The p-modular descent algebras

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Abstract. The concept of descent algebras over a field of characteristic zero is extended to define descent algebras over a field of prime characteristic. Some basic algebraic structure of the latter, including its radical and irreducible modules, is then determined. The decomposition matrix of the descent algebras of Coxeter group types A, B, and D are calculated, and used to derive a description of the decomposition matrix of an arbitrary descent algebra. The Cartan matrix of a variety of descent algebras over a finite field is then obtained.

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1. Introduction

For the past 10 years, descent algebras have been studied intensively, not only for their elegant structure [1, 3, 4, 5, 6, 7, 8, 9, 12], but also for their role as the dual of $Qsym$, the Hopf algebra of quasi-symmetric functions [15, 16, 19]. In each of these instances work has focused on the situation where the underlying field has characteristic zero. Lately, however, there has been some interest in the case where the field involved is finite [2, 20]. This paper will build upon this latter research, but first we recall the definition of a descent algebra.

Let W be a Coxeter group with generating set S of fundamental reflections. If L is any subset of S let W_L be the subgroup generated by L. W_L is called a *standard parabolic* subgroup of W and any subgroup conjugate to a standard parabolic subgroup is said to be parabolic. Let X_L be the (unique) set of minimal length representatives of the left

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cosets of W_L in W. Note that $X_L^{-1} = \{g^{-1} | g \in X_L\}$ is then a set of (minimal length) representatives for the right cosets of W_L and that $X_K^{-1} \cap X_L$ is a set representatives for the double cosets corresponding to W_K, W_L.

With this in mind, Solomon proved the following remarkable theorem:

Theorem 1 *(Theorem 1 [21]) For every subset K of S let*

$$x_K = \sum_{w \in X_K} w.$$

Then

$$x_J x_K = \sum a_{JKL} x_L$$

where a_{JKL} is the number of elements $g \in X_J^{-1} \cap X_K$ such that $g^{-1} W_J g \cap W_K = W_L$ with $L = g^{-1} J g \cap K$.

It immediately follows that the set of all x_K form a basis for a (non-commutative) algebra Σ_W over the field of rationals, and it is this algebra that we today call the *descent algebra* of W.

Solomon himself began the study of this algebra by determining its radical, $\mathrm{rad}(\Sigma_W)$, and some properties of $\Sigma_W/\mathrm{rad}(\Sigma_W)$. To describe some of his results let χ_K be the permutation character of W acting on the right cosets of W_K and let G_W be the Z-module generated by all χ_K. Note that each generalised character in G_W has integer values on the elements of W.

Theorem 2 *(Theorem 3 [21])*

1. $\mathrm{rad}(\Sigma_W)$ *is spanned by all differences $x_J - x_K$ where J and K are conjugate subsets of S.*
2. *The linear map θ defined by the images $\theta(x_K) = \chi_K$ is an algebra homomorphism, and* $\ker\theta = \mathrm{rad}(\Sigma_W)$.

Now that we have some details on descent algebras over the rationals, let us consider a p-modular analogue.

Since the structure constants a_{JKL} are integers the Z-module $\mathcal{Z}_W$ spanned by all x_K is a subring of Σ_W, and for any prime p, $p\mathcal{Z}_W$ is an ideal of $\mathcal{Z}_W$. We define the *p-modular descent algebra* of W to be $\Sigma(W,p) = \mathcal{Z}_W/p\mathcal{Z}_W$. Clearly, $\Sigma(W,p)$ is an algebra over $\mathcal{F}_p$, the field of characteristc p, and is the p-modular version of the descent algebra described in Theorem 1.

Let ρ_1 be the natural projection $\mathcal{Z}_W \to \Sigma(W,p)$ and let $\overline{x}_J = \rho_1(x_J)$. Then

$$\overline{x}_J\overline{x}_K = \sum \overline{a}_{JKL}\overline{x}_L$$

where $\overline{a}_{JKL}$ is the image of a_{JKL} in $\mathcal{F}_p$. Furthermore let ρ_2 be the map defined on G_W which reduces character values modulo p, and let $G(W,p)$ be the image of ρ_2.

The map $\phi : \Sigma(W,p) \to G(W,p)$ defined by

$$\phi(\rho_1(x)) = \rho_2(\theta(x)) \text{ for all } x \in \mathcal{Z}_W$$

is clearly well-defined and is an algebra homomorphism. In section 3 we shall give the analogue of Theorem 2 and describe the radical of $\Sigma(W,p)$ using the homomorphism ϕ. We build on this result in section 4 by defining the irreducible modules of $\Sigma(W,p)$. From here we go on to give descriptions of the decomposition matrix of the descent algebras of Coxeter group types A, B and D; and every prime, p. We then use this to form a base for the representation theory of $\Sigma(W,p)$.

