Chapter 1

Introduction

"With these dark words begins my tale"
– W. Wordsworth, Force of Prayer

Areas of mathematics that display great elegance have always been a source of fascination and inspiration to those in the subject. Descent algebras are no exception. From their introduction by Solomon, in 1976, [Sol76], they have held the attention of many mathematicians around the globe with their intricate nature (for example [Eti84],[BBHT92],[BL96b], [BL96a], [MR95b],[AvW97]). This interest has been most notable over the past five years, during which it has emerged that not only is this an area of great beauty, but also a bridge linking many mathematical fields, including group theory, combinatorics, character theory, homology, and Lie algebras. Initially, however, progress in this area was impeded by their somewhat complex description [Sol76]. Before we give this description, since descent algebras are associative, let us recall some fundamental theory of associative algebras, [CR62], [Bur93].

An associative algebra, A, is an associative ring, with a vector space structure. We say that A is *semisimple* if the only nilpotent two-sided ideal is the zero ideal. Associative algebras are studied by way of their modules.

Definition 1 *A module is said to be* irreducible *if it contains no proper submodule.*

Definition 2 *A module is said to be* indecomposable *if it cannot be written as the direct sum of two or more proper submodules.*

Modules for an associative algebra A are closely related to representations of A. If M is an A-module, then the associated representation φ is given by: for all $a \in A$, $\varphi(a)$ is the linear transformation on M given by

$$\varphi(a)(m) = ma,$$

where m belongs to M. Similarly representations of A give rise to A-modules. A module which is irreducible (or indecomposable) gives rise to a representation which is also termed irreducible (or indecomposable).

Lemma 1 *If A is semisimple, then the irreducible and indecomposable modules of A coincide.*

Unfortunately descent algebras are not semisimple so it is important to distinguish the indecomposable modules from the irreducible modules.

Definition 3 *The* radical *of A, denoted by* $\mathrm{rad}(A)$, *is the unique maximal nilpotent two-sided ideal in A. The* nilpotency index *of* $\mathrm{rad}(A)$ *is the smallest $k \in \mathbb{N}$ such that* $(\mathrm{rad}(A))^k = 0$.

The radical of A is an important ideal of A, and knowledge of it is useful, since the factor ring $A/\mathrm{rad}(A)$ has the following property.

Lemma 2 $A/\mathrm{rad}(A)$ *is semisimple.*

Finally we recall that every associative algebra A can be written as a direct sum

$$A = e_1 A \oplus e_2 A \oplus e_3 A \oplus \dots \tag{1.1}$$

where $e_1, e_2, e_3, \dots$ are idempotents, $\sum_i e_i = 1$, the idempotents are orthogonal ($e_i e_j = 0$ if $i \neq j$), and the idempotents are minimal (no e_i is a sum of orthogonal idempotents). In the direct sum (1.1) each $e_i A$ is an indecomposable A-module, called a principal indecomposable module, and it determines, and is determined by the irreducible A-module $e_i A / e_i \mathrm{rad}(A)$. In this way a one-to-one correspondence between principal indecomposable, and irreducible modules is established.

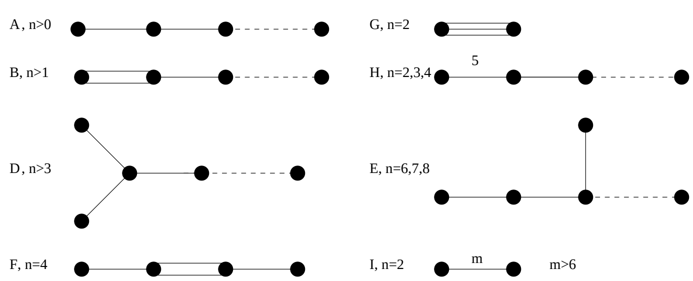
Descent algebras are defined from Coxeter groups, and so we now review some background theory of Coxeter groups.

The general theory of geometrical groups, that is groups of transformations of a plane, or space, was developed in the late 19th century by Schwarz [Sch73], Klein, von Dyck [vD82], and Poincaré. This was built upon by Goursat in 1889, [Gou89], who enumerated all finite reflection groups that exist in four dimensional Euclidean space. Little more was achieved concerning finite reflection groups until 1928, when Cartan considered the n-dimensional scenario in [Car28]. In this work he put forward the n-dimensional finite reflection groups that he had calculated, which were then proved by Coxeter to be an exhaustive list in 1934, [Cox34]. From here on, the set of all finite reflection groups became known as Coxeter groups, although the sufficiency of the abstract relations given below was proved by Witt in 1941, [Wit41], and the use of graphs was a tool of both Witt and Dynkin, as well as Coxeter (for more details consult the historical remarks in [Cox63]). Since their enumeration, Coxeter groups have been a valuable asset in many areas, most notably semisimple Lie theory (see [Bou68]), and the classification of finite simple groups (see [Gor94]). However, it is their role in the generation of descent algebras that we are interested in.

Let W be a group with generating set S. For all s_i, s_j in S, let $m(s_i, s_j)$ denote the order of $s_i s_j$ in W. We say that W is a Coxeter group if W is subject only to the set of relations

$$(s_i s_j)^{m(s_i,s_j)} = 1$$

for all s_i, s_j in S, where $m(s_i, s_i) = 1$ and $m(s_i, s_j) = m(s_j, s_i) \geq 2$ for $s_i \neq s_j$ in S. To further clarify the concept of a Coxeter group, let W be a finite irreducible Coxeter group, that is W cannot be written as a direct product of other Coxeter groups, and let S contain n generators. If an undirected graph, G, is drawn with S as its vertex set, with a single edge between nodes s_i and s_j if $m(s_i, s_j) = 3$, a double edge if $m(s_i, s_j) = 4$, a triple edge if $m(s_i, s_j) = 6$, and an edge labelled $k \notin \{1, 2, 3, 4, 6\}$ if $m(s_i, s_j) = k$, then G will always be one of the following types of graph



Each element, w, in W can be written as a product of generators, the smallest number of which is called the *length* of w, and is denoted by $l(w)$. For geometrical reasons the elements of S are called fundamental reflections (for a more detailed exposition on Coxeter groups, consult [Hum90]).

Let W be a Coxeter group, with generating set S, of fundamental reflections. If J is any subset of S, let W_J be the subgroup generated by J. Let X_J be the unique set of minimal length representatives of the left cosets of W_J in W. Note that $X_J^{-1} = \{x^{-1} | x \in X_J\}$ is the unique set of minimal length representatives for the right cosets of W_J. Then the following theorem holds.

Theorem 1 *(Solomon) For every subset K of S, let*

$$\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.$$

Then for subsets J and K in S

$$\mathcal{X}_J \mathcal{X}_K = \sum a_{JKL} \mathcal{X}_L$$

where a_{JKL} is the number of elements $x \in X_J^{-1} \cap X_K$ such that $x^{-1}W_Jx \cap W_K = W_L$, with $L = x^{-1}Jx \cap K$.

From this it follows that the set of all $\mathcal{X}_K$ is a basis for an algebra, Σ_W, over the field of rationals with integer structure constants a_{JKL}. It was this set of algebras, one for each Coxeter group W, that came to be known collectively as the *descent algebras of Coxeter groups*.

After this initial description in 1976, little more was achieved, until 1984, when Etienne, [Eti84], proved a result from which the existence of the descent algebra of the Coxeter groups of type A, or the *symmetric groups*, could be deduced. This was followed by Atkinson in 1985 who reproved Theorem 1 using yet another approach, [Atk86]. 1985 also saw a simple proof of Theorem 1 produced by Garsia and Remmel for the case of the symmetric groups, Section 5 [GR85]. However, this time the proof had implications. By their clever use of shuffle products in their proof, Garsia and Remmel showed how the structure constants a_{JKL} given in Theorem 1 could be obtained from matrices. More precisely, a_{JKL} is the number of matrices whose matrix entries satisfy various conditions related to J, K and L. This was one of the first instances of what we shall therefore call a "matrix interpretation" of Theorem 1, and it was this interpretation that was to start a revival of interest in the subject, by making it more accessible to manipulation.

Indeed, it was only four years before an extensive paper by Garsia and Reutenauer was published, revealing much of the underlying structure of the descent algebra of the symmetric groups laid bare by the matrix interpretation. It was also at this point that the term "descent algebra" was coined, since each of the summands, σ, of a given $\mathcal{X}_K$ satisfies

$$i^\sigma > (i+1)^\sigma$$

for all i belonging to a certain fixed set of integers. Alternatively, the ordered list $[\ldots,(i-1)^\sigma, i^\sigma, (i+1)^\sigma, \ldots]$ is hardly ever strictly increasing, and the sequence of list values decreases, or "descends" in specific places. Other results in this paper included how the descent algebra of the symmetric groups acts on Lie monomials, and an explicit description of the radical (Section 2). In addition, a complete set of indecomposable representations of the descent algebra, a basis of idempotents, and a natural basis of nilpotent elements for the radical were also identified. In 1991, these properties were added to by Atkinson, [Atk92], whose results included the construction of the irreducible representations of this descent algebra, and the nilpotency index of the radical.

1992 brought a shift in focus within the subject, as efforts were concentrated on investigating the descent algebras of other Coxeter groups. In particular, the descent algebra of the Coxeter groups of type B, or the *hyperoctahedral groups*, was studied in detail in [BB92a], and [Ber92]. In the first of these works, the Bergeron brothers introduced a matrix interpretation, in the style of Garsia and Remmel, for this family of descent algebras. In the second, by again extending previous work done, this time by Garsia and Reutenauer, they derived a set of minimal idempotents (Section 2), and a basis of nilpotent elements for the radical. Their work on idempotents was further extended in [BB92b], in which they were used to obtain an analogue of the Stirling numbers of the first kind. With the help of the computing package MAPLE, 1992 also saw a unification of work on the multiplicative structure of the descent algebras studied in detail so far, [BB92c]. Moreover, in [BBHT92], an understanding of the multiplicative structure of other descent algebras, including a complete decomposition of the descent algebra of the Coxeter groups of type I, or *dihedral groups* was realised, as was a basis of minimal idempotents for the descent algebra of any Coxeter group, lifted from the parabolic Burnside ring into the descent algebra (Section 7). More recently interest has turned away from the core subject of descent algebras, to work related to the area, for example, the use of descent algebras with non-commutative symmetric functions ([GKL$^+$95], [Ung95]); the extension of descent algebras to Hochschild homology ([Ber95a], [Ber95b]); the Fourier-transform of descent algebras ([BF95]); the dual of a descent algebra ([MR95a]; and the advent of two other descent algebras: an analogous commutative "descent algebra" ([Cel95a], [Cel95b]), and the descent algebra of a bi-algebra ([Pat94]). However, there are still two key questions to be answered.

1. Is there a convenient way to multiply elements in the descent algebra of the Coxeter groups of type D?
2. How does the structure of a given descent algebra change when the underlying field is of prime characteristic, p?

These are precisely the questions we intend to address in the following chapters.

In Chapter 2 we begin by reproving the existence of a matrix interpretation of the multiplication rule for the descent algebra of the symmetric groups, by combining graph theory and group theory. This approach, distinct from any other used to prove this result, is then built upon, enabling us to obtain the analogous interpretation of the multiplication rule for the descent algebra of the hyperoctahedral groups. From this point we extend our approach once more, and in doing so, derive a matrix interpretation of the multiplication rule for the descent algebra of the Coxeter groups of type D in the style of the well known interpretations developed earlier in the chapter.

To address the second of these problems, we return to Solomon's Theorem in the form seen earlier, and define the descent algebra over a field of prime characteristic, p. Since the results in Solomon's paper, [Sol76], formed a base from which the subject grew, we proceed to derive results analogous to those found in his 1976 paper, in an attempt to form a base within this area. This we do by initially describing a homomorphism from our new algebra, the *p-modular descent algebra* of W, $\Sigma(W, p)$, into the ring of all integral combinations of permutation characters of W, whose values have been reduced modulo p. The homomorphism is then used to describe the radical of $\Sigma(W, p)$ in a manner similar to the description of the radical of Σ_W in Theorem 3 [Sol76]. In addition, the irreducible representations of $\Sigma(W, p)$ are also determined.

With this base established, we proceed to use results obtained in Chapters 2 and 3 to establish results analogous to those found in [GR89], [BB92a] and [Ber92] for the p-modular descent algebras of the symmetric and hyperoctahedral groups. More precisely, we use the matrix interpretation for the symmetric groups case, and our knowledge of the radical of $\Sigma(W, p)$ to obtain an explicit description of the radical for the p-modular descent algebra of the symmetric groups, and its nilpotency index. In addition we are also able to describe explicitly the irreducible representations of this algebra. Using similar techniques, we also give a precise description of the radical, its nilpotency index, and the irreducible representations of the p-modular descent algebra of the hyperoctahedral groups. In an appendix, we summarise our calculations which completely describe the principal indecomposable modules for the algebra $\Sigma(I_{2n}, p)$.

Chapter 2

Interpretations of Solomon's rule

"We want to multiply, are you going to do it?"
– *Powerstation, Some Like it Hot*

The importance of a matrix interpretation of Solomon's multiplication rule for descent algebras, [Sol76], is evident from the flourish of work that has been produced since the interpretation for the symmetric groups (the Coxeter groups of type A) first became widely known, [GR89]. Since then, its ease of comprehension has made it possible to establish many results for this descent algebra, including its action on Lie monomials (Section 2,[GR89]), and the nilpotency index of its radical ([Atk92]).

Such success with the symmetric groups has prompted investigations into whether such interpretations can be found for other infinite Coxeter group families. Consequently, an interpretation has been developed for the descent algebras of the hyperoctahedral groups (the Coxeter groups of type B), [BB92a]. As yet, no formal proof of it has been published, although it is claimed in [BB92a] that the proof is analogous to the proof for the symmetric groups case seen in [GR85]. Since there also now exists a complete description of the multiplication table for the descent algebra of the dihedral groups ([BBHT92]), it only remains to find a matrix interpretation for the Coxeter groups of type D. However, until now, there has been little success in developing such an interpretation for this infinite family.

In this chapter we shall give a new approach to developing the interpretation for the symmetric groups, and build upon it to justify the interpretation for the hyperoctahedral groups. In doing so we will support the claim in [BB92a] that a proof analogous to that for the symmetric groups can be found for this case. By extending this approach once more, we shall conclude by formulating the missing matrix interpretation for the Coxeter groups of type D. At this stage, however, we describe our approach in general terms to show that there is indeed a general framework. Subsequent sections will then give the details which distinguish the three types A, B, and D.

For our group W, we begin by choosing the set of generators, S, to be a set of permutations that act on a set N. For a subset J of S, we define the *graph* $\mathcal{J} = (\mathcal{V}, \mathcal{E})$ as follows. The vertex set $\mathcal{V} = N$, and the edge set $\mathcal{E}$ consists of edges between i and j, if a generator in J switches i and j. In general we shall use roman capitals $J, K, \ldots$ for subsets of S, and their calligraphic counterparts $\mathcal{J}, \mathcal{K}, \ldots$ for their associated graphs.

We then identify the node set of each connected component of $\mathcal{J}$ and order these sets by their least element in the natural way. We label these node sets $\mathcal{J}_i$, where the value of i is determined by a simple algorithm dependent on W. The *ordered representation* of $\mathcal{J}$ is then defined as the ordered list of all $\mathcal{J}_i$ taken in the natural order, that is if $\mathcal{J}_i$ appears in the list before $\mathcal{J}_k$ then $i < k$. We then introduce compositions $\lambda = [\lambda_1, \ldots, \lambda_k]$ of n, where a composition is defined as follows.

