers S if and only if $S_w \in b[\mathcal{I}_S]$. □

Remark 2.1. We note that the relation in Proposition 2.3 (more precisely, its inverse [13, Proposition 7.1]) gives us a way to define a **cd**-index ψ_F for any $F = \sum k_S K_S \in \Pi_{n+1}$ – in fact, for any $F \in \mathcal{Q}_{n+1}$ – by defining $[w] = [w]_F$, for $\deg w = n$, directly from the coefficients k_S. Further, for non-homogeneous $F \in \mathcal{Q}$, we can define $[w]_F$ for all **cd**-words w by $[w]_F = [w]_{F_i}$, where F_i is the homogeneous component of F of degree $\deg w + 1$.

Example 2.1. For $F \in \mathcal{Q}_3$, $F = k_\emptyset K_\emptyset + k_1 K_1 + k_2 K_2 + k_{12} K_{12}$ and so we define $\psi_F = k_\emptyset\, \mathbf{c}^2 + k_1\, \mathbf{d}$. For $F \in \mathcal{Q}_4$, $F = \sum_{S \subset [3]} k_S\, K_S$ and so

$$\psi_F = k_\emptyset\, \mathbf{c}^3 + (k_2 - k_1)\, \mathbf{cd} + k_1\, \mathbf{dc}.$$

Note that in both cases, the values of k_S for non-sparse S are not relevant to the definition of ψ_F. For $F \in \Pi$, these values are determined by the others. For general $F \in \mathcal{Q}$, this is no longer the case since there are no relations on the f_S, and so on the k_S [18, Proposition 1.1].

We now define another set of a_{n+1} elements in $\mathcal{Q}_{n+1}$ indexed by words w of degree n and relate them to the Θ_w.

Definition 2.2. *For w a* **cd***-word of degree n, let*

$$\Psi_w = \sum_{\substack{S \in b[\mathcal{I}^w] \\ |S| = |w|_{\mathbf{d}}}} K_S,$$

where the sum is over only right sparse $S \subset [n]$.

Consider the projection operator $F \mapsto \overline{F}$ on $\mathcal{Q}$ defined by

$$\overline{M}_S = \begin{cases} M_S & \text{if } S \text{ is right sparse} \\ 0 & \text{if not.} \end{cases}$$

Note that if S is not right sparse then $\overline{K}_S = \overline{F}_S = 0$. This projection operator is injective on Π:

Proposition 2.4. *If $F, G \in \Pi$ and $\overline{F} = \overline{G}$ then $F = G$.*

Proof. It is shown in [3] that a consequence of the generalized Dehn-Sommerville relations is that the flag f-vector for Eulerian posets is determined by its values on right sparse subsets. It follows from Proposition 1.3 that this continues to hold for elements of Π. □

Corollary 2.1. *For any $F \in \mathcal{Q}$, there is a unique element $\pi(F) \in \Pi$ such that $\overline{\pi(F)} = \overline{F}$. The corresponding map $\pi : \mathcal{Q} \to \Pi$ is a linear projection.*

Proof. Again, from [3] we have that the right sparse subsets form a basis for the flag f-vectors of Eulerian posets, and so for all of Π. Given an $F \in \mathcal{Q}$, the values of f_S over all right sparse subsets and the generalized Dehn-Sommerville equations determine values for the remaining f_S in such a way as to determine an element of Π. Call this element $\pi(F)$. That $\pi(F)$ is unique follows from Proposition 2.4.

Note that if $F \in \Pi$, $\pi(F) = F$. That the map π is linear follows from its construction. □

Example 2.2. For $F \in \Pi_3$, the generalized Dehn-Sommerville relations imply that $f_2 = f_1$ and $f_{12} = 2\, f_1$. Thus for any $F = f_\emptyset M_\emptyset + f_1\, M_1 + f_2\, M_2 + f_{12}\, M_{12} \in \mathcal{Q}_3$,

$$\pi(F) = f_\emptyset\, M_\emptyset + f_1\,(M_1 + M_2 + 2\, M_{12}).$$

We call $\pi(F) \in \Pi$ the *Eulerian projection* of F. That π is not an algebra map can be seen from the fact that $\pi(M_1^{(2)}) = 0$ but $\pi(M_1^{(2)} \cdot M_1^{(2)}) \neq 0$. Note that for any $F \in \mathcal{Q}$, $[w]_F = [w]_{\pi(F)}$ and so the fibers of π consist of $F \in \mathcal{Q}$ having the same **cd**-index.

The elements $\overline{\Psi}_w$ form a basis for the subspace $\overline{\mathcal{Q}} = \mathrm{span}\{M_S \mid S \text{ right sparse}\} \subset \mathcal{Q}$. We see next that for $F \in \mathcal{Q}$ the coefficients of the expression of $\overline{F}$ in terms of this basis are given by the **cd**-index of F.

Proposition 2.5. *For $F \in \mathcal{Q}_{n+1}$,*

$$\overline{F} = \sum_{\deg w = n} [w]\, \overline{\Psi}_w,$$

where $[w] = [w]_F$.