2. The parabolic table of marks

In this section we recall some preliminary results that we shall need later.

Let W be a Coxeter group with generating set of fundamental reflections S and let $W_1(=1), W_2, \ldots, W_r(=W)$ be representatives of the conjugacy classes of subgroups of W.

The *table of marks* of W is the $r \times r$-matrix

$$M(W) = \left(\left|\mathrm{Fix}_{W/W_i}(W_j)\right|\right)_{i,j=1,\ldots,r}$$

which records for subgroups W_i, W_j of W the number of fixed points of W_j in the action of W on the cosets of W_i , that is the *mark* of W_j on W/W_i. Note that both W_i and W_j only run through the representatives of conjugacy classes of subgroups of W.

It follows that

$$\left|\mathrm{Fix}_{W/W_i}(W_j)\right| = [N_W(W_i) : W_i] \cdot \left|\{W_i^x \mid x \in W,\, W_j \leq W_i^x\}\right|.$$

Thus, with a suitable ordering of the representatives W_i, we see that $M(W)$ is a lower triangular matrix with non-zero entries on the diagonal, and therefore invertible.

The *parabolic table of marks* is a certain principal submatrix $M^c(W)$ of $M(W)$. This submatrix is defined by those rows and columns of $M(W)$ that correspond to the (standard) parabolic subgroups of W. It is known [7] that the set of conjugacy classes of parabolic subgroups is in one-to-one correspondence with the set E of conjugacy classes of subsets of S via the mapping $J \mapsto W_J$. If we define $\beta_{JK} = \left|\mathrm{Fix}_{W/W_J}(W_K)\right|$ for any $J, K \subseteq S$, then it follows

$$M^c(W) = (\beta_{JK})_{J,K \in E}\,.$$

Recall that a Coxeter element, c, of W is simply a product of all the generators in S in any order. Letting c_K be a Coxeter element of W_K and χ_J be the permutation character of W on W/W_J we have the following result [7].

Lemma 1

$$\begin{aligned}
\beta_{JK} &= [N_W(W_J) : W_J] \cdot |\{W_J^w \mid w \in W,\, W_K \leq W_J^w\}| \\
&= \left|\{w \in X_J^{-1} \cap X_K \mid J^w \cap K = K\}\right| \\
&= a_{JKK} \\
&= \chi_J(c_K).
\end{aligned}$$

In particular, $\beta_{JJ} = [N_W(W_J) : W_J] \neq 0$ *and* β_{JJ} *divides* β_{JK} *for every* $K \subseteq S$.

3. The radical of $\Sigma(W,p)$

The main aim of this section is to prove the following p-modular analogue of Theorem 2.

Theorem 3 $\mathrm{rad}(\Sigma(W,p)) = \ker\phi$. *Moreover,* $\mathrm{rad}(\Sigma(W,p))$ *is spanned by all* $\overline{x}_J - \overline{x}_K$ *where* J, K *are conjugate subsets of* S, *together with all* $\overline{x}_J$ *for which* p *divides* $[N_W(W_J) : W_J]$.

However, before we prove this theorem we shall prove two lemmas.

Let r be the number of rows of $M^c(W)$ and let s be the number of rows indexed by subsets J with $p \not| [N_W(W_J) : W_J]$. Let the p-rank of $M^c(W)$ be the rank of $M^c(W)$ modulo p . Note that this is equal to $\dim G(W,p)$.

Lemma 2 *1. $M^c(W)$ is a lower triangular matrix of rank $r = \dim G_W$.*

2. The p-rank of $M^c(W)$ is s.

Proof. The first part follows immediately from comments made in Section 2. For the second part, if p divides a diagonal entry of $M^c(W)$ then, by Lemma 1, p divides every entry of that row. Thus the rank of $M^c(W) \mod p$ (i.e. $\dim G(W,p)$) is the number of non-zero rows in $M^c(W) \mod p$ and this, by Lemma 1 is again s.