Definition 4 *A* composition *of some non-negative integer, n, is an ordered list of positive integers whose sum is n. We denote this by $\lambda \vDash n$.*

The value of n will be $|N|$, or $\frac{1}{2}|N|$ depending on W, and in the latter case we introduce other compositions too, including compositions of $m \leq n$. In this case if $\lambda \vDash m$, then λ_0 denotes $n - m$. It transpires that compositions of various forms correspond naturally with subsets of S, and that if λ is a composition that corresponds to a subset J, then the isomorphism type of W_J can be described by a group $\boldsymbol{W_\lambda}$ in all cases we wish to study.

Taking J and K to be any subsets of S, and $x \in W$, we let $\mathcal{J}x$ denote the image of the graph $\mathcal{J}$ under x; that is (i^x, j^x) is an edge in $\mathcal{J}x$ if and only if (i, j) is an edge in $\mathcal{J}$. We let $\mathcal{J} \cap \mathcal{K}$ be the graph with vertex set N whose edges are those present in both $\mathcal{J}$ and $\mathcal{K}$. From here we show that each node set $\mathcal{J}_i x \cap \mathcal{K}_j$ gives us a connected component of $\mathcal{J}x \cap \mathcal{K}$, and determine the order in which these node sets must be taken to yield the ordered representation of $\mathcal{J}x \cap \mathcal{K}$.

Having done this we are able to place all the values $|\mathcal{J}_i x \cap \mathcal{K}_j|$ into a matrix such that when the non-zero entries are read by row, we obtain a composition, η, such that $\boldsymbol{W_\eta}$ is isomorphic to $W_{x^{-1}Jx \cap K}$. By observing that each matrix corresponds to some $x \in X_J^{-1} \cap X_K$, and changing $\mathcal{X}_J$ to B_λ, where J and λ correspond, we can restate Solomon's Theorem in terms of basis elements B_λ, and matrices; that is arrive at a matrix interpretation.

With this method in mind, we shall begin by recalling a version Solomon's Theorem, from which we will develop each interpretation:

For every subset K of S, let

$$\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.$$

Then for subsets J and K in S

$$\mathcal{X}_J \mathcal{X}_K = \sum_{x \in X_J^{-1} \cap X_K} \mathcal{X}_{x^{-1}Jx \cap K}$$

We observe that to get the original form seen in Chapter 1, we just group together the summands for each fixed $L = x^{-1}Jx \cap K$.

2.1 The rule for the symmetric groups

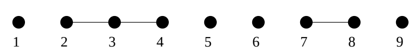
Let us take W to be the symmetric group S_n, that is the Coxeter group of type A whose graph is on $n-1$ nodes. Then we can take $N = \{1, \ldots, n\}$, and S to be the set of $n-1$ transpositions $s_1, s_2, \ldots, s_{n-1}$, where $s_i = (i, i+1)$.

If J is a subset of S, then the graph $\mathcal{J} = (N, \mathcal{E})$ of J has vertex set $\{1, \ldots, n\}$, and edge set $\mathcal{E} = \{(i, i+1) \in J\}$. Suppose now that $\mathcal{J}$ has r connected components. A node set is associated with each component, and we can order these sets by their least elements in a natural way. Once ordered, we can label them $\mathcal{J}_1, \ldots, \mathcal{J}_r$ such that $1 \in \mathcal{J}_1$, and define the ordered representation of $\mathcal{J}$ to be the ordered list

$$(\mathcal{J}_1, \ldots, \mathcal{J}_r).$$

Note that if $u \in \mathcal{J}_i$ and $v \in \mathcal{J}_j$, and $i < j$, then $u < v$.

Example 1 In S_9. If $J = \{(2,3), (3,4), (7,8)\}$, then $\mathcal{J}$ is



with

$$\mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2,3,4\}, \mathcal{J}_3 = \{5\}, \mathcal{J}_4 = \{6\}, \mathcal{J}_5 = \{7,8\}, \mathcal{J}_6 = \{9\}.$$

The ordered representation of $\mathcal{J}$ is

$$(\{1\}, \{2,3,4\}, \{5\}, \{6\}, \{7,8\}, \{9\}).$$

Let us define $\mathcal{W}_{\mathcal{J}_i}$ to be the subgroup of S_n that fixes all points outside of $\mathcal{J}_i$, and

$$\mathcal{W}_{\mathcal{J}} = \mathcal{W}_{\mathcal{J}_1} \times \ldots \times \mathcal{W}_{\mathcal{J}_r}.$$

Observe that $\mathcal{W}_{\mathcal{J}} = W_J$. If we let λ be a composition of n, with components $\{\lambda_i\}_{i=1}^{r}$, and define

$$\boldsymbol{S_\lambda} = S_{\lambda_1} \times \ldots \times S_{\lambda_r},$$

then $\boldsymbol{S_\lambda} \cong \mathcal{W}_{\mathcal{J}}$, where the node sets $\mathcal{J}_i$ of $\mathcal{J}$ satisfy

$$|\mathcal{J}_i| = \lambda_i. \tag{2.1}$$

Moreover, $\boldsymbol{S_\lambda}$ is equal to $\mathcal{W}_{\mathcal{J}}$ if S_{λ_i} is regarded as the symmetric group on the points of $\mathcal{J}_i$.

The ordered representation of $\mathcal{J}x \cap \mathcal{K}$, where $x \in X_J^{-1} \cap X_K$, is given by the following lemma.

Lemma 3 *Let J and K be subsets of S, and let $x \in X_J^{-1} \cap X_K$. Let the ordered representation of $\mathcal{J}$ be $(\mathcal{J}_1, \ldots, \mathcal{J}_r)$, and $\mathcal{K}$ be $(\mathcal{K}_1, \ldots, \mathcal{K}_s)$. Then the ordered representation of $\mathcal{J}x \cap \mathcal{K}$ is*

$$(\mathcal{J}_1x \cap \mathcal{K}_1, \mathcal{J}_2x \cap \mathcal{K}_1, \ldots, \mathcal{J}_rx \cap \mathcal{K}_1, \mathcal{J}_1x \cap \mathcal{K}_2, \mathcal{J}_2x \cap \mathcal{K}_2 \ldots, \mathcal{J}_rx \cap \mathcal{K}_s)$$

with empty sets removed.

PROOF To prove that

$$(\mathcal{J}_1x \cap \mathcal{K}_1, \mathcal{J}_2x \cap \mathcal{K}_1, \ldots, \mathcal{J}_rx \cap \mathcal{K}_1, \mathcal{J}_1x \cap \mathcal{K}_2, \mathcal{J}_2x \cap \mathcal{K}_2 \ldots, \mathcal{J}_rx \cap \mathcal{K}_s)$$

is the ordered representation of $\mathcal{J}x \cap \mathcal{K}$, it is sufficient to prove the following two statements

1. The elements of each set are less than those that appear in any set later in the list.
2. Each non-empty set of nodes $\mathcal{J}_qx \cap \mathcal{K}_m$ is indeed the node set of a connected component of $\mathcal{J}x \cap \mathcal{K}$.

To prove the first statement we must show that

1. The elements in $\mathcal{J}_qx \cap \mathcal{K}_m$ are less than those in $\mathcal{J}_qx \cap \mathcal{K}_{m+1}$.
2. The elements in $\mathcal{J}_qx \cap \mathcal{K}_m$ are less than those in $\mathcal{J}_{q+1}x \cap \mathcal{K}_m$.

The first of these two cases follows immediately, since all nodes in $\mathcal{K}_m$ are less than those in $\mathcal{K}_{m+1}$ by definition.

To prove the second we need only show that if the node $i \in \mathcal{J}_qx \cap \mathcal{K}_m$ and $j \in \mathcal{J}_{q+1}x \cap \mathcal{K}_m$, then $i < j$. To do this we shall first prove that all nodes in $\mathcal{K}_m$ appear in increasing order in the list $(1^x 2^x \ldots n^x)$.

From the definition of X_K as a set of minimal length coset representatives, it follows that if $x \in X_K$ then $l(xk) > l(x)$ for all $k \in K$. Alternatively, we can say that if $x \in X_K$ then $l(kx^{-1}) > l(x^{-1})$

for all $k \in K$. In S_n, $l(x)$ is the number of inversions in x, that is the number of $h < l$ for which $l^x < h^x$ (Section 1.6, [Hum90]). Hence it follows that for all $k = (h, h+1) \in K$ we have $h^{x^{-1}} < (h+1)^{x^{-1}}$, since kx^{-1}, x^{-1} differ only in the reversing of h and $h+1$. From this we can deduce that $h, h+1$ appear in increasing order in the list

$$(1^x 2^x \dots n^x).$$

It follows that all the nodes in $\mathcal{K}_m$ appear in increasing order in $(1^x 2^x \dots n^x)$. Now suppose that $i \in \mathcal{J}_q x \cap \mathcal{K}_m$, $j \in \mathcal{J}_{q+1} x \cap \mathcal{K}_m$. Then $i^{x^{-1}} = u \in \mathcal{J}_q$, and $j^{x^{-1}} = v \in \mathcal{J}_{q+1}$. It follows that $u < v$, and so u^x appears before v^x in $(1^x 2^x \dots n^x)$. However, $u^x = i$ and $v^x = j$, and since we know that the nodes of $\mathcal{K}_m$ appear in increasing order in $(1^x 2^x \dots n^x)$, it follows that $i < j$.

The second statement will follow if we can prove the following assertions.

1. The sets $\mathcal{J}_q x \cap \mathcal{K}_m$ are all disjoint.
2. No edge in $\mathcal{J} x \cap \mathcal{K}$ connects nodes in different subsets $\mathcal{J}_q x \cap \mathcal{K}_m$ and $\mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$.
3. For every $i, i+1 \in \mathcal{J}_q x \cap \mathcal{K}_m$, an edge exists in $\mathcal{J} x \cap \mathcal{K}$ between i, $i+1$.

Again, the first assertion follows since all $\mathcal{J}_q$ and $\mathcal{K}_m$ are disjoint and x is a bijection from N to itself.

To prove the second assertion, let (u, v) be an edge in $\mathcal{J} x \cap \mathcal{K}$, such that $u \in \mathcal{J}_q x \cap \mathcal{K}_m$ and $v \in \mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$. We know that $\mathcal{J}_q$ and $\mathcal{J}_{q'}$ are node sets of connected components of $\mathcal{J}$, so $\mathcal{J}_q x$ and $\mathcal{J}_{q'} x$ must be node sets of connected components of $\mathcal{J} x$. Hence, $q = q'$. Similarly, $\mathcal{K}_m$ and $\mathcal{K}_{m'}$ are node sets of connected components of $\mathcal{K}$, and so $m = m'$.

For the third assertion, let $i, i+1 \in \mathcal{J}_q x \cap \mathcal{K}_m$ and let $i^{x^{-1}} = u$, and $(i+1)^{x^{-1}} = u + l$. Since we know from the proof of statement 1 case 2 that all $i \in \mathcal{K}_m$ appear in increasing order in the list $(1^x 2^x \dots n^x)$, we can deduce that $l \geq 1$. From the proof of statement 1 case 2 we can also deduce that because X_J^{-1} is defined as a set of minimal length right coset representatives, we have that $v^x < (v+1)^x$ for all $(v, v+1) \in J$.

Therefore, since $u, u+l \in \mathcal{J}_q$, we have that $u + k \in \mathcal{J}_q$ for all $k = 0, \dots, l$ such that

$$u^x < (u+1)^x < \dots < (u+l-1)^x < (u+l)^x.$$

However, $u^x = i$, $(u+l)^x = i+1$, so it follows that $l = 1$, and so, by definition $(u, u+l)$ is an edge in $\mathcal{J}$. Therefore, since $u^x = i$ and $(u+l)^x = i+1$, it follows that $(i, i+1)$ is an edge in $\mathcal{J} x \cap \mathcal{K}$, and we are done. $\square$

As a consequence of Lemma 2 [Sol76], and Lemma 3

$$\begin{aligned} x^{-1}W_J x \cap W_K &= W_{x^{-1}Jx\cap K} \\ &= \mathcal{W}_{\mathcal{J}x\cap\mathcal{K}} \\ &= \mathcal{W}_{\mathcal{J}_1 x\cap\mathcal{K}_1} \times \ldots \times \mathcal{W}_{\mathcal{J}_r x\cap\mathcal{K}_s} \\ &= (x^{-1}\mathcal{W}_{\mathcal{J}_1} x \cap \mathcal{W}_{\mathcal{K}_1}) \times \ldots \times (x^{-1}\mathcal{W}_{\mathcal{J}_r} x \cap \mathcal{W}_{\mathcal{K}_s}) \\ &\cong (x^{-1}S_{\lambda_1} x \cap S_{\mu_1}) \times \ldots \times (x^{-1}S_{\lambda_r} x \cap S_{\mu_s}) \end{aligned}$$

where λ and μ are suitable compositions of n determined by J, K respectively, according to condition (2.1). Note that the final isomorphism symbol is an equality if $x^{-1}S_{\lambda_i}x \cap S_{\mu_j}$ is regarded as the group of permutations on $\mathcal{J}_i x \cap \mathcal{K}_j$.

Let

$$z_{ij} = |\mathcal{J}_i x \cap \mathcal{K}_j|$$

then, by Theorem 1.3.10 [JK81], we have a bijective mapping

$$\zeta : x \mapsto (z_{ij})$$

from $X_J^{-1} \cap X_K$ into the set of $s \times r$ matrices with non-negative integer entries, $\boldsymbol{z} = (z_{ij})$, which satisfy

$$\sum_i z_{ij} = \lambda_j, \; \sum_j z_{ij} = \mu_i.$$

Observe that reading the non-zero entries of the matrix $\boldsymbol{z}$ by row give a composition, η, of n. We say that η is the reading word of $\boldsymbol{z}$, and note that $\boldsymbol{S_\eta}$ is isomorphic to $W_{x^{-1}Jx\cap K}$. We also observe that each matrix corresponds to one $x \in X_J^{-1} \cap X_K$, given in Solomon's Theorem. Therefore, if we now rename the basis elements such that $\mathcal{X}_J$ becomes B_λ, where the components of λ in order are the sizes of the node sets of $\mathcal{J}$ taken in the natural order, we can recast Solomon's Theorem in terms of compositions and matrices as follows.

2.1.1 The matrix interpretation

Theorem 2 *For every composition μ of n, let X_μ be the unique set of minimal length left coset representatives of $S_n/\boldsymbol{S_\mu}$. Let*

$$B_\mu = \sum_{\sigma \in X_\mu} \sigma.$$

If λ, μ are compositions of n, then

$$B_\lambda B_\mu = \sum_{\boldsymbol{z}} B_\eta$$

where the sum is over all matrices $\boldsymbol{z} = (z_{ij})$ with non-negative integer entries that satisfy

1. $\sum_i z_{ij} = \lambda_j$,

2. $\sum_j z_{ij} = \mu_i$.

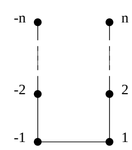
For each matrix, $\boldsymbol{z}$*,* η *is the reading word of* $\boldsymbol{z}$*.*

This is precisely the classical matrix interpretation of Solomon's Theorem for the symmetric groups, for instance Proposition 1.1, [GR89].