Proof. By Proposition 1.4, we can write $F = \sum_S k_S K_S$ and so

$$\overline{F} = \sum_{S \text{ sparse}} k_S \overline{K}_S = \sum_{S \text{ sparse}} \left(\sum_{\substack{w : |w|_{\mathbf{d}} = |S| \\ S_w \in b[\mathcal{I}_S]}} [w] \right) \overline{K}_S, \tag{2.18}$$

by Proposition 2.3, where the sum is over w of degree n and $[w] = [w]_F$. Here *sparse* means right sparse. When $|S| = |w|_{\mathbf{d}} = |S_w|$, we have $S_w \in b[\mathcal{I}_S]$ if and only if $S \in b[\mathcal{I}^w]$, so (2.18) becomes

$$\overline{F} = \sum_{\deg w = n} [w] \left(\sum_{\substack{S \text{ sparse} \\ |S| = |w|_{\mathbf{d}} \\ S \in b[\mathcal{I}^w]}} \overline{K}_S \right) = \sum_{\deg w = n} [w]\, \overline{\Psi}_w. \tag{2.19}$$

□

2.3. Θ_w **and the c-2d-index.** We determine the relationship between Ψ_w and Θ_w and thereby a formula for the representation of $F \in \Pi$ in terms of the Θ_w.

Proposition 2.6. *For any* **cd***-word w,*

$$\overline{\Psi}_w = \frac{1}{2^{|w|_{\mathbf{d}}+1}}\, \overline{\Theta}_w.$$

Proof. Suppose w has degree $n \geq 0$. By (2.14) we have

$$\overline{\Theta}_w = 2 \sum_{\substack{S \text{ sparse} \\ S \in b[\mathcal{I}^w]}} 2^{|S|} M_S, \tag{2.20}$$

where all $S \subset [n]$ and sparse means right sparse.

Using (1.9), we write

$$\overline{\Psi}_w = \sum_{\substack{S \text{ sparse} \\ |S| = |w|_{\mathbf{d}} \\ S \in b[\mathcal{I}^w]}} \overline{K}_S = \frac{1}{2^{|w|_{\mathbf{d}}}} \sum_{\substack{S \text{ sparse} \\ |S| = |w|_{\mathbf{d}} \\ S \in b[\mathcal{I}^w]}} \left(\sum_{S \subset R \subset [n]} 2^{|R|}\, \overline{M}_R \right). \tag{2.21}$$

Now suppose $S \neq S'$ are both right sparse, $|S| = |S'| = |w|_{\mathbf{d}}$ and $S, S' \in b[\mathcal{I}^w]$. Then any $R \supset S \cup S'$ is not right sparse and so $\overline{M}_R = 0$. Thus, combining (2.20) and (2.21) we get

$$\overline{\Psi}_w = \frac{1}{2^{|w|_{\mathbf{d}}}} \sum_{\substack{S \text{ sparse} \\ S \in b[\mathcal{I}^w]}} 2^{|S|}\, M_S = \frac{1}{2^{|w|_{\mathbf{d}}+1}}\, \overline{\Theta}_w.$$

□

Following [14], for any $F \in \mathcal{Q}$, we call the quantities

$$[[w]] = \frac{1}{2^{|w|_{\mathbf{d}}}}\,[w] \tag{2.22}$$

the coefficients of the **c**-2**d**-*index* of F, where $[w] = [w]_F$. When $F \in \Pi$, these coefficients provide a representation of F in terms of the basis elements Θ_w.

Theorem 2.1. *For $F \in \Pi$*

$$F = \frac{1}{2}\,\sum_w [[w]]\;\Theta_w.$$

Proof. We know from Proposition 2.5 and Proposition 2.6 that

$$\overline{F} = \sum_w [w]\;\overline{\Psi}_w = \frac{1}{2}\,\sum_w [[w]]\;\overline{\Theta}_w.$$

But if

$$F' = \frac{1}{2}\,\sum_w [[w]]\;\Theta_w,$$

then $\overline{F} = \overline{F'}$ and so $F = F'$ by Proposition 2.4. ◻

We restate the theorem explicitly in the poset case as

Corollary 2.2. *If P is any Eulerian poset, then*

$$F(P) = \sum_w \frac{1}{2^{|w|_{\mathbf{d}}+1}}\;[w]_P\;\Theta_w,$$

where the $[w]_P$ are the coefficients of the **cd***-index of P.*

In [14, 15] it is shown that any zonotope Z has an integral **c**-2**d**-index, and so if $\mathcal{F}(Z)$ is the lattice of faces of Z, then if we let $F(Z) = F(\mathcal{F}(Z))$, we have

Corollary 2.3. *For a zonotope Z,*

$$2F(Z) = \sum_w [[w]]\Theta_w$$

is in the $\mathbb{Z}$-span of the Θ_w in Π.

This remains true for the dual face lattice of any oriented matroid [14]. Since the **cd**-indices of P and P^* are related by

$$[w]_{P^*} = [w^*]_P \tag{2.23}$$

(see [4, §3]), we have

Corollary 2.4. *If P is the face lattice of any hyperplane arrangement or, more generally, oriented matroid, the quasisymmetric function $F(P)$ has a half-integral representation in terms of the Θ_w.*

Remark 2.2. It follows from Theorem 2.1 that one can also view the elements Θ_w as the limit as $m \to \infty$ of $2\left(\frac{2}{m}\right)^{|w|_{\mathbf{d}}}\;F(P_{w,m})$, where $P_{w,m}$ is the poset defined in the proof of [31, Proposition 1.2]. Another possible approach to Theorem 2.1 could be made via [6, Proposition 2.9] (see Proposition 2.1 and the comments preceding it).

3. The Stembridge map

We describe in this section an algebra map defined by Stembridge in [33, Theorem 3.1(c)]. It is most natural when viewing the algebras $\mathcal{Q}$ and Π as arising from ordinary and enriched P-partitions of labeled posets. In this case, for a given labeled poset, the map sends the quasisymmetric function in $\mathcal{Q}$ obtained via the ordinary theory to that in Π obtained via the enriched theory. In the case of a labeled chain, it relates bases for these algebras in a simple manner that will serve as our definition.