Lemma 3 *1. $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ is commutative.*

2. Every nilpotent element of $\Sigma(W,p)$ lies in $\mathrm{rad}(\Sigma(W,p))$.

Proof. Let θ_1 be the restriction of θ to $\mathcal{Z}_W$. Then θ_1 maps $\mathcal{Z}_W$ onto the commutative ring G_W. By Theorem 2, the kernel of θ_1 is the Z-module $\mathcal{R}_W$ spanned by all $x_J - x_K$ where J and K are conjugate subsets of S, and is a nilpotent ideal of $\mathcal{Z}_W$. In particular $\rho_1(\mathcal{R}_W)$ is a nilpotent ideal of $\Sigma(W,p)$, and therefore $\rho_1(\mathcal{R}_W) \subseteq \mathrm{rad}(\Sigma(W,p))$. Hence there exists an ideal $\mathcal{S}_W$ of Σ_W, the pre-image of $\mathrm{rad}(\Sigma(W,p))$, such that $\mathcal{R}_W \subseteq \mathcal{S}_W$ and $\mathcal{S}_W/p\mathcal{Z}_W \cong \mathrm{rad}(\Sigma(W,p))$. Since $\Sigma(W,p) \cong \mathcal{Z}_W/p\mathcal{Z}_W$, $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p)) \cong \mathcal{Z}_W/\mathcal{S}_W$ is a homomorphic image of $\mathcal{Z}_W/\mathcal{R}_W \cong G_W$. Since the latter ring is commutative the first part follows.

If x is any nilpotent element of $\Sigma(W,p)$ then the coset $x + \mathrm{rad}(\Sigma(W,p))$ is a nilpotent element in the commutative semi-simple algebra

$$\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$$

and so is zero. Therefore $x \in \mathrm{rad}(\Sigma(W,p))$ proving the second part.

With these two lemmas we are now in a position to prove Theorem 3.

Proof. First we note that $\mathrm{rad}(\Sigma(W,p)) \subseteq \ker\phi$. This is because the image of ϕ is an algebra of functions defined over a field and is therefore semi-simple. Consequently the two-sided nilpotent ideal $\phi(\mathrm{rad}(\Sigma(W,p)))$ must be zero.

Now we prove that, if $p|[N_W(W_J):W_J]$, then $\overline{x}_J \in \mathrm{rad}(\Sigma(W,p))$. From the definition of a_{JKL} in Theorem 1, $\overline{a}_{JKL}=0$ unless $L \subseteq K$ and, by Lemma 1, $\overline{a}_{JKK}=0$ also. Thus $\overline{x}_J\overline{x}_K$ is a linear combination of elements $\overline{x}_L$ with $L \subset K$ (and so $|L| \leq |K|-1$). Now, by induction, it follows that $\overline{x}^t_J\overline{x}_K$ is a linear combination of elements $\overline{x}_L$ with $|L| \leq |K|-t$ and so $\overline{x}_J^{|K|+1}\overline{x}_K=0$ for all K. In particular $\overline{x}_J$ is nilpotent and so $\overline{x}_J \in \mathrm{rad}(\Sigma(W,p))$ by Lemma 3.

The elements $\overline{x}_J-\overline{x}_K$ where J and K are conjugate subsets of S are all nilpotent and, by Lemma 3, lie in $\mathrm{rad}(\Sigma(W,p))$. They span a space U of dimension $\dim \mathrm{rad}(\Sigma_W)=\dim \Sigma_W - \dim G_W = 2^{n-1}-r$. In addition there are $r-s$ elements $\overline{x}_J$ corresponding to those rows of $M^c(W)$ for which $p|[N_W(W_J):W_J]$ which also lie in $\mathrm{rad}(\Sigma(W,p))$. These, together with U, span a space of dimension $2^{n-1}-r+(r-s)=2^{n-1}-\dim G(W,p)=\dim\ker\phi$. Hence $\dim \mathrm{rad}(\Sigma(W,p)) \geq \dim\ker\phi$.

This proves that $\ker\phi = \mathrm{rad}(\Sigma(W,p))$ as required and that it is spanned by the desired set of elements.

4. Representation theory of $\Sigma(W,p)$

Some representation theory of Σ_W has been studied, both in the general case, [7], and in the specific cases of the Coxeter groups of type A [12], and B, [5, 6]. In this section we show how the representation theory of $\Sigma(W,p)$ depends on that of Σ_W. Specifically, we shall be interested in the composition factors of the principal indecomposable modules for each of Σ_W and $\Sigma(W,p)$. To study this we need to make the following observation: a representation of Σ_W over $\mathcal{F}_p$ necessarily has $p\mathcal{Z}_W$ in its kernel and so induces a representation of $\Sigma(W,p)$; moreover, every representation of $\Sigma(W,p)$ arises in this way. Therefore we may study the representation theory of $\Sigma(W,p)$ by examining the p-modular representations of Σ_W. We do this in the manner pioneered in group theory: by relating the representations in characteristic zero to those in characteristic p.