2.2 The rule for the hyperoctahedral groups

The definition we use when W is the Coxeter group of type B, B_n, is as follows. Let $N = \{-n, \ldots, -1, 1, \ldots, n\}$, then B_n is the group generated by the set $S = \{s_i\}_{i=0}^{n-1}$ that acts on N such that $s_i = (-i-1, -i)(i, i+1)$ for $i \neq 0$, and $s_0 = (-1, 1)$. We observe from this definition that if $\sigma \in B_n$ then $(-i)^\sigma = -(i^\sigma)$, for all $i \in N$.

Let J be any subset of S, then the graph, $\mathcal{J} = (N, \mathcal{E})$, of J is the graph whose node set is N, and whose edge set is $\mathcal{E} = \{(i,j) | \text{some generator in } J \text{ switches } i \text{ and } j\}$. Note from the description of our generators, it follows that if an edge joins i and j in $\mathcal{J}$, then there also exists an edge in $\mathcal{J}$ between $-i$ and $-j$. Therefore, $\mathcal{J}$ will always be a subgraph of



such that "vertical" components of $\mathcal{J}$ are isomorphic, via the bijection on the nodes that maps node i to $-i$.

Each connected component of $\mathcal{J}$ has an associated node set, and we can order these according to their least elements. Once ordered, we can also label them

$$\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots$$

in a natural way such that

1. If -1 and 1 are contained in the same component, then we label that component $\mathcal{J}_0$.

2. Otherwise we label the components $\mathcal{J}_i, i \neq 0$, such that $1 \in \mathcal{J}_1$, and then set $\mathcal{J}_0 = \{\}$.

We then define the ordered representation of $\mathcal{J}$ to be the ordered list

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots).$$

Observe that if -1 and 1 do not belong to the same connected component then $\mathcal{J}_0 = \{\ \}$, that is *one entry in the ordered representation of $\mathcal{J}$ may be the empty set*. Also observe that since the vertical components of $\mathcal{J}$ are isomorphic, by our labelling we have that $|\mathcal{J}_{-i}| = |\mathcal{J}_i|$.

Define $\mathcal{W}_{\mathcal{J}_i}$ to be the group of all permutations on N that fix all points outside $\mathcal{J}_i$, and $\mathcal{W}_{\mathcal{J}}$ to be the subgroup of

$$\ldots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \ldots$$

such that if $\sigma \in \mathcal{W}_{\mathcal{J}}$ then $(-i)^\sigma = -(i^\sigma)$ for all $i \in N$. We shall write $|_B$ to denote this, that is

$$\mathcal{W}_{\mathcal{J}} = (\ldots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \ldots)|_B$$

and note that $W_J = \mathcal{W}_{\mathcal{J}}$.

Now let us consider the set of compositions $\{\lambda | \lambda \vDash m, m \leq n\}$. Define

$$\begin{aligned} \boldsymbol{B_\lambda} &= B_{n-m} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \\ &= B_{\lambda_0} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \end{aligned}$$

where $\lambda \vDash m \leq n$, with components $\{\lambda_i\}_{i=1}^k$. Note that when $m = 0$, λ is the empty composition, which we shall denote by [ ]. Recall that B_0 is the trivial group, and $B_1 \cong S_2$. Let us also note that

$$\begin{aligned} \boldsymbol{B_\lambda} &= B_{n-m} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \\ &\cong (\mathcal{W}_{\mathcal{J}_{-k}} \times \ldots \times \mathcal{W}_{\mathcal{J}_k})|_B \\ &= \mathcal{W}_{\mathcal{J}} \end{aligned}$$

where the node sets $\mathcal{J}_i$ of $\mathcal{J}$ satisfy the conditions

$$|\mathcal{J}_{-i}| = |\mathcal{J}_i| = \lambda_i \text{ for } i > 0, \quad \frac{1}{2}|\mathcal{J}_0| = \lambda_0 = n - m. \tag{2.2}$$

Definition 5 *Let $\sigma \in B_n$, then $b(\sigma)$, is the value of the sum*

$$\sum_{\substack{0<i<j\leq n \\ j^\sigma < i^\sigma}} 1 - \sum_{\substack{1\leq i\leq n \\ i^\sigma<0}} i^\sigma.$$

Theorem 3 *If $\sigma \in B_n$, then $l(\sigma) = b(\sigma)$.*

PROOF We can prove that $l(\sigma) \leq b(\sigma)$ by an induction on $b(\sigma)$. Clearly the inequality holds when $b(\sigma) = 0$, since then σ is the identity permutation. Let $b(\sigma) > 0$, then we can find $s \in S$ such that $b(s\sigma) < b(\sigma)$ since

- Suppose that $1^\sigma, 2^\sigma, \ldots, n^\sigma$ is not increasing, then there exists an $i \in \{1, 2, \ldots, n-1\}$ such that $i^\sigma > (i+1)^\sigma$, and so $b(s_i\sigma) < b(\sigma)$.

- Suppose that that $1^\sigma, 2^\sigma, \ldots, n^\sigma$ is increasing, then as $b(\sigma) > 0$ it follows that 1^σ is negative, and $b(s_0\sigma) < b(\sigma)$.

Therefore, by our inductive hypothesis $l(s\sigma) \leq b(s\sigma) < b(\sigma)$, and so $l(\sigma) \leq l(s\sigma) + 1 \leq b(\sigma)$.

We can also prove that $b(\sigma) \leq l(\sigma)$, and therefore our result, by an induction on $l(\sigma)$. The inequality holds when $l(\sigma) = 0$ since in this case σ is the identity permutation. For $l(\sigma) > 0$, let $\sigma = s_{i_1}s_{i_2}\ldots s_{i_k}$, where $l(\sigma) = k$, and $s_{i_j} \in S$ for all $j = 1, \ldots k$. Then $l(\sigma s_{i_k}) = l(\sigma) - 1$. By induction, $b(\sigma s_{i_k}) \leq l(\sigma s_{i_k})$, however, since each $s_i \in S$ either switches $i, i+1$ and $-i, -i-1$, or switches -1 and 1,it follows that $b(\sigma s_{i_k}) = b(\sigma) \pm 1$. Therefore,

$$b(\sigma) \leq b(\sigma s_{i_k}) + 1 \leq l(\sigma s_{i_k}) + 1 = l(\sigma)$$

This concludes our second inductive proof. □

We can now describe the ordered representation of $\mathcal{J}x \cap \mathcal{K}$, for $x \in X_J^{-1} \cap X_K$.

Lemma 4 *Let J and K be subsets of S, and let $x \in X_J^{-1} \cap X_K$. If the ordered representation of $\mathcal{J}$ is $(\mathcal{J}_{-r}, \ldots, \mathcal{J}_0, \ldots, \mathcal{J}_r)$, and $\mathcal{K}$ is $(\mathcal{K}_{-s}, \ldots, \mathcal{K}_0, \ldots, \mathcal{K}_s)$. Then the ordered representation of $\mathcal{J}x \cap \mathcal{K}$ is*

$$(\mathcal{J}_{-r}x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r+1}x \cap \mathcal{K}_{-s}, \ldots, \mathcal{J}_1 x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r}x \cap \mathcal{K}_{-s+1}, \ldots, \mathcal{J}_r x \cap \mathcal{K}_s)$$

with empty sets, apart from $\mathcal{J}_0 x \cap \mathcal{K}_0$ if empty, removed.

PROOF As with the previous section, our result will follow if we can show that in

$$(\mathcal{J}_{-r}x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r+1}x \cap \mathcal{K}_{-s}, \ldots, \mathcal{J}_1 x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r}x \cap \mathcal{K}_{-s+1}, \ldots, \mathcal{J}_r x \cap \mathcal{K}_s)$$

the following statements hold

1. A node in any set is less than any node belonging to any other set that appears later in the list.
2. Each non-empty set $\mathcal{J}_q x \cap \mathcal{K}_m$ in the list is the node set of a connected component of $\mathcal{J}x \cap \mathcal{K}$.

The first statement will follow if we can show that

1. If $i \in \mathcal{J}_q x \cap \mathcal{K}_m$ and $j \in \mathcal{J}_q x \cap \mathcal{K}_{m+1}$, then $i < j$.
2. If $i \in \mathcal{J}_q x \cap \mathcal{K}_m$ and $j \in \mathcal{J}_{q+1} x \cap \mathcal{K}_m$, then $i < j$.

Clearly the first case holds since all nodes in $\mathcal{K}_m$ are less than those in $\mathcal{K}_{m+1}$ by definition.

To prove the second case, we need to show that all nodes in $\mathcal{K}_m$ appear in increasing order in

$$((-n)^x \dots (-1)^x 1^x \dots n^x).$$

By the definition of X_K as a set of minimal length coset representatives, it follows that if $x \in X_K$ then $l(xk) > l(x)$, and hence $l(kx^{-1}) > l(x^{-1})$, for all $k \in K$. However, by Theorem 3, in B_n, $l(x) = b(x)$. Therefore, if $k = (-h-1, -h)(h, h+1), h > 0$, then

$$(-h-1)^{x^{-1}} < (-h)^{x^{-1}}, h^{x^{-1}} < (h+1)^{x^{-1}},$$

since kx^{-1}, x^{-1} differ only in the inverting of $-h-1$ and $-h$; h and $h+1$. Hence $h, h+1$ appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$. If $k = (-1, 1)$, then kx^{-1}, x^{-1} differ only in the inverting of -1 and 1, so -1 and 1 appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$. Therefore, we can conclude that the nodes of $\mathcal{K}_m$ appear in increasing order in

$$((-n)^x \dots (-1)^x 1^x \dots n^x).$$

Let $i^{x^{-1}} = u \in \mathcal{J}_q$, and $j^{x^{-1}} = v \in \mathcal{J}_{q+1}$. By definition $u < v$, and so u^x appears before v^x in $((-n)^x \dots (-1)^x 1^x \dots n^x)$. Since we also know that the nodes of $\mathcal{K}_m$ appear in increasing order in this list, it follows that $i < j$.

The second statement will follow if we can show that the following assertions hold.

1. The sets $\mathcal{J}_q x \cap \mathcal{K}_m$ are disjoint,
2. No edge of $\mathcal{J}x \cap \mathcal{K}$ connects two nodes in two different sets, $\mathcal{J}_q x \cap \mathcal{K}_m$ and $\mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$.
3. (a) For all $i, i+1 \in \mathcal{J}_q x \cap \mathcal{K}_m$, there exists an edge between i and $i+1$ in $\mathcal{J}x \cap \mathcal{K}$.
   (b) If $-1, 1 \in \mathcal{J}_0 x \cap \mathcal{K}_0$, then there exists an edge between -1 and 1 in $\mathcal{J}x \cap \mathcal{K}$.

The first assertion holds since all the $\mathcal{J}_q$ and $\mathcal{K}_m$ are disjoint, and x is a bijection of N into itself.

For the second assertion, let (u, v) be an edge of $\mathcal{J}x \cap \mathcal{K}$, such that $u \in \mathcal{J}_q x \cap \mathcal{K}_m$ and $v \in \mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$. Since $\mathcal{J}_q$ and $\mathcal{J}_{q'}$ are node sets of connected components of $\mathcal{J}$, it follows that $\mathcal{J}_q x$ and $\mathcal{J}_{q'} x$ are node sets of connected components of $\mathcal{J}x$, and $q = q'$. Similarly, since $\mathcal{K}_m$ and $\mathcal{K}_{m'}$ are node sets of connected components of $\mathcal{K}$, it follows that $m = m'$.

The third assertion can be proved by the following. The proof of statement 1 case 2 yields that since X_J^{-1} is a set of minimal length right coset representatives of W_J, then

$$(-v-1)^x < (-v)^x, v^x < (v+1)^x \tag{2.3}$$

for all $(-v-1, -v)(v, v+1) \in J$, and $(-1)^x < 1^x$ if $(-1, 1) \in J$.

CASE a) Let $i^{x^{-1}} = u$, and $(i+1)^{x^{-1}} = u+l$. By the proof of statement 1 case 2, we know that all $i \in \mathcal{K}_m$ appear in increasing order in $((-n)^x \ldots (-1)^x 1^x \ldots n^x)$, and so $l \geq 1$. Since $u, u+l \in \mathcal{J}_q$, it follows that $u+k \in \mathcal{J}_q$ for all $k = 0, \ldots, l$ (unless $u+k=0$). Hence, for all $u+k \in \mathcal{J}_q$, by (2.3)

$$u^x < (u+1)^x < \ldots < (u+l-1)^x < (u+l)^x.$$

However, $u^x = i$, $(u+l)^x = i+1$, therefore it follows that $l=1$, and an edge exists between i and $i+1$ in $\mathcal{J}x \cap \mathcal{K}$.

CASE b) If $-1, 1 \in \mathcal{J}_0 x \cap \mathcal{K}_0$, then for some $-u, u \in \mathcal{J}_0$, $(-u)^x = -1$, $u^x = 1$. However, by the proof of statement 1 case 2, nodes -1 and 1 appear in increasing order in

$$((-n)^x \ldots (-1)^x 1^x \ldots n^x)$$

so $-u < u$. By (2.3) and the comment that follows on $(-1,1) \in J$, we have that

$$(-u)^x < (-u+1)^x < \ldots < (-1)^x < 1^x < \ldots < (u-1)^x < u^x.$$

However, $(-u)^x = -1$, and $u^x = 1$, hence it follows that $u = 1$, and an edge exists between -1 and 1 in $\mathcal{J}x \cap \mathcal{K}$. This now proves the second statement. □

From Lemma 2 [Sol76] and the above lemma, we can now deduce that if $x \in X_J^{-1} \cap X_K$, then

$$\begin{aligned}
x^{-1}W_J x \cap W_K &= W_{x^{-1}Jx \cap K} \\
&= \mathcal{W}_{\mathcal{J}x \cap \mathcal{K}} \\
&= [\mathcal{W}_{\mathcal{J}_{-r}x \cap \mathcal{K}_{-s}} \times \ldots \times \mathcal{W}_{\mathcal{J}_r x \cap \mathcal{K}_s}]|_B \\
&= [(x^{-1}\mathcal{W}_{\mathcal{J}_{-r}} x \cap \mathcal{W}_{\mathcal{K}_{-s}}) \times \ldots \times (x^{-1}\mathcal{W}_{\mathcal{J}_r} x \cap \mathcal{W}_{\mathcal{K}_s})]|_B \\
&= [(x^{-1}S^-_{\lambda_r} x \cap S^-_{\mu_s}) \times \ldots \times (x^{-1}S^+_{\lambda_r} x \cap S^+_{\mu_s})]|_B
\end{aligned}$$

where λ and μ are compositions that correspond to $\mathcal{J}$ and $\mathcal{K}$ via (2.2). The superscript $-$ or $+$ on the symmetric group, say S_{λ_i}, simply denotes that for the given $i > 0$, S_{λ_i} acts on $\mathcal{J}_{-i}$ or $\mathcal{J}_i$ respectively: for example, $x^{-1}S^-_{\lambda_i} x \cap S^-_{\mu_j}$ is the group of all permutations acting on

$$\mathcal{J}_{-i}x \cap \mathcal{K}_{-j}.$$

Lemma 5 *Let $x \in X_J^{-1} \cap X_K$. Then $x' \in \mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$ if and only if $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_i x' \cap \mathcal{K}_j|$ for all i and j, and $x' \in B_n$.*

PROOF If $x' \in \mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$, say $x' = \sigma x \tau$, $\sigma \in \mathcal{W}_\mathcal{J}$, $\tau \in \mathcal{W}_\mathcal{K}$, then for each i,

$$\mathcal{J}_i x' = \mathcal{J}_i \sigma x \tau = \mathcal{J}_i x \tau.$$

so for each i, j we have

$$\mathcal{J}_i x' \cap \mathcal{K}_j = \mathcal{J}_i x\tau \cap \mathcal{K}_j = [\mathcal{J}_i x \cap \mathcal{K}_j]\tau.$$

Since τ is a bijection, we have proved half the lemma.