For $S \subset [n]$, define

$$\Lambda(S) = \{i \in S \mid i \neq 1, i-1 \notin S\}. \tag{3.24}$$

For any S, $\Lambda(S)$ is clearly left sparse. If one writes S as a unique union of minimally many intervals, $\Lambda(S)$ will consist of the first element of each such interval, excluding the element 1. For example, $\Lambda(\{1,2,3,5,8,9\}) = \{5,8\}$. If one thinks of S as the descent set of some permutation π in the symmetric group $\mathcal{S}_{n+1}$, then $\Lambda(S)$ consists of those descents that are preceded by ascents, that is, the *peaks* of π.

We define the map $\vartheta : \mathcal{Q} \to \Pi$ by $\vartheta(F_S) = \Theta_{\Lambda(S)}$ for any $S \subset [n]$, $n \geq 0$, where Θ_S is labeled as in the original definition (1.2). It is proved in [33] that ϑ is an algebra map. It arises naturally as the map that associates the weight enumerator of all P-partitions of a labeled poset with that of all enriched P-partitions of the same poset. See [33] for details.

3.1. **A random walk on peak sets.** As a linear map, the restriction $\vartheta : \Pi_{n+1} \longrightarrow \Pi_{n+1}$ can be written, for $S \subset [n]$,

$$\vartheta(\Theta_S) = 2^{|S|+1} \sum_{T, \overline{T} \in b[\mathcal{I}^S]} \Theta_{\Lambda(T)}, \tag{3.25}$$

using Proposition 2.2. Equivalently, we can write

$$\vartheta(\Theta_w) = 2^{|w|_{\mathbf{d}}+1} \sum_u \eta_{u,w} \Theta_u, \tag{3.26}$$

where

$$\eta_{u,w} = \#\{T \subset [n] \mid T, \overline{T} \in b[\mathcal{I}^w]; \Lambda(T) = S_u\}. \tag{3.27}$$

If we let w and u be $\mathbf{cd}$-words of degree n, $S_w = \{w_1 < w_2 < \cdots < w_l\}$, $S_u = \{u_1 < u_2 < \cdots < u_m\}$, $u_0 = 0$ and $u_{m+1} = n+2$, then we have (assuming the empty product to be 1)

Proposition 3.1. *For* $\mathbf{cd}$*-words* w *and* u*,*

$$\eta_{u,w} = \begin{cases} 0 & \text{if } |S_w \cap (u_i, u_{i+1})| > 1 \text{ for some } i, \\ \prod (u_{i+1} - u_i - 1) & \text{otherwise,} \end{cases}$$

where the product is taken over all i*,* $0 \leq i \leq m$*, such that* $S_w \cap (u_i, u_{i+1}) = \emptyset$*.*

Proof. Let $\mathcal{A} = \{T \subset [n] \mid T, \overline{T} \in b[\mathcal{I}^w]; \Lambda(T) = S_u\}$. For disjoint intervals I, J of natural numbers, we will write $I \prec J$ whenever $x < y$ for all $x \in I$ and $y \in J$. Consider the partition $S_w \cup S_u = I_1 \cup \cdots \cup I_r$ into maximal intervals, where $I_1 \prec \cdots \prec I_r$. Since S_w and S_u are both sparse sets, consecutive elements in each I_i alternate between the two sets. It is then easy to see that for all $T \in \mathcal{A}$, $T \cap I_i = S_u \cap I_i$ for every i. In particular, $T \cap I_i$ does not depend on T, and so we only need to consider the possible elements of T outside of $S_w \cup S_u$.

Let $[n+1]\backslash(S_w \cup S_u) = J_1 \cup \cdots \cup J_s$ be a partition into maximal intervals. An interval J is said to have *type* xy, where $x, y \in \{w, u\}$, if $I_i \prec J \prec I_{i+1}$ for some i, and the last element of I_i is in S_x and the first element of I_{i+1} is in S_y. For the sake of the argument, if an interval has more than one possible type, we always choose the unique type which favors u. If $\{0\} \prec J \prec I_1$ then J will be given type ux, where x depends on the first element of I_1, and if $I_r \prec J \prec \{n+2\}$, then J will be given type yu, where y depends on the last element in I_r. Every J_i now has a unique type.

The condition $|S_w \cap (u_i, u_{i+1})| > 1$ for some i is equivalent to the existence of some J_k of type ww. In this case, for any $T \in \mathcal{A}$, there exists some w_j such that $\Lambda(T) \cap [w_j, w_{j+1}] = \emptyset$ and T contains exactly one element in each interval $[w_j - 1, w_j]$ and $[w_{j+1} - 1, w_{j+1}]$. This is clearly impossible, so in this case $\mathcal{A} = \emptyset$.

Suppose now that no J_i has type ww. It is straightforward to verify that for any $T \in \mathcal{A}$, if J_i has type uw then $T \cap J_i = J_i$; if J_i has type wu then $T \cap J_i = \emptyset$; and if J_i has type uu then $J_i = (u_j, u_{j+1})$ for some u_j, and $T \cap J_i = [u_j + 1, u_j + t]$ for some $0 \le t \le u_{j+1} - u_j - 2$. (We set $[u_j + 1, u_j] = \emptyset$.) Thus, every $T \in \mathcal{A}$ is determined only by $T \cap J_i$ for all intervals J_i of type uu. These are precisely the intervals (u_j, u_{j+1}), $0 \le j \le m$, such that $S_w \cap (u_j, u_{j+1}) = \emptyset$. This shows that our formula is an upper bound for η_{wu}.