More precisely we will be relating the representations in characteristic zero to those in characteristic p via a *decomposition matrix*, $\mathcal{D}$, whose entries d_{ij} are the multiplicity of the modular irreducible representation $\overline{\beta}_j$ as a composition factor of $\overline{\alpha}_i$, where $\overline{\alpha}_i$ is the modular representation determined by the irreducible representation α_i.

This approach is tractable because the irreducible representations are all 1-dimensional. In fact, since $\Sigma_W/\mathrm{rad}(\Sigma_W)$ and $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ are commutative of dimensions r and s (the rank and p-rank of $M^c(W)$ respectively) Σ_W has r 1-dimensional irreducible representations over a field of characteristic zero and s 1-dimensional irreducible representations over a field of characteristic p. It follows (see 54.16, [11]) that the multiplicities of the principal indecomposable modules as direct summands in the regular representation of both Σ_W and $\Sigma(W,p)$ are all 1.

In addition, we can explicitly describe the irreducible representations. As in Section 2 let E denote the set of representatives of the subsets of S that index the rows and columns of $M^c(W)$. For each $K \in E$ define the map $\lambda_K : \Sigma_W \to \mathbb{Q}$ by

$$\lambda_K(x) = \theta(x)(c_K) \text{ for all } x \in \Sigma_W.$$

Since θ is a homomorphism it follows readily that λ_K is also a homomorphism, and therefore a 1-dimensional representation of Σ_W. Note that λ_K is completely determined by its values on basis elements x_J, that $\lambda_K(x_J) = \theta(x_J)(c_K) = \chi_J(c_K) = \beta_{JK}$, and these values of λ_K form the column of the matrix $M^c(W)$ indexed by K. In particular, $\lambda_K|_{\mathcal{Z}_W}$ takes integer values and reducing these values modulo p yields the irreducible representations in a field of characteristic p. Since we know from Lemma 2 that the p-rank of $M^c(W)$ is s we have the following.

Lemma 4 *1. The columns of $M^c(W)$ define the irreducible representations of Σ_W.*

2. The columns of $M^c(W)$ modulo p define the irreducible representations of $\Sigma(W,p)$ and $M^c(W)$ modulo p has precisely s distinct columns.

As a result of this lemma the set E indexes the irreducible representations of Σ_W. We now select a subset $F \subseteq E$ to index the irreducible representations of $\Sigma(W,p)$. In principle any subset that indexes s distinct columns of $M^c(W) \mod p$ will suffice but we shall make a specific choice so that our results are easier to state. From the proof of Lemma 4 , it follows that in $M^c(W) \mod p$ there are exactly s non-zero rows and we let $F \subseteq E$ index this set of rows. Since $M^c(W) \mod p$ is lower triangular of rank p, F also indexes a set of distinct columns of $M^c(W)$ mod p. We define a matrix $\mathcal{D} = (d_{KL})$ whose rows and columns are indexed by the members of E and F respectively. If $K \in E, L \in F$ then $d_{KL} = 1$ if columns K and L of $M^c(W)$ are equal modulo p, and

$d_{KL} = 0$ otherwise. By the previous lemma, the sets E and F index the irreducible representations of Σ_W and $\Sigma(W, p)$ respectively and, since $\mathcal{D}$ determines the structure of each irreducible representation of Σ_W when reduced modulo p, we have:

Proposition 1 *$\mathcal{D}$ is the decomposition matrix of the algebra Σ_W.*

Every $w \in W$ can be written as a product $w1.w2$ where the order of $w1$ is a power of p, and of $w2$ is coprime to p. The element $w2$ is known as the p-regular part of w, and using this terminology we can state the following straightforward result, whose converse is often true.

Proposition 2 *Let $K \in E, L \in F$ head columns of the matrix $M^c(W)$. Then, if c_K and c_L have conjugate p-regular parts, $d_{KL} = 1$.*

Proof. By the arguments in §82 of [11] every character χ_J takes equal values modulo p on c_K and c_L. Thus, $\lambda_K = \lambda_L \mod p$ and so $d_{KL} = 1$.

In the remainder of this section we shall consider descent algebras according to their Coxeter group type to obtain the following description of the decomposition matrix of an arbitrary descent algebra.