Conversely, $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_i x' \cap \mathcal{K}_j|$ implies that for a fixed j, subsets $\{\mathcal{J}_i x \cap \mathcal{K}_j\}_i$ and $\{\mathcal{J}_i x' \cap \mathcal{K}_j\}_i$ form two dissections of $\mathcal{K}_j$ into subsets which can be collected naturally into pairs of subsets of equal order. Hence, for each $j > 0$, there exists a $\tau_j \in \mathcal{W}_{\mathcal{K}_j}$ which satisfies

$$[\mathcal{J}_i x \cap \mathcal{K}_j]\tau_j = \mathcal{J}_i x' \cap \mathcal{K}_j \text{ for all } i.$$

Choose $\tau_{-j} \in \mathcal{W}_{\mathcal{K}_{-j}}$ such that for all $h \in \mathcal{K}_j$, $-h \in \mathcal{K}_{-j}$, $(-h)^{\tau_{-j}} = -(h^{\tau_j})$. Then since $x' \in B_n$, we know that for each $j > 0$, τ_{-j} satisfies

$$[\mathcal{J}_i x \cap \mathcal{K}_{-j}]\tau_{-j} = \mathcal{J}_i x' \cap \mathcal{K}_{-j}.$$

Since $x' \in B_n$, we also know that we can choose a $\tau_0 \in \mathcal{W}_{\mathcal{K}_0}$ such that for all $h, -h \in \mathcal{K}_0$, $(-h)^{\tau_0} = -(h^{\tau_0})$, and τ_0 satisfies

$$[\mathcal{J}_i x \cap \mathcal{K}_0]\tau_0 = \mathcal{J}_i x' \cap \mathcal{K}_0.$$

With this construction,

$$\tau = \ldots \tau_{-2}\tau_{-1}\tau_0\tau_1\tau_2 \ldots \in (\ldots \times \mathcal{W}_{\mathcal{K}_{-2}} \times \mathcal{W}_{\mathcal{K}_{-1}} \times \mathcal{W}_{\mathcal{K}_0} \times \mathcal{W}_{\mathcal{K}_1} \times \mathcal{W}_{\mathcal{K}_2} \times \ldots)|_B = \mathcal{W}_{\mathcal{K}}.$$

Moreover, τ satisfies

$$\mathcal{J}_i x\tau = \mathcal{J}_i x' \text{ for all } i,$$

so there exists a $\sigma \in \mathcal{W}_{\mathcal{J}}$ such that $x' = \sigma x \tau$ as stated. □

Let $z_{ij} = |\mathcal{J}_i x \cap \mathcal{K}_j|$. Consider the map

$$\zeta : x \mapsto (z_{ij})$$

from $X_J^{-1} \cap X_K$ into the set of all $(2s+1) \times (2r+1)$ matrices

$$\begin{pmatrix} z_{(-s)(-r)} & \cdots & z_{(-s)0} & \cdots & z_{(-s)r} \\ \vdots & & \vdots & & \vdots \\ z_{0(-r)} & \cdots & z_{00} & \cdots & z_{0r} \\ \vdots & & \vdots & & \vdots \\ z_{s(-r)} & \cdots & z_{s0} & \cdots & z_{sr} \end{pmatrix}$$

with non-negative integer entries that satisfy

$$\begin{array}{ll} \sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0, \\ \sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0, \end{array}$$

and

$$z_{ij} = z_{(-i)(-j)}.$$

Then ζ is surjective by the following argument. Let $\boldsymbol{z} = (z_{ij})$ be a matrix such that

$$\begin{array}{ll} \sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0, \\ \sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0, \end{array}$$

and

$$z_{ij} = z_{(-i)(-j)}.$$

If we can find a $\sigma \in B_n$ such that $z_{ij} = |\mathcal{J}_i\sigma \cap \mathcal{K}_j|$ for all i, j, then by Lemma 5 we can find an element $x \in X_J^{-1} \cap X_K$ in the same double coset as σ such that $z_{ij} = |\mathcal{J}_i x \cap \mathcal{K}_j|$ for all i, j. From this the surjectivity of ζ will follow.

Define σ by first defining it on $\mathcal{J}_1, \ldots, \mathcal{J}_r$ such that

$$|\mathcal{J}_i\sigma \cap \mathcal{K}_j| = z_{ij}$$

for all j. Then by putting $(-h)^\sigma = -(h^\sigma)$ for $h \in \mathcal{J}_i$ for $i = 1, \ldots r$, and using $z_{ij} = z_{(-i)(-j)}$, we shall also have that $|\mathcal{J}_i\sigma \cap \mathcal{K}_j| = z_{ij}$ for all $i < 0$ and all j. Now σ has been defined on all $\mathcal{J}_i$ where $i \neq 0$.

To define σ on $\mathcal{J}_0$, we first define it so that $|\mathcal{J}_0\sigma \cap \mathcal{K}_j| = z_{0j}$ for all $j > 0$, and then extend this definition to all $j < 0$, where we require $(-h)^\sigma = -(h^\sigma)$ for all $h \in \mathcal{J}_0$. To define σ on $\mathcal{J}_0$ and $\mathcal{K}_0$, let $\mathcal{K}_0^+$ be the set of all $k \in \mathcal{K}_0$ such that $k > 0$, and let $\mathcal{K}_0^-$ be the set of all $k \in \mathcal{K}_0$ such that $k < 0$. From the conditions $\sum_{i=-s}^{s} z_{i0} = 2\lambda_0, \sum_{j=-r}^{r} z_{0j} = 2\mu_0,$ and $z_{ij} = z_{(-i)(-j)}$, we can deduce that z_{00} is even. Define σ such that $|\mathcal{J}_0\sigma \cap \mathcal{K}_0^+| = \frac{1}{2} z_{00}$, and then extend this definition to $\mathcal{K}_0^-$ where we require $(-h)^\sigma = -(h^\sigma)$ for all $h \in \mathcal{J}_0$. By construction $\sigma \in B_n$, and so ζ is surjective.

By Lemma 5, we also have that ζ is injective, and so ζ is a bijection from $X_J^{-1} \cap X_K$ into the set of all $(2s+1) \times (2r+1)$ matrices

$$\begin{pmatrix} z_{(-s)(-r)} & \cdots & z_{(-s)0} & \cdots & z_{(-s)r} \\ \vdots & & \vdots & & \vdots \\ z_{0(-r)} & \cdots & z_{00} & \cdots & z_{0r} \\ \vdots & & \vdots & & \vdots \\ z_{s(-r)} & \cdots & z_{s0} & \cdots & z_{sr} \end{pmatrix}$$

with non-negative integer entries that satisfy

$$\begin{array}{ll} \sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0, \\ \sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0, \end{array}$$

and

$$z_{ij} = z_{(-i)(-j)}.$$

Let $\boldsymbol{z} = (z_{ij})$ be the matrix into which $x \in X_J^{-1} \cap X_K$ maps. We observe that reading the non-zero matrix entries by row gives a composition of $2n$, with components $\eta_{-l}, \ldots, \eta_0, \ldots, \eta_l$ where $\eta_0 = z_{00}$ if $z_{00} \neq 0$. If $z_{00} = 0$ then the composition has components $\eta_{-l}, \ldots, \eta_{-1}, \eta_1, \ldots, \eta_l$ where η_1 is the first non-zero matrix entry read by row after z_{00}. Note that with each of these compositions $\eta_i = \eta_{-i}$, since $z_{ij} = z_{(-i)(-j)}$. Also observe that the composition of $2n$ is such that

$$(S_{\eta_{-l}} \times \ldots \times S_{\eta_{-1}} \times B_{\frac{1}{2}\eta_0} \times S_{\eta_1} \times \ldots \times S_{\eta_l})|_B,$$

or

$$(S_{\eta_{-l}} \times \ldots \times S_{\eta_{-1}} \times S_{\eta_1} \times \ldots \times S_{\eta_l})|_B$$

if $z_{00} = 0$, is isomorphic to $\boldsymbol{B_\eta}$, for some composition $\eta \models m \leq n$, with components $\eta_1, \ldots \eta_l$. Moreover, $\boldsymbol{B_\eta}$ is isomorphic to $W_{x^{-1}Jx \cap K}$. In addition, we note that each matrix corresponds to one $x \in X_J^{-1} \cap X_K$. Thus, if we recode the basis elements given by Solomon such that $\mathcal{X}_J$ now becomes B_λ, where λ is obtained from $\mathcal{J}$ via (2.2), we may now give a matrix interpretation of Solomon's Theorem for the descent algebra of B_n.

Proposition 1 [1] *For every composition μ of an integer $m \leq n$, let X_μ be the unique set of minimal length left coset representatives of $B_n/\boldsymbol{B_\mu}$. Let*

$$B_\mu = \sum_{\sigma \in X_\mu} \sigma.$$

If $\lambda = [\lambda_1, \ldots, \lambda_r], \mu = [\mu_1, \ldots, \mu_s]$ are compositions of $l, m \leq n$ respectively, then

$$B_\lambda B_\mu = \sum_{\boldsymbol{z}} B_\eta$$

where the sum is over all matrices $\boldsymbol{z} = (z_{ij})$, where $i = -s, \ldots, s$ and $j = -r, \ldots, r$, with non-negative integer entries that satisfy

1. $\sum_{i=-s}^{s} z_{ij} = \lambda_{|j|},\ j \neq 0,$
2. $\sum_{i=-s}^{s} z_{i0} = 2(n-l),$
3. $\sum_{j=-r}^{r} z_{ij} = \mu_{|i|},\ i \neq 0,$
4. $\sum_{j=-r}^{r} z_{0j} = 2(n-m),$
5. $z_{ij} = z_{(-i)(-j)}.$

[1] Unfortunately, due to the intersection of standard notations, the proliferation of "B"s with different meanings in this proposition is unavoidable.

For each matrix $\boldsymbol{z}$, $\eta = [z_{01}, \ldots, z_{0r}, z_{1(-r)}, \ldots, z_{sr}]$, *with zero entries omitted.*

However, we can further simplify this adaptation of Solomon's Theorem by making the following observation.

Let $\lambda = [\lambda_1, \ldots, \lambda_r] \vDash l \leq n$, and $\mu = [\mu_1, \ldots, \mu_s] \vDash m \leq n$, then in view of Proposition 1 to obtain η we only need to obtain the matrix entries $z_{01}, \ldots, z_{0r}, z_{1(-r)}, \ldots, z_{sr}$. Therefore, it would be useful to reformulate conditions 1 to 4 of Proposition 1 such that only these entries are involved. This we can do using the equation $z_{ij} = z_{(-i)(-j)}$to obtain

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + z_{i(-j)}) = \lambda_j, \ j \neq 0$,
2. $\frac{1}{2}z_{00} + \sum_{i=1}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + z_{i(-j)}) = \mu_i, \ i \neq 0$,
4. $\frac{1}{2}z_{00} + \sum_{j=1}^{r} z_{0j} = n - m.$

Thus, instead of using $(2s+1) \times (2r+1)$ matrices such that conditions 1 to 5 of Proposition 1 hold, we can use "partial matrices" of the form

$$P = \begin{pmatrix} & & z_{00} & \cdots & z_{0r} \\ z_{1(-r)} & \cdots & z_{10} & \cdots & z_{1r} \\ \vdots & & \vdots & & \vdots \\ z_{s(-r)} & \cdots & z_{s0} & \cdots & z_{sr} \end{pmatrix}$$

where

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + z_{i(-j)}) = \lambda_j, \ j \neq 0$,
2. $z_{00} + \sum_{i=1}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + z_{i(-j)}) = \mu_i, \ i \neq 0$,
4. $z_{00} + \sum_{j=1}^{r} z_{0j} = n - m.$

Now consider "templates" of the form

$$\boldsymbol{t} = \begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0r} \\ & y_{11} & y_{12} & \cdots & y_{1r} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{s1} & y_{s2} & \cdots & y_{sr} \\ z_{s0} & z_{s1} & z_{s2} & \cdots & z_{sr}. \end{pmatrix}$$

Every partial matrix P corresponds to a template $\boldsymbol{t}$ by the correspondence

$$\begin{aligned} z_{ij} &\leftrightarrow z_{ij} \\ z_{i(-j)} &\leftrightarrow y_{ij} \end{aligned}$$

for $i = 0, 1, \ldots, s$, $j = 0, 1, \ldots, r$ when mapping z_{ij}, and $i = 1, \ldots, s$, $j = 0, 1, \ldots, r$ when mapping $z_{i(-j)}$. Our conditions on P become

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + y_{ij}) = \lambda_j,\ j \neq 0$,
2. $z_{00} + \sum_{i=1}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + y_{ij}) = \mu_i,\ i \neq 0$,
4. $z_{00} + \sum_{j=1}^{r} z_{0j} = n - m$

for $\boldsymbol{t}$. Proposition 1 can now be further simplified by being rewritten in terms of templates.

2.2.1 The matrix interpretation

Consider "templates" with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0r} \\ & y_{11} & y_{12} & \cdots & y_{1r} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{s1} & y_{s2} & \cdots & y_{sr} \\ z_{s0} & z_{s1} & z_{s2} & \cdots & z_{sr} \end{pmatrix}$$

where

1. All entries in a template are non-negative integers.
2. The y-lines do not have entries in column 0.

Let the above template establish generic names for template entries.

Definition 6 *We define the* reading word, $r(\boldsymbol{t})$, *of a given template* $\boldsymbol{t}$, *to be*

$$[z_{01}, z_{02}, \ldots, z_{0r}, y_{1r}, \ldots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \ldots, z_{1r}, \ldots, z_{s0}, z_{s1}, z_{s2}, \ldots, z_{sr}]$$

with zero entries omitted.

Theorem 4 *For every composition μ of an integer $m \leq n$,let X_μ be the unique set of minimal length left coset representatives of $B_n/\boldsymbol{B_\mu}$. Let*

$$B_\mu = \sum_{\sigma \in X_\mu} \sigma.$$

If λ, μ are compositions of $l, m \leq n$ respectively, then

$$B_\lambda B_\mu = \sum_{\boldsymbol{t}} B_\eta$$

where the sum is over all templates $\boldsymbol{t}$ that satisfy

1. $z_{0j} + \sum_{i \neq 0}(y_{ij} + z_{ij}) = \lambda_j, \ j \neq 0,$
2. $\sum_i z_{i0} = n - l,$
3. $z_{i0} + \sum_{j \neq 0}(y_{ij} + z_{ij}) = \mu_i, \ i \neq 0,$
4. $\sum_j z_{0j} = n - m.$

For each template, $\boldsymbol{t}$, $\eta = r(\boldsymbol{t})$.

This is precisely Theorem 1 in [BB92a] (which was stated there without proof).

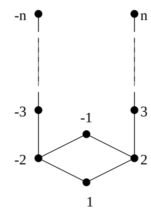
2.3 The rule for the Coxeter groups of type D

In the last application of our framework, let us take W to be the Coxeter group of type D, D_n. For this group we shall take the set upon which S acts to be $N = \{-n, \ldots, -1, 1, \ldots, n\}$, and S to be the set $\{s_i\}_{i=1}^{n-1} \cup \{s'_1\}$, where $s_i = (-i-1, -i)(i, i+1)$, and $s'_1 = (-2, 1)(-1, 2)$. Note that from our definition of D_n, it follows that if $\sigma \in D_n$, then

1. $(-i)^\sigma = -(i^\sigma)$, for all $i \in N$.
2. The parity of σ is even i.e. the multiplicity of negative numbers in $\{1^\sigma, \ldots, n^\sigma\}$ is even.

