Corrigendum to "Coincidences among skew Schur functions", Adv. Math. 216 (2007) 118-152. ⋆

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Abstract

We provide a rigorous proof that if D is a connected skew diagram, then the skew Schur function s_D is irreducible.

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We provide a proof for the second assertion of [2, Corollary 6.3], which is both of independent interest and an important reduction in determining when two skew Schur functions are equal. More precisely, we prove

Theorem 1 *For D a connected skew diagram, the skew Schur function s_D is irreducible considered as an element of $\mathbb{Z}[h_1, h_2, \ldots]$.*

PROOF. We will induct on the number $\ell := \ell(D)$ of nonempty rows in the connected skew diagram D. The base case $\ell = 1$ is trivial, as then $s_D = h_{|D|}$ where $|D|$ is the number of cells of D.

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Thus, in the inductive step one may assume $\ell \geq 2$, and assume for the sake of contradiction that s_D is reducible. Express $D = \lambda/\mu$ with $|\lambda|$ minimal, so that, in particular, $\ell(D) = \ell(\lambda) =: \ell$ and $\mu_\ell = 0$. Let $L = \lambda_1 + \ell - 1$, so that by [2, Proposition 6.2(i)] the $\ell \times \ell$ Jacobi-Trudi matrix J for s_D expresses

$$s_D = s \cdot h_L + r, \tag{1}$$

in which both r, s involve only the variables $h_1, h_2, \ldots, h_{L-1}$.

We claim that neither r nor s is the zero polynomial. For r, note that [2, Proposition 6.2(ii)] implies that r must contain the monomial $h_{r_1} \cdots h_{r_\ell}$ with coefficient $+1$ where $r_1, \ldots, r_\ell$ are the lengths of the rows of λ/μ. For s, note that s is $(-1)^{\ell-1}$ times the determinant of the $(\ell-1)\times(\ell-1)$ complementary minor to h_L in J, and the complementary minor is the Jacobi-Trudi matrix for $s_{\hat{\lambda}/\hat{\mu}} = s_{\hat{D}}$, where $\hat{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$, $\hat{\mu} = (\mu_1+1, \mu_2+2, \ldots, \mu_\ell+1)$. Observe that $\hat{D}$ is obtained from D by removing the northwesternmost ribbon from the northwest border of the connected skew diagram D (in english notation).

Thus, (1) shows that s_D is *linear* as a polynomial in h_L. Since we are assuming s_D is reducible, this means s_D must have at least one nontrivial irreducible factor, call it f, which is of degree zero in h_L. This factor f must therefore also divide r, and hence also divide s.

Denote by J^A_B the submatrix obtained from J by removing its rows indexed by the subset A and columns indexed by the subset B. Then the *Lewis Carroll* or *Dodgson condensation* or *Desnanot-Jacobi adjoint matrix* identity [1, Theorem 3.12] asserts that

$$\det J^{1,\ell}_{1,\ell} \cdot \det J = \det J^1_1 \cdot \det J^\ell_\ell - \det J^1_\ell \cdot \det J^\ell_1. \tag{2}$$

Note that the left side of (2) is divisible by f since $\det J = s_D$, and the second term on the right side of (2) is also divisible by f, since $\det J^1_\ell$ is the same as the minor determinant appearing in s in (1). Therefore, the first term on the right of (2) is divisible by f, implying that one of its factors $\det J^1_1$ or $\det J^\ell_\ell$ must be divisible by f. However, one can check that these last two determinants are the Jacobi-Trudi determinants for the skew diagrams E, F obtained from D by removing its first, last row, respectively. Since E, F are connected skew diagrams with fewer rows than D, both s_E, s_F are irreducible by the inductive hypothesis. Hence either $f = s_E$ or $f = s_F$. But since f divides s, its degree satisfies

$$\deg(f) \leq \deg(s) = |D| - (\lambda_1 + \ell - 1)$$

and this last quantity is strictly less than both

$$\begin{aligned} &\deg(s_E) = |D| - (\lambda_1 - \mu_1), \text{ and} \\ &\deg(s_F) = |D| - (\lambda_\ell - \mu_\ell) \end{aligned}$$

since $\ell \geq 2$. This contradicts having either $f = s_E$ or $f = s_F$, ending the proof. □

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References

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