Theorem 4 *Let w' be the p-regular part of $w \in W$ and let $\rightarrow_p$ be the relation on E defined by $J \rightarrow_p K$ if $\langle w' \rangle^c$ is conjugate to W_K and $\langle w \rangle^c$ is conjugate to W_J. Let $K \in E$ and $L \in F$. Then $d_{KL} = 1$ if and only if K and L lie in the same class of the equivalence generated by $\rightarrow_p$.*

This theorem is proved by inspecting the various Coxeter group types. The decomposition matrices of the exceptional types E, F, G, and H are given in [13] and a routine check confirms that they satisfy the theorem. The same is true for the type I which was extensively analysed in [22]. For the remaining types, the families A, B, D, we require a more detailed analysis to which the next three subsections are devoted. In each of these cases we obtain a description of the decomposition matrix (Theorems 5, 6, 7); from these characterisations it is easy to see that they too satisfy Theorem 4.

However before we begin, let us recall some notions that will be of use later. A composition of n where $n \geq 0$ is a sequence of positive integers whose sum is n. The integers in the sequence are called components, and by convention the empty sequence is the unique composition of 0. If the sequence is non-increasing then we call it a partition consisting of parts rather than components. We say that two compositions determine the same partition if they each form the same partition when their components are listed in strictly non-increasing order. Let $\pi(n)$ denote the number of partitions of n and $\pi(n,p)$ the number of partitions of n in which no part has multiplicity p or more, then we have the following.

Lemma 5 *(p.41 [18]) $\pi(n,p)$ is the number of partitions of n consisting of parts not divisible by p.*

4.1. Representation theory of $\Sigma(A_{n-1},p)$

In this subsection we let $W = A_{n-1}$ which is isomorphic to the symmetric group S_n acting on $\{1,2,\dots,n\}$ with generating set $S = \{(i,i+1)|i = 1,\dots,n-1\}$. If $K \subseteq S$ then the Coxeter element c_K has cycles on sets $[u..v]$ of consecutive integers. The ordered list of cycle lengths, that is one cycle appears before another if it permutes integers with smaller values, determines and is determined by K. Therefore the subsets of S can be naturally parameterised by compositions of n. The following lemma and corollary are consequences of this parameterisation and Lemmas 2 and 4, which we leave to the reader to verify.

Lemma 6
1. *If $K, L \subseteq S$ then K is conjugate to L if and only if the corresponding compositions determine the same partition of n.*
2. *If $K \subseteq S$ and its corresponding composition has a_i components equal to i (for $i = 1,\dots,n$) then*

$$[N(W_K) : W_K] = a_1!\, a_2! \cdots a_n!$$

Corollary 1
1. *$r = \pi(n)$.*
2. *$s = \pi(n,p)$.*

We are now ready to describe the decomposition matrix of $\Sigma_{A_{n-1}}$.

Theorem 5 *Let $K \in E$, and $L \in F$, then $d_{KL} = 1$ if and only if c_K and c_L have conjugate p-regular parts.*

Proof. There are two equivalence relations δ_1, δ_2 on the set E (which indexes the columns of $M^c(W)$):

$$(K,J) \in \delta_1 \text{ if } \lambda_K = \lambda_J \mod p$$

$(K, J) \in \delta_2$ if c_K, c_J have conjugate p-regular parts.

We have seen from Proposition 2 that $\delta_2 \subseteq \delta_1$. However, by Lemma 4 the number of δ_1-equivalence classes is s which by Corollary 1 is $\pi(n,p)$. By Lemma 5 this is also the number of partitions with no part divisible by p which is the number of equivalence classes of δ_2. Hence $\delta_1 = \delta_2$ and the theorem follows.

4.2. Representation theory of $\Sigma(B_n, p)$

In this section we let $W = B_n$. For convenience we shall represent B_n as a permutation group on $\{\pm 1, \dots, \pm n\}$ with block system $\{i, -i\}_{i=1}^n$ on which it acts as the full symmetric group with kernel of order 2^n. In this case the set of Coxeter generators $S = \{s_0, s_1, \dots, s_{n-1}\}$ is defined as $s_0 = (-1, 1)$ and $s_i = (i, i+1)(-i, -i-1), 1 \leq i \leq n-1$.