Example 2 The multiplicity of negative numbers is 2 in $\{-3, 2, 4, -5, 6, 1\}$.

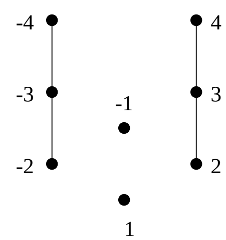
Let $\mathcal{J}$ be any subset of S. Then the graph $\mathcal{J} = (N, \mathcal{E})$ of J is the graph with vertex set N, and edge set $\mathcal{E} = \{(i,j) | (i,j)(-i,-j) \in J\}$. Observe from this definition that $\mathcal{J}$ will always be a subgraph of



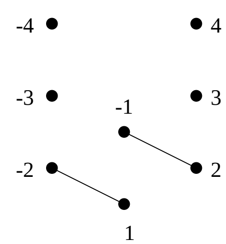
and that the vertical chains of $\mathcal{J}$ are isomorphic under the bijection that maps node i to node $-i$.

Example 3 We give examples of graphs associated with D_4 which we shall use to illustrate some of the definitions to be made below.

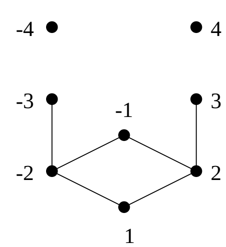
1. Let $J = \{(-3,-2)(2,3),(-4,-3)(3,4)\}$. Then the graph, $\mathcal{J}$, of J is



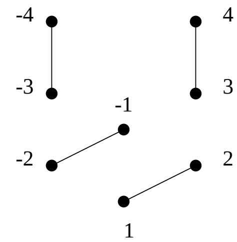
2. Let $K = \{(-2,1)(-1,2)\}$. Then the graph, $\mathcal{K}$, of K is



3. Let $L = \{(-2,1)(-1,2),(-2,-1)(1,2),(-3,-2)(2,3)\}$. Then the graph, $\mathcal{L}$, of L is



4. Let $M = \{(-2,-1)(1,2),(-4,-3)(3,4)\}$. Then the graph, $\mathcal{M}$, of M is



Each connected component of $\mathcal{J}$ has an associated node set. These node sets can be ordered by their least elements in a natural way, and then labelled

$$\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots$$

using the following algorithm:

1. If -1 and 1 are contained in the same component, then we label that component $\mathcal{J}_0$.

2. If -1 is not contained in the same component as 1, but is in the same component as 2, we label the components $\mathcal{J}_i, i \neq 0$, such that $-1 \in \mathcal{J}_1$, and then set $\mathcal{J}_0 = \{\}$.

3. If 1 is not contained in the same component as -1, but is in the same component as 2, we label the components $\mathcal{J}_i, i \neq 0$, such that $1 \in \mathcal{J}_1$, and then set $\mathcal{J}_0 = \{\}$.

4. Otherwise we label the components $\mathcal{J}_i, i \neq 0$, such that $-1 \in \mathcal{J}_{-1}$, $1 \in \mathcal{J}_1$, and then set $\mathcal{J}_0 = \{\}$.

From this algorithm we observe that if -1 and 1 do not belong to the same component, then we introduce an empty node set. In addition, since the vertical chains of $\mathcal{J}$ are isomorphic, by our labelling it follows that $|\mathcal{J}_i| = |\mathcal{J}_{-i}|$.

Let $\mathcal{G}(n)$ denote the set of all graphs defined from the subsets of S in the manner defined above. It is convenient to divide $\mathcal{G}(n)$ into four disjoint classes called, for reasons which will become clear later, $\mathcal{G}_{<n}, \mathcal{G}'_n, \mathcal{G}_n$, and $\mathcal{G}_1$. Every graph of $\mathcal{G}(n)$ belongs to exactly one of the classes depending on how its components have been labelled:

1. $\mathcal{J} \in \mathcal{G}_{<n}$ if and only if $\mathcal{J}_0 \neq \{\ \}$.

2. $\mathcal{J} \in \mathcal{G}'_n$ if and only if $-1, 2 \in \mathcal{J}_1$.

3. $\mathcal{J} \in \mathcal{G}_n$ if and only if $1, 2 \in \mathcal{J}_1$.

4. $\mathcal{J} \in \mathcal{G}_1$ otherwise.

An ordered representation of a graph $\mathcal{J} \in \mathcal{G}(n)$ is then defined to be an ordered list of node sets of $\mathcal{J}$ such that if $i \in \mathcal{J}_q$, $j \in \mathcal{J}_t$, and $\mathcal{J}_t$ appears later in the list than $\mathcal{J}_q$, then $i < j$ unless $i = 1, j = -1$. In addition, $\mathcal{J}_0$ appears between $\mathcal{J}_{-1}$ and $\mathcal{J}_1$. From this definition, we can deduce that for all $\mathcal{J} \in \mathcal{G}(n) \backslash \mathcal{G}_1$, the ordered representation of $\mathcal{J}$ is unique, and is the ordered list

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots).$$

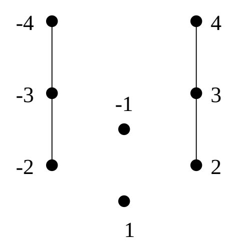
However, if $\mathcal{J} \in \mathcal{G}_1$, then there are two ordered representations of $\mathcal{J}$, namely

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots)$$

and

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_1, \mathcal{J}_0, \mathcal{J}_{-1}, \mathcal{J}_2, \mathcal{J}_3 \ldots)$$

Example 4 1. Recall $\mathcal{J}$,



The node sets of $\mathcal{J}$, when ordered by their least elements, are

$$\{-4, -3, -2\}, \{-1\}, \{1\}, \{2, 3, 4\}.$$

By our labelling algorithm these node sets are labelled

$$\mathcal{J}_{-2} = \{-4, -3, -2\}, \mathcal{J}_{-1} = \{-1\}, \mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2, 3, 4\}$$

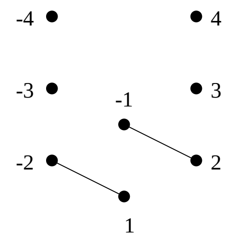
and $\mathcal{J}_0 = \{\}$. Note that $\mathcal{J} \in \mathcal{G}_1$. The ordered representations of $\mathcal{J}$ are

$$(\{-4, -3, -2\}, \{-1\}, \{\ \}, \{1\}, \{2, 3, 4\})$$

and

$$(\{-4, -3, -2\}, \{1\}, \{\ \}, \{-1\}, \{2, 3, 4\}).$$

2. Recall $\mathcal{K}$,



The node sets of $\mathcal{K}$, when ordered by their least elements, are

$$\{-4\}, \{-3\}, \{-2, 1\}, \{-1, 2\}, \{3\}, \{4\}.$$

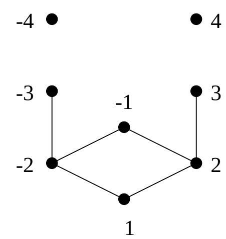
By our labelling algorithm these node sets are labelled

$$\mathcal{K}_{-3} = \{-4\}, \mathcal{K}_{-2} = \{-3\}, \mathcal{K}_{-1} = \{-2, 1\}, \mathcal{K}_1 = \{-1, 2\}, \mathcal{K}_2 = \{3\}, \mathcal{K}_3 = \{4\}$$

and $\mathcal{K}_0 = \{\}$. Note that $\mathcal{K} \in \mathcal{G}'_n$. The ordered representation of $\mathcal{K}$ is

$$(\{-4\}, \{-3\}, \{-2, 1\}, \{\,\}, \{-1, 2\}, \{3\}, \{4\}).$$

3. Recall $\mathcal{L}$,



The node sets of $\mathcal{L}$, when ordered by their least elements, are

$$\{-4\}, \{-3, -2, -1, 1, 2, 3\}, \{4\}.$$

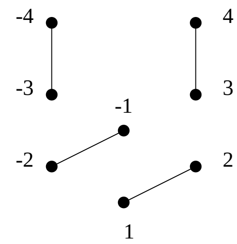
By our labelling algorithm these node sets are labelled

$$\mathcal{L}_{-1} = \{-4\}, \mathcal{L}_0 = \{-3, -2, -1, 1, 2, 3\}, \mathcal{L}_1 = \{4\}.$$

Note that $\mathcal{L} \in \mathcal{G}_{<n}$. The ordered representation of $\mathcal{L}$ is

$$(\{-4\}, \{-3, -2, -1, 1, 2, 3\}, \{4\}).$$

4. Recall $\mathcal{M}$,



The node sets of $\mathcal{M}$, when ordered by their least elements, are

$$\{-4,-3\},\{-2,-1\},\{1,2\},\{3,4\}.$$

By our labelling algorithm these node sets are labelled

$$\mathcal{M}_{-2}=\{-4,-3\},\mathcal{M}_{-1}=\{-2,-1\},\mathcal{M}_{1}=\{1,2\},\mathcal{M}_{2}=\{3,4\},$$

and $\mathcal{M}_0=\{\}$. Note that $\mathcal{M}\in\mathcal{G}_n$. The ordered representation of $\mathcal{M}$ is

$$(\{-3,-4\},\{-2,-1\},\{1,2\},\{3,4\}).$$

Let us define $\mathcal{W}_{\mathcal{J}_i}$ to be the group of all permutations on N that fix all nodes outside $\mathcal{J}_i$. Define $\mathcal{W}_{\mathcal{J}}$ to be the subgroup of

$$\ldots\times\mathcal{W}_{\mathcal{J}_{-2}}\times\mathcal{W}_{\mathcal{J}_{-1}}\times\mathcal{W}_{\mathcal{J}_{0}}\times\mathcal{W}_{\mathcal{J}_{1}}\times\mathcal{W}_{\mathcal{J}_{2}}\times\ldots$$

whose elements satisfy

1. For all $\sigma\in\mathcal{W}_J$, $(-i)^\sigma=-(i^\sigma)$ for all $i\in N$, and
2. For all $\sigma\in\mathcal{W}_J$, the parity of σ is even.

We write $|_D$ to denote the enforcement of these two conditions, that is

$$\mathcal{W}_{\mathcal{J}}=(\ldots\times\mathcal{W}_{\mathcal{J}_{-2}}\times\mathcal{W}_{\mathcal{J}_{-1}}\times\mathcal{W}_{\mathcal{J}_{0}}\times\mathcal{W}_{\mathcal{J}_{1}}\times\mathcal{W}_{\mathcal{J}_{2}}\times\ldots)|_D$$

and observe that $W_J=\mathcal{W}_{\mathcal{J}}$.

We now define four sets of compositions which will eventually be used to index the graphs defined above, and the basis elements of the descent algebra for D_n.

- $\mathcal{C}_{<n}=\{\lambda|\lambda\vDash m, m\leq n-2\}$,
- $\mathcal{C}_1=\{\lambda|\lambda\vDash n,\lambda_1=1\}$,
- $\mathcal{C}_n=\{\lambda|\lambda\vDash n,\lambda_1\geq 2\}$,

- $\mathcal{C}'_n = \{\lambda' | \lambda \vDash n, \lambda_1 \geq 2\}$.

Note that $\mathcal{C}'_n$ differs from $\mathcal{C}_n$ only in that its elements are given as compositions with a prime qualifier (as in λ'). For those familiar with partition theory, note that this qualifier does *not* denote conjugation of partitions. Informally, we shall refer to the members of $\mathcal{C}'_n$ as "primed" compositions; elements of $\mathcal{C}_{<n}$, $\mathcal{C}_1$, and $\mathcal{C}_n$ will be called "plain" compositions. We also define $\mathcal{C}(n)$ to be the union of these four sets.

We now see the reason for the somewhat mysterious terminology for the four subclasses of $\mathcal{G}(n)$. There is a one-to-one correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$ in which $\mathcal{G}_{<n}$ corresponds to $\mathcal{C}_{<n}$; $\mathcal{G}'_n$ to $\mathcal{C}'_n$; $\mathcal{G}_n$ to $\mathcal{C}_n$; and $\mathcal{G}_1$ to $\mathcal{C}_1$. This correspondence is defined as follows.

The composition, $\lambda \vDash m \leq n$, with components $\{\lambda_i\}_{i>0}$, corresponds to the graph $\mathcal{J}$, with node sets $\{\mathcal{J}_i\}_i$, if

1. $|\mathcal{J}_{-i}| = |\mathcal{J}_i| = \lambda_i$ for $i > 0$.
2. $\frac{1}{2}|\mathcal{J}_0| = \lambda_0 = n - m$.
3. (a) If$-1, 1 \in \mathcal{J}_0$ then $\lambda \in \mathcal{C}_{<n}$,
   (b) If $-1, 2 \in \mathcal{J}_1$ then $\lambda \in \mathcal{C}'_n$,
   (c) If $1, 2 \in \mathcal{J}_1$ then $\lambda \in \mathcal{C}_n$,
   (d) Otherwise $\lambda \in \mathcal{C}_1$.

Example 5 We take again our four graphs from Example 3.

1. For $\mathcal{J}$ we recall its node sets are

$$\mathcal{J}_{-2} = \{-4, -3, -2\}, \mathcal{J}_{-1} = \{-1\}, \mathcal{J}_0 = \{\}, \mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2, 3, 4\}.$$

   Hence $\mathcal{J}$ corresponds to the composition λ such that

$$|\mathcal{J}_{-1}| = |\mathcal{J}_1| = \lambda_1 = 1$$
$$|\mathcal{J}_{-2}| = |\mathcal{J}_2| = \lambda_2 = 3$$
$$\frac{1}{2}|\mathcal{J}_0| = \lambda_0 = n - m = 0.$$

   Since $-1 \in \mathcal{J}_{-1}$, $1 \in \mathcal{J}_1$ then $\lambda \in \mathcal{C}_1$. Hence, we have that $\lambda = [1, 3]$.

2. For $\mathcal{K}$ we recall that its node sets are

$$\mathcal{K}_{-3} = \{-4\}, \mathcal{K}_{-2} = \{-3\}, \mathcal{K}_{-1} = \{-2, 1\}, \mathcal{K}_0 = \{\}, \mathcal{K}_1 = \{-1, 2\}, \mathcal{K}_2 = \{3\}, \mathcal{K}_3 = \{4\}.$$

Hence $\mathcal{K}$ corresponds to the composition μ such that

$$|\mathcal{K}_{-1}| = |\mathcal{K}_1| = \mu_1 = 2$$
$$|\mathcal{K}_{-2}| = |\mathcal{K}_2| = \mu_2 = 1$$
$$|\mathcal{K}_{-3}| = |\mathcal{K}_3| = \mu_3 = 1$$
$$\tfrac{1}{2}|\mathcal{K}_0| = \mu_0 = n - m = 0.$$

Since $-1, 2 \in \mathcal{K}_1$, then $\mu \in \mathcal{C}'_n$. Hence, we have that $\mu = [2, 1, 1]'$.

