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Abstract

We provide a rigorous proof that if $D$ is a connected skew diagram, then the skew Schur function $s_D$ is irreducible.

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We provide a proof for the second assertion of [2, Corollary 6.3], which is both of independent interest and an important reduction in determining when two skew Schur functions are equal. More precisely, we prove

\textbf{Theorem 1} For $D$ a connected skew diagram, the skew Schur function $s_D$ is irreducible considered as an element of $\mathbb{Z}[h_1, h_2, \ldots]$. 

\textbf{Proof.} We will induct on the number $\ell := \ell(D)$ of nonempty rows in the connected skew diagram $D$. The base case $\ell = 1$ is trivial, as then $s_D = h_{|D|}$ where $|D|$ is the number of cells of $D$.

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Thus, in the inductive step one may assume \( \ell \geq 2 \), and assume for the sake of contradiction that \( s_D \) is reducible. Express \( D = \lambda/\mu \) with \( |\lambda| \) minimal, so that, in particular, \( \ell(D) = \ell(\lambda) =: \ell \) and \( \mu_\ell = 0 \). Let \( L = \lambda_1 + \ell - 1 \), so that by [2, Proposition 6.2(i)] the \( \ell \times \ell \) Jacobi-Trudi matrix \( J \) for \( s_D \) expresses

\[
s_D = s \cdot h_L + r,
\]

in which both \( r, s \) involve only the variables \( h_1, h_2, \ldots, h_{L-1} \).

We claim that neither \( r \) nor \( s \) is the zero polynomial. For \( r \), note that [2, Proposition 6.2(ii)] implies that \( r \) must contain the monomial \( h_{r_1} \cdots h_{r_\ell} \) with coefficient \(+1\) where \( r_1, \ldots, r_\ell \) are the lengths of the rows of \( \lambda/\mu \). For \( s \), note that \( s \) is \((-1)^{\ell-1}\) times the determinant of the \((\ell-1) \times (\ell-1)\) complementary minor to \( h_L \) in \( J \), and the complementary minor is the Jacobi-Trudi matrix for \( s_{\hat{\lambda}/\hat{\mu}} = s_D \), where \( \hat{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell) \), \( \hat{\mu} = (\mu_1 + 1, \mu_2 + 2, \ldots, \mu_\ell + 1) \). Observe that \( \hat{D} \) is obtained from \( D \) by removing the northwesternmost ribbon from the northwest border of the connected skew diagram \( D \) (in English notation).

Thus, (1) shows that \( s_D \) is linear as a polynomial in \( h_L \). Since we are assuming \( s_D \) is reducible, this means \( s_D \) must have at least one nontrivial irreducible factor, call it \( f \), which is of degree zero in \( h_L \). This factor \( f \) must therefore also divide \( r \), and hence also divide \( s \).

Denote by \( J_A^B \) the submatrix obtained from \( J \) by removing its rows indexed by the subset \( A \) and columns indexed by the subset \( B \). Then the Lewis Carroll or Dodgson condensation or Desnanot-Jacobi adjoint matrix identity [1, Theorem 3.12] asserts that

\[
\det J_{1,\ell}^1 \cdot \det J = \det J_1^1 \cdot \det J_{\ell}^\ell - \det J_{1}^1 \cdot \det J_\ell^\ell.
\]

Note that the left side of (2) is divisible by \( f \) since \( \det J = s_D \), and the second term on the right side of (2) is also divisible by \( f \), since \( \det J_1^1 \) is the same as the minor determinant appearing in \( s \) in (1). Therefore, the first term on the right of (2) is divisible by \( f \), implying that one of its factors \( \det J_1^1 \) or \( \det J_\ell^\ell \) must be divisible by \( f \). However, one can check that these last two determinants are the Jacobi-Trudi determinants for the skew diagrams \( E, F \) obtained from \( D \) by removing its first, last row, respectively. Since \( E, F \) are connected skew diagrams with fewer rows than \( D \), both \( s_E, s_F \) are irreducible by the inductive hypothesis. Hence either \( f = s_E \) or \( f = s_F \). But since \( f \) divides \( s \), its degree satisfies

\[
\deg(f) \leq \deg(s) = |D| - (\lambda_1 + \ell - 1)
\]

and this last quantity is strictly less than both

\[
\deg(s_E) = |D| - (\lambda_1 - \mu_1), \text{ and } \deg(s_F) = |D| - (\lambda_\ell - \mu_\ell)
\]
since $\ell \geq 2$. This contradicts having either $f = s_E$ or $f = s_F$, ending the proof. □

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References