Let $K \subseteq S$ and consider the Coxeter element c_K. If c_K has a cycle $(a, b, \dots)$ consisting of positive elements (a *positive cycle*) then it will also have a corresponding *negative cycle* $(-a, -b, \dots)$. Furthermore, at most one cycle of c_K can contain both positive and negative elements, and such a cycle is present if and only if $s_0 \in K$. Note, we may write

$$c_K = x_0 x_1 \tag{1}$$

where x_0 is the cycle containing both positive and negative elements, or 1 if there is no such cycle and x_1 is the product of all the other cycles where positive and negative cycles are placed in matching pairs. Note that x_0 commutes with x_1. Each positive cycle is on some range $[u..v]$ of consecutive integers and the list of lengths of positive cycles taken in the natural order determines and is determined by K. In this way the subsets of S can be parameterised by compositions of integers $m, 0 \leq m \leq n$. The following lemma and corollary are a consequence of the results of [17].

Lemma 7 *1. If $K, L \subseteq S$ then K is conjugate to L if and only if the corresponding compositions determine the same partition.*

2. If $K \subseteq S$ and the corresponding composition is a composition of m with a_i components of size i and t components in all then

$$[N(W_K) : W_K] = 2^t a_1!\, a_2! \cdots a_m!$$

Corollary 2 *1. $r = \sum_{m=0}^{n} \pi(m)$.*

2. If $p \neq 2$ *then* $s = \sum_{m=0}^{n} \pi(m,p)$.

Let K be one of the subsets indexing the rows and columns of $M^c(W)$ and $c_K = x_0 x_1$ as in Equation 1. If x_1 is a p-regular element we say that K is a *p-special* subset of S. Since the order of x_1 is the lowest common multiple of its cycle lengths, K is p-special if and only if the partition corresponding to K has no part divisible by p. By Lemma 5 and Corollary 2, there are precisely s p-special subsets when $p \neq 2$.

Lemma 8 *If* $K \subseteq S$ *then there exists a p-special* $K_1 \subseteq S$ *such that* c_K *and* c_{K_1} *have conjugate p-regular parts.*

Proof. Let $c_K = x_0 x_1$ as in Equation 1 and let x_2 be the p-regular part of x_1. Since x_2 is a power of x_1, its cycles also come in matching positive, negative pairs. Therefore x_2 is conjugate, via a permutation in the centraliser of x_0, to a Coxeter element x_3 with the same properties as x_2. But then $x_0 x_3$ is also a Coxeter element c_{K_1} whose p-regular part is conjugate to that of $x_0 x_1$.

Lemma 9 *If* $p \neq 2$ *the columns of* $M^c(W)$ *which are indexed by the p-special subsets provide a full set of irreducible representations of* $\Sigma(B_n, p)$.

Proof. By the last lemma the columns of $M^c(W) \mod p$ indexed by p-special subsets contain a full set of distinct columns and since there are s such columns they must yield a complete set of irreducible representations of $\Sigma(B_n, p)$.

We are now ready to describe the decomposition matrix of Σ_{B_n}.

Theorem 6 *Let* $K \in E$, *and* $L \in F$. *If* $p \neq 2$ *then* $d_{KL} = 1$ *if and only if* c_K *and* c_L *have conjugate p-regular parts. If* $p = 2$ *then* $F = \{S\}$ *and* $d_{KS} = 1$ *for all* K.

Proof. Suppose first that $p \neq 2$. Proposition 2 has proved one implication already. For the other, suppose $d_{KL} = 1$ and let K_1, L_1 be the p-special subsets, guaranteed by Lemma 8, such that K, K_1 have conjugate p-regular parts and L, L_1 have conjugate p-regular parts. Then, by Proposition 2, $d_{K_1 L_1} = 1$ and Lemma 9 shows that $K_1 = L_1$.

If $p = 2$, Lemma 7 implies that the only $K \in E$ for which 2 does not divide $[N(W_K) : W_K]$ is the one with $t = 0$, namely $K = S$. Therefore

$\Sigma(B_n, 2)$ has just one irreducible representation and so $d_{KL} = 1$ for all $K \in E$, $L \in F = \{S\}$.

4.3. Representation theory of $\Sigma(D_n, p)$.

In this section we let $W = D_n$ and note that it can be considered as a normal subgroup of index 2 in the Coxeter group of type B, $\hat{W} = B_n$. In this case its set of Coxeter generators is $S = \{u, s_1, \ldots, s_{n-1}\}$ where $s_0, s_1, \ldots, s_{n-1}$ are as in the previous subsection, and $u = s_0 s_1 s_0 = (-1, 1)(1, 2)(-1, -2)(-1, 1) = (-1, 2)(1, -2)$.