3. For $\mathcal{L}$ we recall its node sets are

$$\mathcal{L}_{-1} = \{-4\}, \mathcal{L}_0 = \{-3, -2, -1, 1, 2, 3\}, \mathcal{L}_1 = \{4\}.$$

Hence $\mathcal{L}$ corresponds to the composition η such that

$$|\mathcal{L}_{-1}| = |\mathcal{L}_1| = \eta_1 = 1$$
$$\tfrac{1}{2}|\mathcal{L}_0| = \eta_0 = n - m = 3.$$

Since $-1, 1 \in \mathcal{L}_0$, then $\eta \in \mathcal{C}_{<n}$. Hence, we have that $\eta = [1]$.

4. For $\mathcal{M}$ we recall its node sets are

$$\mathcal{M}_{-2} = \{-4, -3\}, \mathcal{M}_{-1} = \{-2, -1\}, \mathcal{M}_0 = \{\}, \mathcal{M}_1 = \{1, 2\}, \mathcal{M}_2 = \{3, 4\}.$$

Hence $\mathcal{M}$ corresponds to the composition κ such that

$$|\mathcal{M}_{-1}| = |\mathcal{M}_1| = \kappa_1 = 2$$
$$|\mathcal{M}_{-2}| = |\mathcal{M}_2| = \kappa_2 = 2$$
$$\tfrac{1}{2}|\mathcal{M}_0| = \kappa_0 = n - m = 0.$$

Since $1, 2 \in \mathcal{M}_1$, then $\kappa \in \mathcal{C}_n$. Hence, we have that $\kappa = [2, 2]$.

Let $\lambda \in \mathcal{C}(n)$, and let λ have components $\{\lambda_i\}_{i=1}^k$. Then we define

$$\begin{aligned} \boldsymbol{D_\lambda} &= D_{\lambda_0} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \\ &= D_{n-m} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \end{aligned}$$

and observe that

$$\begin{aligned} \boldsymbol{D_\lambda} &= D_{n-m} \times S_{\lambda_1} \times \ldots \times S_{\lambda_k} \\ &\cong (\mathcal{W}_{\mathcal{J}_{-k}} \times \ldots \times \mathcal{W}_{\mathcal{J}_k})|_D \\ &= \mathcal{W}_{\mathcal{J}} \end{aligned}$$

where λ corresponds to $\mathcal{J}$ via the correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$ given earlier. Recall that D_0, is isomorphic to the trivial group, $D_1 \cong S_1 \times S_1$, $D_2 \cong S_2 \times S_2$, and $D_3 \cong S_4$.

Definition 7 *Let $\sigma \in D_n$, then $d(\sigma)$, is the value of the sum*

$$\sum_{\substack{0<i<j\leq n\\ j^\sigma<i^\sigma}} 1 - \sum_{\substack{1\leq i\leq n\\ i^\sigma<0}} (i^\sigma+1).$$

Theorem 5 *If $\sigma \in D_n$, then $l(\sigma) = d(\sigma)$.*

PROOF As in the previous section, we can prove this result using inductions to show that $l(\sigma) \leq d(\sigma)$, and $d(\sigma) \leq l(\sigma)$.

To show that $l(\sigma) \leq d(\sigma)$, we do an induction on $d(\sigma)$. Observe that the inequality holds when $d(\sigma) = 0$, since then σ is the identity permutation. Let $d(\sigma) > 0$, then we can find $s \in S$ such that $d(s\sigma) < d(\sigma)$, since

- If $1^\sigma, 2^\sigma, \ldots, n^\sigma$ is not increasing then there exists $i \in \{1, \ldots, n-1\}$ such that $(i+1)^\sigma < i^\sigma$, and so $d(s_i\sigma) < d(\sigma)$.

- If $1^\sigma, 2^\sigma, \ldots, n^\sigma$ is increasing then since $d(\sigma) > 0$ and the parity of σ is even, it follows that 1^σ and 2^σ are negative. Say, $1^\sigma = -k, 2^\sigma = -j$ where $0 < j < k$, then in $s'_1\sigma$, $1^{s'_1\sigma} = j, 2^{s'_1\sigma} = k$. Therefore the sum

$$-\sum_{\substack{1\leq i\leq n\\ i^\sigma<0}} (i^\sigma+1)$$

when calculated for $s_i\sigma$ will be $j + k - 2$ less than when this sum is calculated for σ, but the sum

$$\sum_{\substack{0<i<j\leq n\\ j^\sigma<i^\sigma}} 1$$

will be $j + k - 3$ more. Hence it follows that $d(s'_1\sigma) < d(\sigma)$.

Therefore, by our induction hypothesis on $d(\sigma)$, we have that $l(s\sigma) \leq d(s\sigma) < d(\sigma)$, and therefore $l(\sigma) \leq l(s\sigma) + 1 \leq d(s\sigma) + 1 \leq d(\sigma)$.

To show that $d(\sigma) \leq l(\sigma)$, we perform an induction on $l(\sigma)$. We note that the inequality holds when $l(\sigma) = 0$, since then σ is the identity permutation. Let $\sigma = s_{i_1}s_{i_2}\ldots s_{i_k}$, where $k = l(\sigma)$, and $s_{i_j} \in S$ for all $j = 1, \ldots k$. Then $l(\sigma s_{i_k}) = l(\sigma) - 1$. By our induction hypothesis, it follows that $d(\sigma s_{i_k}) \leq l(\sigma s_{i_k})$. However, $s_{i_k} \in S$ either switches $i, i+1$ and $-i, -i-1$; or switches -1 and 2, and -2 and 1. That is,

$$\sum_{\substack{0<i<j\leq n\\ j^{\sigma s_{i_k}}<i^{\sigma s_{i_k}}}} 1$$

is either one more or one less than

$$\sum_{\substack{0<i<j\leq n\\ j^\sigma<i^\sigma}} 1;$$

or

$$-\sum_{\substack{1\leq i\leq n\\ i^{\sigma s_{i_k}}<0}} (i^{\sigma s_{i_k}}+1)$$

is one more or one less than

$$-\sum_{\substack{1\leq i\leq n\\ i^{\sigma}<0}} (i^{\sigma}+1).$$

Hence, it follows that $d(\sigma s_{i_k}) = d(\sigma) \pm 1$. Therefore,

$$d(\sigma) \leq d(\sigma s_{i_k}) + 1 \leq l(\sigma s_{i_k}) + 1 = l(\sigma).$$

With this second inequality now proved, our result follows. □

Lemma 6 *Let J and K be subsets of S, and let $x \in X_J^{-1} \cap X_K$. Let the ordered representation of $\mathcal{J}$ be $(\mathcal{J}_{-r}, \ldots, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_r)$, and $\mathcal{K}$ be $(\mathcal{K}_{-s}, \ldots, \mathcal{K}_{-1}, \mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_s)$. Then an ordered representation of $\mathcal{J}x \cap \mathcal{K}$ is*

$$(\mathcal{J}_{-r}x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r+1}x\cap\mathcal{K}_{-s}, \ldots, \mathcal{J}_1x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r}x\cap\mathcal{K}_{-s+1}, \ldots, \mathcal{J}_0x\cap\mathcal{K}_0, \ldots, \mathcal{J}_rx\cap\mathcal{K}_s)$$

or

$$(\mathcal{J}_{-r}x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r+1}x\cap\mathcal{K}_{-s}, \ldots, \mathcal{J}_1x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r}x\cap\mathcal{K}_{-s+1}, \ldots, \{-1\}, \{\,\}, \{1\}, \ldots, \mathcal{J}_rx\cap\mathcal{K}_s)$$

if $\mathcal{J}_0x \cap \mathcal{K}_0 = \{-1, 1\}$, with empty sets, not including $\mathcal{J}_0x \cap \mathcal{K}_0$ if empty, removed.

PROOF To prove that

$$(\mathcal{J}_{-r}x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r+1}x\cap\mathcal{K}_{-s}, \ldots, \mathcal{J}_1x\cap\mathcal{K}_{-s}, \mathcal{J}_{-r}x\cap\mathcal{K}_{-s+1}, \ldots, \mathcal{J}_rx\cap\mathcal{K}_s)$$

is an ordered representation of $\mathcal{J}x \cap \mathcal{K}$, we must show that

1. Any node, i, in any set is less than any node, j, that appears in any other set later in the list for all $\{i, j\} \neq \{-1, 1\}$.
2. Each non-zero set $\mathcal{J}_qx \cap \mathcal{K}_m \neq \{-1, 1\}$ in the list is the node set of a connected component of $\mathcal{J}x \cap \mathcal{K}$.

The first statement will hold if we can prove that

1. If $i \in \mathcal{J}_qx \cap \mathcal{K}_m$ and $j \in \mathcal{J}_qx \cap \mathcal{K}_{m+t}$, then $i < j$.

2. If $i \in \mathcal{J}_q x \cap \mathcal{K}_m$ and $j \in \mathcal{J}_{q+t} x \cap \mathcal{K}_m$, then $i < j$.

for $\{i, j\} \neq \{-1, 1\}$, and $t > 0$.

The first case holds since $\mathcal{K}_m$ and $\mathcal{K}_{m+t}$ are terms in the ordered representation of $\mathcal{K}$.

The second case will follow if we can show that the nodes of $\mathcal{K}_m$ appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$, apart from maybe -1 and 1. By our definition of X_K as a set of minimal length coset representatives, it follows that for all $k \in K$, $l(xk) > l(x)$. Therefore, it follows that $l(kx^{-1}) > l(x^{-1})$, for all $k \in K$.

However, by Lemma 5, we know that $l(x) = d(x)$. Hence, if $k = (-h-1, -h)(h, h+1), h > 0$, then it follows that

$$(-h-1)^{x^{-1}} < (-h)^{x^{-1}}, h^{x^{-1}} < (h+1)^{x^{-1}}$$

since kx^{-1}, x^{-1} differ only in the switching of $-h-1$ and $-h$; and the switching of h and $h+1$. From this it follows that the numbers $h, h+1$ and $-h-1, -h$ appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$.

If k is not of this form, then $k = (-1, 2)(-2, 1)$, and it follows that

$$(-2)^{x^{-1}} < 1^{x^{-1}}, (-1)^{x^{-1}} < 2^{x^{-1}}.$$

since kx^{-1}, x^{-1} differ only in the switching of -1 and 2; and the switching of -2 and 1. Therefore, -1 and 2, and -2 and 1 appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$.

From this we can deduce that all nodes of $\mathcal{K}_m$ appear in increasing order in

$$((-n)^x \dots (-1)^x 1^x \dots n^x),$$

apart from maybe -1 and 1.

Let $u = i^{x^{-1}} \in \mathcal{J}_q$, and $v = j^{x^{-1}} \in \mathcal{J}_{q+t}$, then by definition $u < v$, unless $u = 1$, $v = -1$. Since we know that all the nodes in $\mathcal{K}_m$ appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$, apart from maybe -1 and 1, it follows that $i < j$, unless $u = 1$, $v = -1$.

However, we need not worry about $u = 1 \in \mathcal{J}_{-1}$, and $v = -1 \in \mathcal{J}_1$ for the following reason. Since $1^x, (-1)^x \in \mathcal{K}_m$ it follows that $\mathcal{K}_m = \mathcal{K}_0$. However, the nodes of $\mathcal{K}_0$ appear in increasing order in $((-n)^x \dots (-1)^x 1^x \dots n^x)$, with the possible exception of -1 and 1, which may be switched. Since $(|(-n)^x| \dots |(-1)^x||1^x| \dots |n^x|)$ is symmetrical about its mid-point, it follows that $\{i, j\} = \{(-1)^x, 1^x\} = \{-1, 1\}$, which we do not need to consider.

To prove the second statement, we must justify the following assertions:

1. The sets $\mathcal{J}_q x \cap \mathcal{K}_m$ are all disjoint.

2. No edge of $\mathcal{J}x \cap \mathcal{K}$ can connect two nodes in different sets, $\mathcal{J}_q x \cap \mathcal{K}_m$ and $\mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$.

3. (a) For all $i, i+1 \in \mathcal{J}_q x \cap \mathcal{K}_m$, there exists an edge between i and $i+1$ in $\mathcal{J}x \cap \mathcal{K}$.

   (b) If $-1, 2 \in \mathcal{J}_q x \cap \mathcal{K}_m$, then there exists an edge between -1 and 2 in $\mathcal{J}x \cap \mathcal{K}$.

   (c) If $-2, 1 \in \mathcal{J}_q x \cap \mathcal{K}_m$, then there exists an edge between -2 and 1 in $\mathcal{J}x \cap \mathcal{K}$.

Since all $\mathcal{J}_q$ and $\mathcal{K}_m$ are disjoint, and x is a bijection from N into itself, the first assertion follows.

To justify the second assertion, let (u, v) be an edge of $\mathcal{J}x \cap \mathcal{K}$ with $u \in \mathcal{J}_q x \cap \mathcal{K}_m$, and $v \in \mathcal{J}_{q'} x \cap \mathcal{K}_{m'}$. Since $\mathcal{J}_q$ and $\mathcal{J}_{q'}$ are nodes sets of connected components of $\mathcal{J}$, $\mathcal{J}_q x$ and $\mathcal{J}_{q'} x$ are nodes sets of connected components of $\mathcal{J}x$, and therefore $q = q'$. Similarly, since $\mathcal{K}_m$ and $\mathcal{K}_{m'}$ are node sets of connected components of $\mathcal{K}$, $m = m'$.

For the third assertion, observe that the proof of statement 1 case 2 yields that since X_J^{-1} is defined as a set of coset representatives of minimal length,

$$(-z)^x < (-y)^x, y^x < z^x \tag{2.4}$$

for all $(-y, -z)(y, z) \in J$ with $0 < y < z$.

CASE a) Let $i^{x^{-1}} = u$, and $(i+1)^{x^{-1}} = v$. From the proof of statement 1 case 2 we know that $i, i+1$ appear in increasing order in

$$((-n)^x \ldots (-1)^x 1^x \ldots n^x),$$

and so $u < v$. Since $u, v \in \mathcal{J}_q$, by definition we have that all $w \in N$ between u and v also belong to $\mathcal{J}_q$. By this and (2.4) we can deduce that since $u^x = i$ and $v^x = i+1$, there is no such w, and so an edge exists between u and v. Therefore, an edge exists between nodes i and $i+1$ in $\mathcal{J}x \cap \mathcal{K}$.

CASE b) If $-1, 2 \in \mathcal{J}_q x \cap \mathcal{K}_m$, then there exists $u, v \in \mathcal{J}_q$ such that $u^x = -1$, and $v^x = 2$. We can conclude that $u < v$ since the proof of statement 1 case 2 tells us that -1 and 2 appear in order in $((-n)^x \ldots (-1)^x 1^x \ldots n^x)$. Again since, $u, v \in \mathcal{J}_q$, all $w \in N$ between u and v also belong to $\mathcal{J}_q$. This and (2.4) implies that if no edge exists between u and v then $w^x = 1$. But $(-u)^x = 1$, and since $u < v$ then either $v < -u$ or $-u < u$. Since x is a bijection on N, it follows that no such w can exist, and so an edge exists between -1 and 2 in $\mathcal{J}x \cap \mathcal{K}$.

CASE c) Similarly, if $-2, 1 \in \mathcal{J}_q x \cap \mathcal{K}_m$, then there exists $u, v \in \mathcal{J}_q$ such that $u^x = -2$, and $v^x = 1$. From the proof of statement 1 case 2 we again deduce that $u < v$ since $-2, 1$ appear in increasing order in $((-n)^x \ldots (-1)^x 1^x \ldots n^x)$. Now by (2.4) our only possibility for an edge not to exist between u and v is if there is some $w \in N$ between u and v, such that $w^x = -1$; but since $(-u)^x = -1$, and $u < v$ then either $v < -u$ or $-u < u$. As x is a bijection from N into itself, this w cannot arise, so an edge exists between -2 and 1 in $\mathcal{J}x \cap \mathcal{K}$, and the assertion follows.