For the reverse inequality, suppose that $T \subset [n]$ satisfies $T \cap I_i = S_u \cap I_i$ for all i; $T \cap J_i = J_i$ for all J_i of type uw; $T \cap J_i = \emptyset$ for all J_i of type wu; and for all $J_i = (u_j, u_{j+1})$ of type uu, there exists a $0 \le t \le u_{j+1} - u_j - 2$ such that $T \cap J_i = [u_j + 1, u_j + t]$. We first show that $T, \overline{T} \in b[\mathcal{I}^w]$. This is trivial if $S_w = \emptyset$, so let $w_j \in S_w$, and let I_k be the interval containing w_j. Suppose that $w_j \in S_u$. In this case $w_j \in T$, and if $w_j - 1 \in I_k$, then $w_j - 1 \notin T$ because $T \cap I_k$ is sparse. If $w_j - 1 \notin I_k$, then $w_j - 1 \in J_i$ for some J_i of type uu or wu, and so $w_j - 1 \notin T$. Now suppose that $w_j \notin S_u$. In this case $w_j \notin T$, and if $w_j - 1 \in I_k$, then $w_j - 1$ is in S_u and hence T. If $w_j - 1 \notin I_k$ then $w_j - 1$ is in some J_i of type uw (no J_i has type ww), which implies $w_j - 1 \in T$. In either case, $|T \cap [w_j - 1, w_j]| = 1$.

It remains to prove that $\Lambda(T) = S_u$. One can use a similar argument to show that if $u_j \in S_u$, then $u_j - 1 \notin T$. Therefore, $\Lambda(T) \supset S_u$. Let $x \in \Lambda(T)$, so that $x \in T$ and $x - 1 \notin T$. If $x \in I_i$ for some i, then $x \in S_u$ since $T \cap I_i \subset S_u$. If $x \in J_i$ for some i, then J_i must have type uw or uu. In both cases, $x - 1 \in T$, a contradiction. This completes the proof. ☐

Corollary 3.1. *The transformation ϑ is indecomposable on Π_{n+1}.*

Proof. Let Γ be the directed graph on the **cd**-words of degree n defined by $(u, w) \in \Gamma$ whenever $\eta_{u,w} > 0$. Then $\mathbf{c}^n$ is a sink in Γ, *i.e.*, every node has an arc pointing to $\mathbf{c}^n$. Further $(u, w) \in \Gamma$ whenever w is obtained from u by replacing a $\mathbf{c}^2$ by $\mathbf{d}$. Thus every w is in a directed cycle with the word $\mathbf{c}^n$, and the assertion follows. ☐

As in the proof of Proposition 2.2, we have that

$$\#\{T \subset [n] \mid T, \overline{T} \in b[\mathcal{I}^S]\} = 2^{n-|S|},$$

so every column of the matrix representing ϑ has sum 2^{n+1}. Thus we can conclude that 2^{n+1} is an eigenvalue of the linear map ϑ. Since ϑ is indecomposable, the Perron-Frobenius theory of nonnegative matrices (see, e.g., [8, Chapter 16]) implies that 2^{n+1} is the largest eigenvalue of ϑ on the finite-dimensional space Π_{n+1}. It has multiplicity 1; the corresponding eigenvector p_{n+1} is nonnegative up to scaling.

These assertions are, in fact, all verified in the next subsection. The coefficients of this eigenvector have a particularly interesting interpretation.

Proposition 3.2. *The distribution of peak sets in the symmetric group S_{n+1} gives the nonnegative eigenvector $p_{n+1} \in \Pi_{n+1}$ of ϑ corresponding to the eigenvalue 2^{n+1}. That is, if*

$$p_{n+1} = \sum_{S \subset [n] \text{ left sparse}} p_S \; \Theta_S,$$

where p_S is the number of permutations in S_{n+1} with peak set S, then

$$\vartheta(p_{n+1}) = 2^{n+1} p_{n+1}.$$

Proof. From the interpretation of the multiplication of the generators Θ_w in terms of shuffles of sequences with peak sets given by S_w [33, (3.1)], it follows that $p_{n+1} = (\Theta_{\mathbf{1}})^{n+1}$, where $\Theta_{\mathbf{1}}$ is the unique generator in degree 1 corresponding to the empty **cd**-word **1**. That is, $(\Theta_{\mathbf{1}})^{n+1}$ gives the distribution of peak sets in S_{n+1}. It is easy to check that $\vartheta(\Theta_{\mathbf{1}}) = 2\Theta_{\mathbf{1}}$, and since ϑ is an algebra map, we have

$$\vartheta((\Theta_{\mathbf{1}})^{n+1}) = 2^{n+1}(\Theta_{\mathbf{1}})^{n+1}.$$

□

See [33, p.784] for an expression for the coefficients of p_{n+1} in terms of peak sets of shifted standard Young tableaux. In fact, p_{n+1} is the unique nonnegative eigenvector of ϑ, since eigenvectors corresponding to any other eigenvalue must have coefficients (in terms of the Θ_w) that sum to 0. This is so since the vector of ones is an eigenvector for the transpose of the matrix of ϑ, and eigenvectors for a matrix and its transpose corresponding to distinct eigenvalues must be orthogonal.

One way to interpret Proposition 3.2 is that the operator $\frac{1}{2^{n+1}}\vartheta$ defines a random walk on the family of left sparse subsets of $[n]$ with stationary distribution given by the probability distribution of peak sets in a random permutation in S_{n+1}. We conjecture that this random walk is a specialization of a random walk on S_{n+1} with uniform stationary distribution. We have checked this through S_4; in fact, in each case it suffices to take a specialization of a random walk on the braid arrangement defined in [11].