For any $K \subseteq S$ the parabolic subgroup W_K is isomorphic to $W_0 \times W_1$ where W_0 is a Coxeter group of type D, say D_{n_0} for some $n_0 \leq n$, $n_0 \neq 1$ and $W_1 = \langle K_1 \rangle$ for some $K_1 \subseteq \{s_{n_0}, \ldots, s_{n-1}\}$. Here the group D_2 is $\langle u, s_1 \rangle$ and isomorphic to the group $A_1 \times A_1$ and the group D_3 is $\langle u, s_1, s_2 \rangle$ and isomorphic to a A_3. If $n_0 = 0$ then W_1 is a subgroup of either the group W' generated by $S' = \{s_1, s_2, \ldots, s_{n-1}\}$ or the group W'' generated by $S'' = \{u, s_2, s_3, \ldots, s_{n-1}\}$ which are both isomorphic to A_{n-1}.

Thus to each subset $K \subseteq S$ there is associated via W_1 a composition of $m \leq n$. Each composition occurs this way, except those of $n - 1$. Conversely, for each composition λ of $m \neq n - 1$, there is a unique $K \subseteq S$, unless λ is a composition of n with $\lambda_1 > 1$. In that case there are two subsets with that label, each containing exactly one of s_1 and u.

Consider $K, L \subseteq S$. Then K and L are conjugate in W if and only if their corresponding compositions determine the same partition, unless this partition is a partition of n with all parts even. In that case K and L are conjugate only if they both lie in S' or both in S''.

Consider $W_K \cong W_0 \times W_1$ with $W_0 = D_{n_0}$ for some $n_0 \geq 2$. Then there is a parabolic subgroup $\hat{W}_K \cong \hat{W}_0 \times W_1$ of $\hat{W}$ where $\hat{W}_0 = B_{n_0}$, a Coxeter group of type B. We have $W_K = \hat{W}_K \cap W$ and $[\hat{W}_K : W_K] = 2$. Also $[N_{\hat{W}}(\hat{W}_K) : N_W(W_K)] = 2$ whence β_{KK} is computed from the partition corresponding to K in the same way as in the case for type B.

Now let $n_0 = 0$ and let W_K be a subgroup of W' with corresponding partition μ. Then W_K is a parabolic subgroup of both W and $\hat{W}$. We have $N_{\hat{W}}(W_K) \subseteq W$ if and only if all parts of μ are even. We therefore obtain the following formula.

Lemma 10 *Let $K \subseteq S$ with corresponding partition μ whose part i has multiplicity m_i. Then*

$$[N_W(W_K) : W_K] = 2^{m_1} m_1! \cdots 2^{m_n} m_n!\, a$$

where $a = 1$ unless μ is a partition of n and has at least one odd part. In that case $a = 1/2$.

Let c_K be a Coxeter element of W_K. Again, we have a unique decomposition $c_K = x_0 x_1$ where $x_0 \in W_0$ and $x_1 \in W_1$. We call K a p-special subset if x_1 is p-regular. By the same argument as for type B, we have that for each $K \subseteq S$ there is a p-special $K_1 \subseteq S$ such that c_K and c_{K_1} have conjugate p-regular parts.

Similar considerations as for type B then lead to the following description of the decomposition matrix of Σ_{D_n}.

Theorem 7 *Let $K \in E$, and $L \in F$. If $p \neq 2$ then $d_{KL} = 1$ if and only if c_K and c_L have conjugate p-regular parts. If $p = 2$ and n is even then we have $F = \{S\}$ and $d_{KS} = 1$ for all $K \in E$; if n is odd then we have $F = \{S', S\}$ and $d_{KL} = 1$ if and only if either $L = S$ and $K \neq S'$ or $L = S'$ and $K = S'$.*

Proof. The theorem for $p \neq 2$ follows as in the case for type B and we now suppose that $p = 2$. By Lemma 10, $\beta_{LL} = [N_W(W_L) : W_L]$ is odd only if all $m_i = 0$ (whence μ is the empty partition corresponding to $L = S$) or, if μ is a partition of n with at least one odd part, at most one $m_i = 1$, and all other $m_i = 0$ (whence n is odd and μ is the partition $[n]$ corresponding to $L = S'$). Thus, if n is even then $F = \{S\}$ and $d_{KL} = 1$ for all $K \in E$.

If n is odd then $F = \{S, S'\}$. Since $\beta_{S'S} = 0$ and $\beta_{S'S'} = 1$ we shall have $d_{KS} = 1$ or $d_{KS'} = 1$ depending on whether $\beta_{S'K}$ is even or odd.