Our last observation is that if $\mathcal{J}_0 x \cap \mathcal{K}_0 = \{-1, 1\}$ then this is clearly not the node set of a connected component of $\mathcal{J} x \cap \mathcal{K}$, but to obtain the ordered representation in this case, we just need to replace $\{-1, 1\}$ by the three sets $\{-1\}, \{\ \}, \{1\}$ (observe that we introduce the empty node set since $-1, 1$ are not in the same connected component). □

Consequently from Lemma 2 [Sol76], and Lemma 6, it follows that if $x \in X_J^{-1} \cap X_K$ then

$$\begin{aligned}
x^{-1} W_J x \cap W_K &= W_{x^{-1} J x \cap K} \\
&= \mathcal{W}_{\mathcal{J} x \cap \mathcal{K}} \\
&= [\mathcal{W}_{(\mathcal{J}_{-r} x \cap \mathcal{K}_{-s})} \times \ldots \times \mathcal{W}_{(\mathcal{J}_r x \cap \mathcal{K}_s)}]|_D \\
&= [(x^{-1} \mathcal{W}_{\mathcal{J}_{-r}} x \cap \mathcal{W}_{\mathcal{K}_{-s}}) \times \ldots \times (x^{-1} \mathcal{W}_{\mathcal{J}_r} x \cap \mathcal{W}_{\mathcal{K}_s})]|_D \\
&= [(x^{-1} S^{-}_{\lambda_r} x \cap S^{-}_{\mu_s}) \times \ldots \times (x^{-1} S^{+}_{\lambda_r} x \cap S^{+}_{\mu_s})]|_D
\end{aligned}$$

where λ and μ are suitable compositions in $\mathcal{C}(n)$, determined by the correspondence between $\mathcal{G}(n)$ and $\mathcal{C}(n)$. As in the previous section, the superscript $-$ or $+$ on a symmetric group S_{λ_i} simply denotes that for the given $i > 0$, S_{λ_i} acts on $\mathcal{J}_{-i}$ or $\mathcal{J}_i$ respectively. Observe that, say, $x^{-1} S^{-}_{\lambda_i} x \cap S^{-}_{\mu_j}$ is the group of all permutations on

$$\mathcal{J}_{-i} x \cap \mathcal{K}_{-j}.$$

Let $z_{ij} = |\mathcal{J}_i x \cap \mathcal{K}_j|$, then we have a mapping

$$\zeta : x \mapsto (z_{ij})$$

from $X_J^{-1} \cap X_K$ into a subset of all $(2s+1) \times (2r+1)$ matrices

$$\begin{pmatrix}
z_{(-s)(-r)} & \cdots & z_{(-s)0} & \cdots & z_{(-s)r} \\
\vdots & & \vdots & & \vdots \\
z_{0(-r)} & \cdots & z_{00} & \cdots & z_{0r} \\
\vdots & & \vdots & & \vdots \\
z_{s(-r)} & \cdots & z_{s0} & \cdots & z_{sr}
\end{pmatrix}$$

with non-negative integer entries that satisfy

$$\begin{array}{ll}
\sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0 \\
\sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0.
\end{array}$$

In addition, since $x \in X_J^{-1} \cap X_K$ is such that $(-k)^x = -(k^x)$, it follows that $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_{-i} x \cap \mathcal{K}_{-j}|$, and so we have the further condition that $z_{ij} = z_{(-i)(-j)}$. However, a complete description of the subset of matrices into which $X_J^{-1} \cap X_K$ maps is also dependent on the parity condition. In fact, this subset is most easily found by working through the possible combinations of $\mathcal{J}, \mathcal{K}$, depending on whether they belong to $\mathcal{G}_1, \mathcal{G}_n, \mathcal{G}'_n$ or $\mathcal{G}_{<n}$. Since there are only 16 different combinations, we shall conduct a case study.

Before we begin, we recall that x has even parity if the number of $i \in N$ such that $0 < i$ and $i^x < 0$ is even. Therefore, to see how the parity condition in D_n affects the matrices we need to study the set $\bigcup_{i,j\geq 0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$. In addition, for $\mathcal{J} \in \mathcal{G}_1$ and $\mathcal{J} \in \mathcal{G}_n$ we have that $\mathcal{J}_0 = \{ \}$, and all $u \in \bigcup_{i>0} \mathcal{J}_i$ are positive. Hence, the following four arguments for $\mathcal{J} \in \mathcal{G}_1$ also hold for $\mathcal{J} \in \mathcal{G}_n$. Therefore, we shall state the arguments for the cases in which $\mathcal{J} \in \mathcal{G}_1$, and then immediately deduce the corresponding results for the four cases in which $\mathcal{J} \in \mathcal{G}_n$.

1. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_1$

   Since both $\mathcal{J}_0$ and $\mathcal{K}_0$ are empty, all $u \in \bigcup_{i>0} \mathcal{J}_i$ are positive, and all $v \in \bigcup_{j>0} \mathcal{K}_{-j}$ are negative, it follows that all $u \in \{1, \ldots, n\}$ such that $u^x < 0$ will appear in the set $\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$. Since the parity of x is even, it follows that $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}|$ is even, that is

   $$\sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$$

   is even. Therefore, we have the extra matrix condition that $\sum_{i,j>0} z_{i(-j)}$ must be even.

2. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_n$

   Again we have that the parity of x is even, $\mathcal{J}_0 = \mathcal{K}_0 = \{\}$, all $u \in \bigcup_{i>0} \mathcal{J}_i$ are positive, and all $v \in \bigcup_{j>0} \mathcal{K}_{-j}$ are negative. Therefore, we again have that all $u \in \{1, \ldots, n\}$ such that $u^x < 0$ appear in $\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$, so $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}|$ must be even. That is

   $$\sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}| = \sum_{i,j>0} z_{i(-j)}$$

   must be even.

3. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}'_n$

   Again we have that $\mathcal{J}_0$ and $\mathcal{K}_0$ are empty, and all $u \in \bigcup_{i>0} \mathcal{J}_i$ are positive. But now $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, 2, 3, \ldots, n\}$. By definition there exists some $u \in \bigcup_{i>0} \mathcal{J}_i$ such that u^x is either -1 or 1. Suppose that first $u^x = -1$. Then since $-1 \notin \bigcup_{j>0} \mathcal{K}_{-j}$ we have that $-1 \notin \bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$. As all other $u \in \bigcup_{i>0} \mathcal{J}_i$ such that $u^x < 0$ are contained in $\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$, it follows that $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ will be one less than the parity of x.

   Now suppose that $u^x = 1$. Then $1 \in \bigcup_{j>0} \mathcal{K}_{-j}$, and so $1 \in \bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$. However, all other $v \in \bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$ are such that $v < 0, 0 < v^{x^{-1}}$, and hence it follows that $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ will be one more than the parity of x.

Since the parity of x is even, it follows that in both these cases, we will have that

$$\sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$$

is odd. This gives the further matrix condition that $\sum_{i,j>0} z_{i(-j)}$ must be odd.

4. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_{<n}$

   We have that $\mathcal{J}_0$ is empty and all $u \in \bigcup_{i>0} \mathcal{J}_i$ are positive. For every $v \in \mathcal{K}_0$ where $v < 0$ we have that $-v \in \mathcal{K}_0$. Therefore we cannot tell how may $v \in \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$ are such that $0 < v^{x^{-1}}$. Since this may be even or odd and the parity of x is even, it follows that the number of $v \in \bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}$ such that $0 < v^{x^{-1}}$ may be even or odd. Therefore, $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ may be even or odd, hence no more conditions will be placed on the matrices.

5. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_1$

   Now we have the further condition that $\sum_{i,j>0} z_{i(-j)}$ will be even.

6. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_n$

   Now we have the further condition that $\sum_{i,j>0} z_{i(-j)}$ will be even.

7. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}'_n$

   Now we have the further condition that $\sum_{i,j>0} z_{i(-j)}$ will be odd.

8. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_{<n}$

   No further conditions will be placed on the matrices.

Before we continue our analysis, we shall define a group automorphism, ψ, on D_n, that will help us to exploit the analysis already completed. Let ψ be the transposition permutation $(-1, 1)$ on N. Then ψ induces an automorphism of D_n which we shall also call ψ. From this definition we can immediately deduce that

$$\psi(i^\sigma) = \psi(i)^{\psi(\sigma)}$$

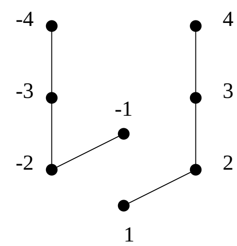
and the action of the group automorphism ψ is to replace s_1 by s'_1, and replace s'_1 by s_1 in every $\sigma \in D_n$, when σ is written as a product of generators.

Let us define $\psi(\mathcal{J})$ to be the graph with vertex set N, and edge set

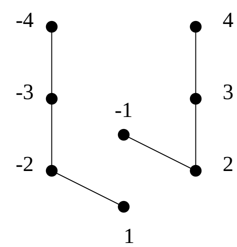
$$\{(i, j) | (i^{(-1,1)}, j^{(-1,1)}) \in \mathcal{J}\},$$

and define $\psi(\mathcal{J}_k)$ to be the set such that if $h \in \psi(\mathcal{J}_k)$ then $h^{(-1,1)} \in \mathcal{J}_k$.

Example 6 In D_4, if $\mathcal{J}$ is the graph



then $\psi(\mathcal{J})$ is the graph



Observe from these definitions that

(a) If $\mathcal{J} \in \mathcal{G}_n$ then $\psi(\mathcal{J}) \in \mathcal{G}'_n$,

(b) If $\mathcal{J} \in \mathcal{G}'_n$ then $\psi(\mathcal{J}) \in \mathcal{G}_n$,

(c) If $\mathcal{J} \in \mathcal{G}_{<n}$ then $\psi(\mathcal{J}) = \mathcal{J}$,

(d) If $\mathcal{J} \in \mathcal{G}_1$ then $\psi(\mathcal{J}) = \mathcal{J}$,

and in the latter case, $\mathcal{J}_1$ is mapped into $\mathcal{J}_{-1}$ and vice versa. We can deduce that since all $x \in X_J^{-1} \cap X_K$ are of minimal length, $x \in X_J^{-1} \cap X_K$ if and only if $\psi(x) \in X_{\psi(J)}^{-1} \cap X_{\psi(K)}$. In addition we can also deduce that

$$\psi(\mathcal{J}_i x \cap \mathcal{K}_j) = \psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_j).$$

With this in mind, let us look at the next four of the remaining 8 cases.

9. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_1$

We know that $\psi(\mathcal{J}) \in \mathcal{G}_n$, $\psi(\mathcal{K}) = \mathcal{K} \in \mathcal{G}_1$, and that $\psi(x) \in X_{\psi(J)}^{-1} \cap X_K$, therefore, since we have dealt with the case $\psi(\mathcal{J}) \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_1$, we know that $\sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_{-j}|$ must be even. Hence, $\sum_{i,j>0} |\psi(\psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_{-j})| = \sum_{i,j>0} |\mathcal{J}_i x \cap \psi(\mathcal{K}_{-j})|$ is even. However, $\psi(\mathcal{K}_{-1}) = \mathcal{K}_1$. Since one of either -1 or 1 belongs to $(-1)^x, 2^x, \ldots, n^x$, and $\bigcup_{i>0} \mathcal{J}_i x = \{(-1)^x, 2^x, \ldots, n^x\}$, it follows that

$$\sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$$

will either be one more, or one less than $\sum_{i,j>0} |\mathcal{J}_i x \cap \psi(\mathcal{K}_{-j})|$. Therefore, we can conclude that when $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_1$, we have the extra matrix condition that $\sum_{i,j>0} z_{i(-j)}$ is odd.

10. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_n$

    Since $\psi(\mathcal{J}) \in \mathcal{G}_n$, $\psi(\mathcal{K}) \in \mathcal{G}'_n$, and $\psi(x) \in X^{-1}_{\psi(J)} \cap X_{\psi(K)}$, we know that

    $$\sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j})|$$

    must be odd. Therefore, $\sum_{i,j>0} |\psi(\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j}))| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ must be odd. Hence we have the extra condition that $\sum_{i,j>0} z_{i(-j)}$ must be odd.

11. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}'_n$

    Since $\psi(\mathcal{J}) \in \mathcal{G}_n$, $\psi(\mathcal{K}) \in \mathcal{G}_n$, and $\psi(x) \in X^{-1}_{\psi(J)} \cap X_{\psi(K)}$, we know that

    $$\sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j})|$$

    is even. Therefore, we know

    $$\sum_{i,j>0} |\psi(\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j}))| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$$

    must be even. Hence we have the extra condition that $\sum_{i,j>0} z_{i(-j)}$ must be even.

12. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_{<n}$

    We have that $\psi(\mathcal{J}) \in \mathcal{G}_n$, $\psi(\mathcal{K}) \in \mathcal{G}_{<n}$, and since there are no further restrictions in this case, it follows that no further conditions will be placed on the matrices when $\mathcal{J} \in \mathcal{G}'_n$, and $\mathcal{K} \in \mathcal{G}_{<n}$.

    The last four cases to consider are those for which $\mathcal{J} \in \mathcal{G}_{<n}$. However, we cannot tell how many $0 < u \in \mathcal{J}_0$ such that $u^x < 0$ lie in $\bigcup_{j>0} \mathcal{J}_0 x \cap \mathcal{K}_{-j}$, since for each $u \in \mathcal{J}_0$, $-u \in \mathcal{J}_0$. Since the number of such u may be even or odd, and the parity of x is even, it follows that $|\bigcup_{i,j>0} \mathcal{J}_i x \cap \mathcal{K}_{-j}| = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ may be even or odd. Hence, no further conditions will be placed on the matrices, irrespective of $\mathcal{K}$.

13. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_1$

    No further conditions will be placed on the matrices.

14. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_n$

    No further conditions will be placed on the matrices.

15. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}'_n$

    No further conditions will be placed on the matrices.

16. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_{<n}$

No further conditions will be placed on the matrices.

We will now prove that we have indeed found all the matrices into which each of the coset representatives map in each of the 16 cases using the following two lemmas.