We give a complete analysis of the spectrum of ϑ in the next subsection. In particular, we show that the eigenvalues of $\frac{1}{2^{n+1}}\vartheta$ on Π_{n+1} are $(\frac{1}{4})^k$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

3.2. **Diagonalization of ϑ.** We describe further the spectrum of ϑ and give a complete set of eigenvectors in Π_{n+1} for each $n \geq 0$. We have already observed that $\Theta_{\mathbf{1}}$ is the unique eigenvector in Π_1, with corresponding eigenvalue $\lambda = 2$. We construct the remaining eigenvectors from $\Theta_{\mathbf{1}}$ by means of two simple operations.

Define the map $L : \mathcal{Q} \to \mathcal{Q}$ by $L(M_S^{(n)}) = M_S^{(n+1)}$ for any $S \subset [n-1]$. We will show that $L^2 = L \circ L$ commutes with ϑ, and so L^2 preserves eigenvectors of ϑ: if $\vartheta(v) = \lambda\ v$ then $\vartheta(L^2(v)) = L^2(\vartheta(v)) = \lambda\ L^2(v)$, showing $L^2(v)$ to be an eigenvector for the same eigenvalue.

Since ϑ is an algebra map, products of eigenvectors in Π are again eigenvectors. In particular, if $\vartheta(v) = \lambda\ v$ then $\Theta_{\mathbf{1}} \cdot v$ is an eigenvector for eigenvalue 2λ. We will consider multiplication by $\Theta_{\mathbf{1}}$ as a linear map on Π, also denoted as $\Theta_{\mathbf{1}}$ when there is no possibility of confusion.

For any $\mathbf{cd}$-word $w = w(\mathbf{c}, \mathbf{d})$, define the operator

$$\widehat{w} : \Pi \longrightarrow \Pi$$

by $\widehat{w} = w(\Theta_{\mathbf{1}}, L^2)$. For example, $\widehat{\mathbf{cdc}} = \Theta_{\mathbf{1}} \circ L^2 \circ \Theta_{\mathbf{1}}$. Note that if $w = \mathbf{1}$ then $\widehat{w}$ is the identity map. It follows from the discussion above that $\widehat{w}$ preserves eigenvectors of ϑ on Π, multiplying the corresponding eigenvalue by the factor $2^{|w|_{\mathbf{c}}}$, where $|w|_{\mathbf{c}}$ is the number of $\mathbf{c}$'s in w.

The main result of this section is

Theorem 3.1. *The map ϑ is diagonalizable on Π. A complete set of eigenvectors is given by*

$$\Omega_w = \widehat{w}(\Theta_{\mathbf{1}}),$$

where w is any $\mathbf{cd}$-word w, $\widehat{w}(\Theta_{\mathbf{1}})$ is the image of $\Theta_{\mathbf{1}}$ under the map $\widehat{w}$. The eigenvalue corresponding to Ω_w is $2^{|w|_{\mathbf{c}}+1}$, and so, on Π_{n+1}, the eigenvalues of ϑ are 2^{n+1-2k}, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

The proof of this result proceeds by a sequence of propositions. The first of these is the commutativity of ϑ and L^2 on $\mathcal{Q}$.

Proposition 3.3. *As maps on $\mathcal{Q}$, $\vartheta \circ L^2 = L^2 \circ \vartheta$, so L^2 preserves eigenvectors of ϑ, as well as their eigenvalues.*

Proof. It is straightforward to verify that for $S \subset [n-1]$

$$L(F_S^{(n)}) = F_S^{(n+1)} - F_{S \cup \{n\}}^{(n+1)},$$

and so

$$L^2(F_S^{(n)}) = F_S^{(n+2)} - F_{S\cup\{n\}}^{(n+2)} - F_{S\cup\{n+1\}}^{(n+2)} + F_{S\cup\{n,n+1\}}^{(n+2)}. \tag{3.28}$$

Similarly, we have

$$L^2(\Theta_w) = \Theta_{w\mathbf{c}^2} - \Theta_{w\mathbf{d}} \tag{3.29}$$

or, equivalently, $L^2(\Theta_T^{(n)}) = \Theta_T^{(n+2)} - \Theta_{T\cup\{n+1\}}^{(n+2)}$ for left sparse $T \subset [n-1]$. Now, using (3.28) and (3.29), one can verify that

$$\vartheta \circ L^2\,(F_S^{(n)}) = L^2 \circ \vartheta\,(F_S^{(n)}) = \Theta_{\Lambda(S)}^{(n+2)} - \Theta_{\Lambda(S)\cup\{n+1\}}^{(n+2)}. \tag{3.30}$$

□

We note that $\vartheta \circ L \neq L \circ \vartheta$; in particular, we have $L \circ \vartheta(M_\emptyset^{(1)}) = 2M_\emptyset^{(2)}$ while $\vartheta \circ L(M_\emptyset^{(1)}) = 0$. Next, we need to show that the eigenspaces induced by L^2 are independent of those induced by $\Theta_{\mathbf{1}}$. This will follow from

Proposition 3.4. *For each $n \geq 0$,*

$$\mathcal{Q}_{n+1} = L(\mathcal{Q}_n) \oplus \Theta_{\mathbf{1}}(\mathcal{Q}_n).$$