To resolve this case we consider the following action on complementary pairs, first as a B_n action. Let $I = \{1, \ldots, n\}$ and let

$$X = \big\{\{P, Q\} \;\big|\; P, Q \subseteq I; P \cup Q = I; P \cap Q = \emptyset\big\}$$

(so we always have $Q = I \setminus P$). Then B_n acts on X as follows. The action of s_i for $i \geq 1$ is induced from its action as $(i, i+1)$ on I and the action of s_0 is given by

$$\{P, Q\}^{s_0} = \{P \perp \{1\}, Q \perp \{1\}\},$$

where $A \perp B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of the sets A, B. Note that, if we define $t_i = s_i \cdots s_1 s_0 s_1 \cdots s_i$ then t_i acts as

$$\{P, Q\}^{t_i} = \{P \perp \{i{+}1\}, Q \perp \{i{+}1\}\},$$

the symmetric difference with $\{i{+}1\}$, and the longest element $w_0 = t_0 t_1 \cdots t_{n-1}$ of $\hat{W}$ acts as the symmetric difference with I whence it fixes every point in X. The complementary pair $\{P, Q\}$ arises from $\{\emptyset, I\}$ by taking symmetric differences with P (or Q). Thus the action of $\hat{W}$ is transitive on all of the 2^{n-1} complementary pairs in X and the stabiliser of $\{\emptyset, I\}$ is $\langle s_1, \ldots, s_{n-1}, w_0\rangle$, a group of index 2^{n-1} in $\hat{W}$.

Now restrict the action to W. Then, as in the case where n is odd we have $w_0 \notin W$, and the stabiliser of $\{\emptyset, I\}$ in W is W', which is of index 2^{n-1} in W. Hence W acts transitively on X and the action is equivalent to the action on the cosets of W'. Therefore we can use it to determine the values $\beta_{S'K}$.

We know that $\beta_{S'K} = 0$ whenever W_K is not conjugate to a subgroup of $W_{S'} = W'$. It remains to investigate the fixed points of parabolic subgroups of $W' \cong A_{n-1}$. Consider s_1 and its fixed points. If $n > 2$ then $\{P, Q\}$ is stable under s_1 if and only if $\{1, 2\} \subseteq P$ or $\{1, 2\} \subseteq Q$. In either case taking symmetric differences with $\{1, 2\}$ yields a different point $\{P', Q'\}$ which is also fixed by s_1. Hence the fixed points of s_1 come in pairs.

A similar argument applies to a Coxeter element c_K of any parabolic subgroup W_K of W' unless $K = S'$. Here we denote by $J \subseteq I$ the set of points moved by c_K. Then we find that $\{P, Q\}$ is stable under c_K if and only if $J \subseteq P$ or $J \subseteq Q$. Again, taking symmetric differences with J produces a different fixed point $\{P', Q'\}$. This shows that $\beta_{S'K}$ is even for all proper parabolic subgroups W_K of W'.

4.4. Cartan matrices

The representation theory of $\Sigma(W, p)$ can now be studied using that of Σ_W. As a first step in this direction we consider the Cartan matrix. First we recall its definition:

Definition 1 *Let $\mathcal{A}$ be any finite dimensional algebra and let $P_1, \ldots, P_r$ be a complete set of principal indecomposable modules for $\mathcal{A}$, with corresponding irreducible modules $T_1, \ldots, T_r$. Let c_{ij} be the multiplicity of*

T_j as a composition factor of P_i. Then the $r \times r$ matrix $C = [c_{ij}]$ is called the Cartan matrix of $\mathcal{A}$.

As a special case of the results of [14] we have:

Theorem 8 *Let C and $\tilde{C}$ be the Cartan matrices of the descent algebra Σ_W and its p-modular counterpart $\Sigma(W,p)$ and let $\mathcal{D}$ be the decomposition matrix of Σ_W. Then $\tilde{C} = \mathcal{D}^T C \mathcal{D}$.*

By Theorems 4 and 8 we immediately see that the Cartan matrix of $\Sigma(W,p)$ can be determined once it is known for Σ_W. The descent algebras of Coxeter group types A and B can therefore be handled by Theorem 5.4 of [12] and Theorem 3.3 of [6] which give the Cartan matrices in characteristic zero. Furthermore, the work of [7] allows the dihedral case to be dealt with, although this case has been solved directly in [22]. However, we have not calculated the Cartan matrix in characteristic zero in any other cases, and such a calculation awaits a more detailed study of these algebras.

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