Proof. Since both L and $\Theta_{\mathbf{1}}$ are injective ($\mathcal{Q}$ has no zerodivisors), it is enough to prove $L(\mathcal{Q}_n) \cap \Theta_{\mathbf{1}}(\mathcal{Q}_n) = \{0\}$. To this end, recall that $\Theta_{\mathbf{1}} = 2M_{(1)}$, where (1) is the unique composition of 1 (see (1.1) and (1.2)). Using the formula [19, Lemma 3.3] for multiplication in the basis M_β, we have

$$M_{(1)} \cdot M_\beta \quad = \quad M_{\beta,1} + \sum_{i=1}^{k} \left(M_{(\beta_1,\ldots,\beta_{i-1},1,\beta_i,\ldots,\beta_k)} + M_{(\beta_1,\ldots,\beta_{i-1},\beta_i+1,\beta_{i+1},\ldots,\beta_k)} \right),$$

where $\beta = (\beta_1, \ldots, \beta_k)$ is any composition of n. Order compositions of $n+1$ first by the number of parts (those with fewer parts are smaller in the order) then lexicographically, *i.e.*,

$$\beta = (\beta_1, \ldots, \beta_k) \prec (\beta'_1, \ldots, \beta'_{k'}) = \beta'$$

if $k < k'$ or $k = k'$ and for some i, $\beta_j = \beta'_j$ for $j < i$ while $\beta_i < \beta'_i$. With this order, the composition $(\beta, 1)$ is the largest index in the right-hand side of the expression for $M_{(1)} \cdot M_\beta$ given above. Since any element of $L(\mathcal{Q}_n)$ involves only combinations of $M_{(\gamma_1,\ldots,\gamma_l)}$, where $\gamma_l > 1$, this shows that $L(\mathcal{Q}_n) \cap \Theta_{\mathbf{1}}(\mathcal{Q}_n) = \{0\}$. □

From this we can conclude immediately

Corollary 3.2. *For each $n \geq 1$,*

$$\Pi_{n+2} = L^2(\Pi_n) \oplus \Theta_{\mathbf{1}}(\Pi_{n+1}).$$

Now we can complete the

Proof of Theorem 3.1. A basis of eigenvectors is constructed inductively, beginning with $\Theta_{\mathbf{1}}$ for Π_1 and $(\Theta_{\mathbf{1}})^2$ for Π_2. If we have constructed a basis for Π_n and Π_{n+1}, then applying L^2 to the former and $\Theta_{\mathbf{1}}$ to the latter yields a basis for Π_{n+2} by Corollary 3.2. The resulting basis consists of all Ω_w, where $\deg w = n+1$. The eigenvalue corresponding to $\Omega_{\mathbf{c}^{n+1}} = \Theta_{\mathbf{1}}^{n+2}$ is 2^{n+2} by Proposition 3.2. Every substitution of a $\mathbf{d}$ for a $\mathbf{c}^2$ divides the eigenvalue by 4. □

Remark 3.1. Note that $\Omega_{\mathbf{c}^n} = \Theta_{\mathbf{1}}^{n+1}$ is the peak set distribution of S_{n+1} as described in Proposition 3.2. It would be interesting to see whether the other eigenvectors Ω_w have similar combinatorial interpretations.

Remark 3.2. In (3.29) we observe $L^2(\Theta_w) = \Theta_{w\mathbf{c}^2} - \Theta_{w\mathbf{d}}$. Similarly, it is straightforward to observe

$$\begin{aligned} \Theta_1(\Theta_w) &= \Theta_{\mathbf{c}w} + \Theta_{w\mathbf{c}} + \sum_{w=w_1\mathbf{c}w_2} \Theta_{w_1\mathbf{d}w_2} \\ &\quad + \sum_{w=w_1\mathbf{d}w_2} (\Theta_{w_1\mathbf{cd}w_2} + \Theta_{w_1\mathbf{dc}w_2}). \end{aligned}$$

For example,

$$\begin{aligned} \Theta_1(\Theta_{\mathbf{cd}}) &= \Theta_{\mathbf{ccd}} + \Theta_{\mathbf{cdc}} + \Theta_{\mathbf{dd}} + \Theta_{\mathbf{ccd}} + \Theta_{\mathbf{cdc}} \\ &= 2\Theta_{\mathbf{c}^2\mathbf{d}} + 2\Theta_{\mathbf{cdc}} + \Theta_{\mathbf{d}^2}. \end{aligned}$$

Remark 3.3. With the basis Ω_w, we can define a new **cd**-index for elements $F \in \Pi$ or for Eulerian posets P, in which the coefficient of the word w is given by the corresponding coefficient of the basis element Ω_w in the expression of F or $F(P)$. This does not appear to have reasonable properties for face posets of polytopes, although it is nonnegative for *simplicial* 3-polytopes.

Remark 3.4. The cone in Π_{n+1} spanned by all Ω_w, $\deg w = n$ is not invariant under the antipode s on Π, as is that spanned by the Θ_w. On the other hand, its extreme rays, and so all its faces, are fixed by the combinatorially interesting map ϑ. It might be useful to have a basis invariant under both s and ϑ. The corresponding index might have some interesting properties.

3.3. **Peaks, hyperplane arrangements and Gorenstein* posets.** It has been pointed out to us by Aguiar and Bergeron (personal communications) that the map ϑ is essentially the map ω of [14]. More precisely, if L is any geometric lattice, let $L_{\hat{0}}$ be the lattice L with a *new* minimal element $\hat{0}$ added. Then $L_{\hat{0}}$ is a graded lattice and so $F(L_{\hat{0}}) \in \mathcal{Q}$.

Proposition 3.5. *For the geometric lattice L of an oriented matroid $\mathcal{O}$,*

$$\vartheta(F(L_{\hat{0}})) = 2\ F(Z),$$

where Z is the dual face lattice of $\mathcal{O}$. In particular, when $\mathcal{O}$ corresponds to an arrangement of hyperplanes, then Z is the face lattice of the associated zonotope.

Proof. If we give the usual R-labeling to L, and label the unique cover relation over $\hat{0}$ by 0, then this follows from the observation of Aguiar-Bergeron and [14, Corollary 3.2]. □

One can view Proposition 3.5 as a complete summary of the relationship between enumerative invariants of chains in a central hyperplane arrangement and those of the associated lattice of intersections, whose study was begun by Zaslavsky in [34].

Since geometric lattices are known to be Cohen-Macaulay posets, that is, the associated complex of chains is a Cohen-Macaulay complex [27], it follows that $L_{\hat{0}}$ is also Cohen-Macaulay and so $F(L_{\hat{0}})$ has a nonnegative representation in the basis $\{F_S\}$ of $\mathcal{Q}$. As a consequence, we get from Proposition 3.5 a special case of [31, Corollary 2.2], namely, we can conclude that arrangements and zonotopes have nonnegative **cd**-indices.

A poset is called *Gorenstein** if it is both Eulerian and Cohen-Macaulay. Such posets include all face posets of spherical complexes. Stanley has conjectured that if P is Gorenstein*, then it has a nonnegative **cd**-index, that is, $[w]_P \geq 0$, for all **cd**-words w [31, Conjecture 2.1]. In light of Theorem 2.1, this amounts to saying that for P Gorenstein*, $F(P)$ must lie in the cone in Π_{n+1} generated by the Θ_w, $\deg w = n$, that is, the nonnegative orthant of Π_{n+1} defined by the basis $\{\Theta_w\}$.

The map ϑ allows us to define a slightly larger simplicial cone than the nonnegative orthant in Π_{n+1} that must contain $F(P)$ for Gorenstein* posets P.

Proposition 3.6. *For Cohen-Macaulay posets P, we always have $\vartheta(F(P)) \geq 0$, that is, $\vartheta(F(P))$ always lies in the cone in Π_{n+1} generated by the Θ_w, $\deg w = n$.*

Proof. By Proposition 1.4, we have $F(P) = \sum h_S F_S$, where $h_S \geq 0$ since P is Cohen-Macaulay [27]. The proposition now follows from the definition of ϑ. □

Considering ϑ restricted to Π_{n+1}, we can view the set $\{\ F \in \Pi_{n+1} \mid \vartheta(F) \geq 0\ \}$ as a simplicial cone in Π_{n+1}. A more explicit description in terms of inequalities on the coefficients $[w]_P$ is given by the rows of the matrix $(\eta_{u,w})$ in (3.27). This cone includes the image of the nonnegative orthant in h-space under the linear map that takes the flag-h vector to the **cd**-index. That this latter cone is given by the inequalities

$$h_T = \sum_{T,\overline{T} \in b[\mathcal{I}^w]} [w] \geq 0 \tag{3.31}$$

follows directly from [31, Proposition 1.3] or from Proposition 2.2. It is straighforward to obtain the inequalities in Proposition 3.6 from those in (3.31): to get the inequality given by row u in $(\eta_{u,w})$, add the expression for h_T over all T for which $\Lambda(T) = S_u$.

Example 3.1. If P is Gorenstein* and the rank of P is 4, then the cone described in Proposition 3.6 is given in **cd**-coordinates by the inequalities

$$\begin{array}{rcl} 4[\mathbf{c}^3] + [\mathbf{cd}] + [\mathbf{dc}] & \geq & 0 \\ 2[\mathbf{c}^3] + 2[\mathbf{cd}] + [\mathbf{dc}] & \geq & 0 \\ 2[\mathbf{c}^3] + [\mathbf{cd}] + 2[\mathbf{dc}] & \geq & 0. \end{array}$$

On the other hand, the nonnegativity of the h_S imply directly that

$$\begin{array}{ccccccccccc} h_2 & = & h_{13} & = & [\mathbf{c}^3] & + & [\mathbf{cd}] & + & [\mathbf{dc}] & \geq 0 \\ h_1 & = & h_{23} & = & [\mathbf{c}^3] & & & + & [\mathbf{dc}] & \geq 0 \\ h_3 & = & h_{12} & = & [\mathbf{c}^3] & + & [\mathbf{cd}] & & & \geq 0 \\ h_\emptyset & = & h_{123} & = & [\mathbf{c}^3] & & & & & \geq 0 \end{array}$$

The second system clearly implies the first.

4. The g-homomorphism

We define an algebra homomorphism from $\mathcal{Q}$ to $\mathbb{Q}[x]$ that extends the definition of the g-polynomial of a graded poset. In the case of the face lattices of (rational) convex polytopes, this polynomial is related to the Poincaré polynomial of the associated toric variety. For all rational polytopes, the g-polynomial is known to have nonngegative coefficients; in the case of simplicial convex polytopes, this fact is known as the *generalized lower bound theorem*. It was proved by Stanley [28, 30] by means of the toric variety associated to a rational polytope. It remains open for nonrational polytopes.

We begin by defining the g-polynomial of a graded poset. For any graded poset P of rank $n+1$ we define two polynomials $f(P,x), g(P,x) \in \mathbb{Q}[x]$ (actually in $\mathbb{Z}[x]$) recursively as follows. If $n+1 = 0$, then $f(P,x) = g(P,x) = 1$. If $n+1 > 0$, then

$$f(P,x) = \sum_{y \in P \setminus \{\hat{1}\}} g([\hat{0}, y], x)(x-1)^{n-r(y)}. \tag{4.32}$$

If $f(P,x) = \sum_{i=0}^{n} \kappa_i x^i$ has been defined, then we define

$$g(P,x) = \kappa_0 + (\kappa_1 - \kappa_0)\, x + \cdots + \left(\kappa_{\lfloor \frac{n}{2} \rfloor} - \kappa_{\lfloor \frac{n}{2} \rfloor - 1}\right) x^{\lfloor \frac{n}{2} \rfloor}. \tag{4.33}$$

For an Eulerian poset P, the vector $(h_0, \ldots, h_n) = (\kappa_n, \ldots, \kappa_1, \kappa_0)$ is what is usually called the *toric h-vector* of P. Since for Eulerian P, $h_i = h_{n-i}$ [30], our definition of $g(P,x)$ agrees with the usual one in the Eulerian case. We note that in [5], this distinction between κ_i and h_i is not made, so their formulas for h_i are, in reality, for h_{n-i}.

Since the coefficients of $g(P,x)$ are integer linear combinations of the quantities $f_S(P)$ (see, for example, [4, Theorem 6], [5, Theorem 3.1] or [18, §4.3]), these necessarily unique expressions can be used to extend this definition to give a linear map

$$g : \mathcal{Q} \longrightarrow \mathbb{Q}[x], \tag{4.34}$$

satisfying $g(F(P)) = g(P,x)$ for any graded poset P. That g is an algebra homomorphism follows from the following observation, which was first noted in [24] in the case of polytope face lattices. Its proof depends on the fact that an interval in a product of posets is the product of intervals from each, and seems not to have appeared in this generality anywhere.

Proposition 4.1. *For graded posets P and Q,*

$$g(P \times Q, x) = g(P, x)g(Q, x).$$

Proof. The conclusion is immediate if $r(P \times Q) = 0$. Otherwise, using (4.32) and induction, we get

$$\begin{aligned}(1-x)f(P \times Q, x) &= g(P,x)\,(1-x)f(Q,x) + (1-x)f(P,x)\,g(Q,x) \\ &\quad - (1-x)f(P,x)\,(1-x)f(Q,x).\end{aligned}$$

By (4.33), $g(P \times Q, x)$ consists of the terms of $(1-x)f(P \times Q, x)$ of degree at most $(r(P) + r(Q) - 1)/2$. Writing $(1-x)f(P,x) = g(P,x) + \tilde{g}(P,x)$, similarly for Q, we note that all the terms of $\tilde{g}(P,x)$ (respectively, $\tilde{g}(Q,x)$) have degree at least $r(P)/2$ (respectively, $r(Q)/2$). Now

$$(1-x)f(P \times Q, x) = g(P,x)\,g(Q,x) - \tilde{g}(P,x)\,\tilde{g}(Q,x),$$

where the last term has only terms of degree at least $(r(P) + r(Q))/2$. The proposition follows. □

Using the fact that $\mathcal{Q}$ is spanned by elements of the form $F(P)$ [18, Proposition 1.1], and recalling that $F(P \times Q) = F(P)F(Q)$ [19], we can conclude

Corollary 4.1. *The map*

$$g : \mathcal{Q} \longrightarrow \mathbb{Q}[x]$$

is an algebra homomorphism.

Proof. We need only check multiplicativity. Suppose $G, H \in \mathcal{Q}$, $G = \sum_i \alpha_i\, F(P_i)$ and $H = \sum_j \beta_j\, F(Q_j)$. Then

$$\begin{aligned}g(GH) &= \sum_{i,j} \alpha_i\beta_j\, g\,(F(P_i)F(Q_j)) \\ &= \sum_{i,j} \alpha_i\beta_j\, g\,(F(P_i \times Q_j)) \\ &= \sum_{i,j} \alpha_i\beta_j\, g(P_i, x)g(Q_j, x) \\ &= g(G)g(H),\end{aligned}$$

by Proposition 4.1 and the fact that $g(F(P)) = g(P, x)$. □

Restricted to Π, there is an explicit formula for g, due essentially to Bayer and Ehrenborg [5]. We follow the development in [5] to express this. Define $p(n,k) = \binom{n}{k} - \binom{n}{k-1}$ and polynomials

$$Q_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k p(n,k) x^k$$

for any n and

$$T_{n+1} = (-1)^{\frac{n}{2}} p\left(n, \frac{n}{2}\right) x^{\frac{n}{2}}$$

for n even. Note that $Q_1 = T_1 = 1$.

Say that a $\mathbf{cd}$-word w is *even* if every element of S_w is even, that is, if $w = \mathbf{c}^{n_1}\mathbf{d}\mathbf{c}^{n_2}\mathbf{d}\cdots\mathbf{c}^{n_k}\mathbf{d}\mathbf{c}^m$, and $n_1,\dots,n_k$ are all even. The following is an interpretation of [5, Theorem 4.2] in our context. It follows since Π is spanned by elements of the form $F(P)$, where P is Eulerian.

Proposition 4.2. *If* $w = \mathbf{c}^{n_1}\mathbf{d}\mathbf{c}^{n_2}\mathbf{d}\cdots\mathbf{c}^{n_k}\mathbf{d}\mathbf{c}^m$, *then*

$$g(\Theta_w) = \begin{cases} 2^{k+1}\ x^k\ Q_{m+1}\ \prod_{j=1}^{k} T_{n_j+1} & \text{if } w \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.1. Note that $g(\Theta_w)$ depends only on the initial and inter-peak distances of the peak set indicated by w, but not on their order, vanishing when any one of these is odd. One could easily describe the kernel of the g map from this. That g is multiplicative on Π is not evident from the expression in Proposition 4.2.

Remark 4.2. Since the basis Ω_w is partially multiplicative, the images $g(\Omega_w)$ should have a simpler expression than that of Proposition 4.2. In particular, since $g(\Theta_{\mathbf{1}}) = 1$, the calculation of $g(\Omega_w)$ is determined entirely by the effect of the map L^2.

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