Lemma 7 *Let $x \in X_J^{-1} \cap X_K$. If $x' \in \mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$ then $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_i x' \cap \mathcal{K}_j|$.*

PROOF Let $x' \in \mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$, say $x' = \sigma x \tau$, $\sigma \in \mathcal{W}_\mathcal{J}$, $\tau \in \mathcal{W}_\mathcal{K}$, then for each i,

$$\mathcal{J}_i x' = \mathcal{J}_i \sigma x \tau = \mathcal{J}_i x \tau.$$

Hence, for each i, j we have

$$\mathcal{J}_i x' \cap \mathcal{K}_j = \mathcal{J}_i x \tau \cap \mathcal{K}_j = [\mathcal{J}_i x \cap \mathcal{K}_j]\tau.$$

Since τ is a bijection, and $\tau \in \mathcal{W}_\mathcal{K}$, the lemma follows. □

Lemma 8 *Let $x \in X_J^{-1} \cap X_K$. If $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_i x' \cap \mathcal{K}_j|$ for all i and j and $x' \in D_n$, then $x' \in \mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$, unless $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$, and $\mathcal{J}_0 x \cap \mathcal{K}_0 = \{\ \}$. In this case x' belongs to either $\mathcal{W}_\mathcal{J} x \mathcal{W}_\mathcal{K}$, or $\mathcal{W}_\mathcal{J} \psi(x) \mathcal{W}_\mathcal{K}$.*

PROOF If $|\mathcal{J}_i x \cap \mathcal{K}_j| = |\mathcal{J}_i x' \cap \mathcal{K}_j|$ then for a fixed j, subsets $\{\mathcal{J}_i x \cap \mathcal{K}_j\}_i$ and $\{\mathcal{J}_i x' \cap \mathcal{K}_j\}_i$ form two dissections of $\mathcal{K}_j$ into subsets which can be collected naturally into pairs of subsets of equal order. For each $j > 0$, find a permutation $\tau_j \in \mathcal{W}_{\mathcal{K}_j}$ which satisfies

$$[\mathcal{J}_i x \cap \mathcal{K}_j]\tau_j = \mathcal{J}_i x' \cap \mathcal{K}_j \text{ for all } i.$$

Choose $\tau_{-j} \in \mathcal{W}_{\mathcal{K}_{-j}}$ such that for all $h \in \mathcal{K}_j, -h \in \mathcal{K}_{-j}$, $(-h)^{\tau_{-j}} = -(h^{\tau_j})$. Then since $x' \in D_n$, we know that for each $j > 0$, τ_{-j} satisfies

$$[\mathcal{J}_i x \cap \mathcal{K}_{-j}]\tau_{-j} = \mathcal{J}_i x' \cap \mathcal{K}_{-j}.$$

For $j = 0$ we would like to find a permutation $\tau_0 \in \mathcal{W}_{\mathcal{K}_0}$, such that for all

$$h, -h \in \mathcal{K}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$$

we have that $(-h)^{\tau_0} = -(h^{\tau_0})$; the number of negatives in $\{1^{\tau_0}, 2^{\tau_0}, \ldots, l^{\tau_0}\}$ is even; and

$$[\mathcal{J}_i x \cap \mathcal{K}_0]\tau_0 = \mathcal{J}_i x' \cap \mathcal{K}_0 \text{ for all } i.$$

If we can find such a permutation, τ_0, then by construction $\tau = \ldots \tau_{-2}\tau_{-1}\tau_0\tau_1\tau_2 \ldots \in (\ldots \times \mathcal{W}_{\mathcal{K}_{-2}} \times \mathcal{W}_{\mathcal{K}_{-1}} \times \mathcal{W}_{\mathcal{K}_0} \times \mathcal{W}_{\mathcal{K}_1} \times \mathcal{W}_{\mathcal{K}_2} \times \ldots)|_D = \mathcal{W}_\mathcal{K}$. In addition, τ satisfies

$$\mathcal{J}_i x \tau = \mathcal{J}_i x' \text{for all } i,$$

and so we can find a $\sigma \in \mathcal{W}_{\mathcal{J}}$ such that $x' = \sigma x \tau$.

Since for all $u \in \mathcal{K}_0$ we have that $-u \in \mathcal{K}_0$ and $x' \in D_n$, we will certainly always be able to find a permutation $\tau_0 \in \mathcal{W}_{\mathcal{K}_0}$, such that for all $h, -h \in \mathcal{K}_0$ we have that $(-h)^{\tau_0} = -(h^{\tau_0})$; and

$$[\mathcal{J}_i x \cap \mathcal{K}_0]\tau_0 = \mathcal{J}_i x' \cap \mathcal{K}_0 \text{ for all } i.$$

Therefore the only problem will be if we can only find τ_0 such that the number of negatives in $\{1^{\tau_0}, 2^{\tau_0}, \ldots, l^{\tau_0}\}$ is odd. Clearly, if $\mathcal{K}_0 = \{\ \}$ this problem cannot occur, that is this problem may only occur if $\mathcal{K}_0 \neq \{\ \}$.

Now since $x \in X_J^{-1} \cap X_K$, we know that $x^{-1} \in X_K^{-1} \cap X_J$. Moreover, since x is a bijection from N to itself, so is x^{-1}. Therefore,

$$|\mathcal{J}_i x \cap \mathcal{K}_j| = |(\mathcal{J}_i x \cap \mathcal{K}_j)x^{-1}| = |\mathcal{J}_i \cap \mathcal{K}_j x^{-1}|.$$

Similarly, $|\mathcal{J}_i x' \cap \mathcal{K}_j| = |\mathcal{J}_i \cap \mathcal{K}_j (x')^{-1}|$, and so $|\mathcal{J}_i \cap \mathcal{K}_j x^{-1}| = |\mathcal{J}_i \cap \mathcal{K}_j (x')^{-1}|$. For a fixed i, the subsets $\{\mathcal{J}_i \cap \mathcal{K}_j x^{-1}\}_j$ and $\{\mathcal{J}_i \cap \mathcal{K}_j (x')^{-1}\}_j$ form two dissections of $\mathcal{J}_i$ into subsets which can be collected naturally into pairs of subsets of equal order. In the same way that we constructed the permutation τ earlier, we can construct a permutation $\sigma = \ldots \sigma_{-2}\sigma_{-1}\sigma_0\sigma_1\sigma_2 \ldots$, such that $\mathcal{K}_j x^{-1}\sigma = \mathcal{K}_j (x')^{-1}$; $\sigma_i \in \mathcal{W}_{\mathcal{J}_i}$; and $\sigma \in \mathcal{W}_{\mathcal{J}}$, unless we can only find permutations σ_0 acting on $\mathcal{J}_0$, such that σ_0 has odd parity. If $\mathcal{J}_0 = \{\ \}$, then this problem cannot arise, and so it follows that we can find a $\sigma \in \mathcal{W}_{\mathcal{J}}$ such that $\mathcal{K}_j x^{-1}\sigma = \mathcal{K}_j (x')^{-1}$, and so we can find some $\tau \in \mathcal{W}_{\mathcal{K}}$, such that $(x')^{-1} = \tau x^{-1}\sigma \in \mathcal{W}_{\mathcal{K}} x^{-1} \mathcal{W}_{\mathcal{J}}$, that is $x' \in \mathcal{W}_{\mathcal{J}} x \mathcal{W}_{\mathcal{K}}$.

Therefore our problem can only arise if $\mathcal{J} \neq \{\ \}$, and $\mathcal{K}_0 \neq \{\ \}$, that is $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$. Let $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$, and $\mathcal{K}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$. As stated above we can find a permutation, τ_0, such that for all $h, -h \in \mathcal{K}_0$ we have that $(-h)^{\tau_0} = -(h^{\tau_0})$ and

$$[\mathcal{J}_i x \cap \mathcal{K}_0]\tau_0 = \mathcal{J}_i x' \cap \mathcal{K}_0 \text{ for all } i.$$

Suppose that the number of negatives in $\{1^{\tau_0}, 2^{\tau_0}, \ldots, l^{\tau_0}\}$ is odd. Then we may set $\tau_0 = (-1, 1)\tau_0'$, where the number of negatives in $\{1^{\tau_0'}, 2^{\tau_0'}, \ldots, l^{\tau_0'}\}$ is even. Clearly if $-1, 1$ belong to the same subset intersection, $\mathcal{J}_i x \cap \mathcal{K}_0$, then since $x \in D_n$ it follows that $-1, 1 \in \mathcal{J}_0 x \cap \mathcal{K}_0$. Moreover, it follows that

$$[\mathcal{J}_i x \cap \mathcal{K}_0]\tau_0' = \mathcal{J}_i x' \cap \mathcal{K}_0 \text{ for all } i.$$

Therefore, for our problem to occur it follows that -1 and 1 do not belong to the same subset intersection, or equivalently, $\mathcal{J}_0 x \cap \mathcal{K}_0 = \{\ \}$. Let $\tau = \ldots \tau_{-2}\tau_{-1}\tau_0\tau_1\tau_2 \ldots$, then $\tau = \ldots \tau_{-2}\tau_{-1}(-1, 1)\tau_0'\tau_1\tau_2 \ldots$. In addition, since $-1, 1 \in \mathcal{K}_0$ it follows that $(-1, 1)$ commutes with all $\tau_{-j}, j > 0$ and so $\tau = (-1, 1)\tau'$, where $\tau' = \ldots \tau_{-2}\tau_{-1}\tau_0'\tau_1\tau_2 \ldots \in (\ldots \times \mathcal{W}_{\mathcal{K}_{-2}} \times \mathcal{W}_{\mathcal{K}_{-1}} \times \mathcal{W}_{\mathcal{K}_0} \times \mathcal{W}_{\mathcal{K}_1} \times \mathcal{W}_{\mathcal{K}_2} \times \ldots)|_D = \mathcal{W}_{\mathcal{K}}$. Moreover, since $\mathcal{J}_i x \tau = \mathcal{J}_i x'$for all i, it follows that

$\mathcal{J}_i x(-1,1)\tau' = \mathcal{J}_i x'$for all i, or equivalently

$$\psi(\mathcal{J}_i x)\tau' = \mathcal{J}_i x' \text{for all } i.$$

However, since $\mathcal{J} \in \mathcal{G}_{<n}$, we have that $\psi(\mathcal{J}_i) = \mathcal{J}_i$ for all i, and so

$$\psi(\mathcal{J}_i x)\tau' = \mathcal{J}_i \psi(x)\tau' = \mathcal{J}_i x' \text{for all } i.$$

Moreover, we can find a $\sigma \in \mathcal{W}_{\mathcal{J}}$ such that $x' = \sigma\psi(x)\tau' \in \mathcal{W}_{\mathcal{J}}\psi(x)\mathcal{W}_{\mathcal{K}}$, and we are done. $\square$

By Lemma 8, we know that ζ is injective, apart from when $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$, $\mathcal{J}_0 x \cap \mathcal{K}_0 = \{\ \}$, when the two double coset representatives x and $\psi(x)$ have the same images under ζ. Hence to show that we have found all the matrices, we only need to show that ζ is surjective.

Let $\boldsymbol{z} = (z_{ij})$ be a matrix which satisfies the conditions

$$\begin{array}{ll} \sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0 \\ \sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0, \\ \multicolumn{2}{c}{z_{ij} = z_{(-i)(-j)}} \end{array}$$

together with any other conditions obtained through the analysis of the above 16 cases.

If we can find a $\sigma \in D_n$ such that $z_{ij} = |\mathcal{J}_i\sigma \cap \mathcal{K}_j|$, for all i, j, then by Lemma 7, we can find an element $x \in X_J^{-1} \cap X_K$, that lies in the same double coset as σ such that $z_{ij} = |\mathcal{J}_i x \cap \mathcal{K}_j|$, for all i, j. From this it will follow that ζ is surjective.

From the argument at the corresponding point of the previous section, we know that from the conditions

$$\begin{array}{ll} \sum_{i=-s}^{s} z_{ij} = \lambda_{|j|}, j \neq 0, & \sum_{j=-r}^{r} z_{ij} = \mu_{|i|}, i \neq 0 \\ \sum_{i=-s}^{s} z_{i0} = 2\lambda_0, & \sum_{j=-r}^{r} z_{0j} = 2\mu_0, \\ \multicolumn{2}{c}{z_{ij} = z_{(-i)(-j)}} \end{array}$$

we may certainly find a σ such that $(-h)^\sigma = -(h^\sigma)$ for all $h \in N$ with the property $z_{ij} = |\mathcal{J}_i\sigma \cap \mathcal{K}_j|$. Let us assume that any σ we deal with from now on has this property. Therefore, to find a $\sigma \in D_n$, we need only check that we can find a σ that also satisfies the parity condition for each of the 16 cases we have just studied. This will include showing that if a σ may not satisfy the parity condition, then we may refine σ if needs be so that it does. In addition we show that in this situation the entries in the matrix remain unchanged.

Again since the same arguments apply to the cases where $\mathcal{J} \in \mathcal{G}_1$ (cases 1 to 4), and $\mathcal{J} \in \mathcal{G}_n$ (cases 5 to 8), we will deduce the results for $\mathcal{J} \in \mathcal{G}_n$ immediately from the results for $\mathcal{J} \in \mathcal{G}_1$.

1. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_1$

Here we have $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$, $\bigcup_{i>0} \mathcal{J}_i = \{1, 2, \ldots, n\}$, $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$. We also have the condition that $\sum_{i,j>0} z_{i(-j)}$ is even. Therefore,

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even, that is the number of negatives in $\{1^\sigma, 2^\sigma, \ldots, n^\sigma\}$ is even, and so σ satisfies the parity condition.

2. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_n$

   Again we have $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$, $\bigcup_{i>0} \mathcal{J}_i = \{1, 2, \ldots, n\}$, $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$, and $\sum_{i,j>0} z_{i(-j)}$ is even. Therefore,

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

   is even, and so the number of negatives in $\{1^\sigma, 2^\sigma, \ldots, n^\sigma\}$ is even. Hence, σ satisfies the parity condition.

3. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}'_n$

   Here we have that $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$, $\bigcup_{i>0} \mathcal{J}_i = \{1, 2, \ldots, n\}$, $\bigcup_{j>0} \mathcal{K}_{-j} = \{1, -2, \ldots, -n\}$, and $\sum_{i,j>0} z_{i(-j)}$ is odd. Hence,

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

   is odd, therefore

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

   is even, that is the number of negatives in $\{1^\sigma, 2^\sigma, \ldots, n^\sigma\}$ is even, and σ satisfies the parity condition.

4. $\mathcal{J} \in \mathcal{G}_1, \mathcal{K} \in \mathcal{G}_{<n}$

   Here we have that $\bigcup_{i>0} \mathcal{J}_i = \{1, 2, \ldots, n\}$. Let $\mathcal{K}_0 = \{-m, \ldots, -1, 1, \ldots, m\}$, and $\bigcup_{j>0} \mathcal{K}_{-j} = \{-m-1, \ldots, -n\}$. Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

   is even. If $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is even, then

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

   is even. However, if $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd then we may refine σ as follows. For some $t \in \{1, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, set $t^\sigma = u$, and $(-t)^\sigma = -u$. Observe that since $-u, u \in \mathcal{K}_0$, the refining of σ leaves all z_{ij} unchanged. However we now have that $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$, and hence $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even.

Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd. Then

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

is odd. If $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd, then

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

is even. However, if $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is even, then if it is greater than 0, we can refine σ by choosing $t \in \{1, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$; otherwise we refine σ by choosing $t \in \{1, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = u$, and setting $t^\sigma = -u$, and $(-t)^\sigma = u$. Having refined σ we have that all z_{ij} remain unchanged, since $-u, u \in \mathcal{K}_0$, but now

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$$

is odd. That is, $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even.

Therefore in either case, we can find a σ such that the parity of σ is even.

5. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_1$

   We can find a σ that satisfies the parity condition.

6. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_n$

   We can find a σ that satisfies the parity condition.

7. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}'_n$

   We can find a σ that satisfies the parity condition.

8. $\mathcal{J} \in \mathcal{G}_n, \mathcal{K} \in \mathcal{G}_{<n}$

   We can find a σ that satisfies the parity condition.

9. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_1$

   We have that $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$, $\bigcup_{i>0} \mathcal{J}_i = \{-1, 2, \ldots, n\}$, $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$. In this case, we also have that $\sum_{i,j>0} z_{i(-j)}$ is odd. Therefore,

   $$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

   is odd, and so

   $$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

   is even. Therefore, the number of negatives in $\{1^\sigma, 2^\sigma, \ldots, n^\sigma\}$ is even, and so σ satisfies the parity condition.

10. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_n$

Again $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$. Since $\sum_{i,j>0} z_{i(-j)}$ is odd, it follows that

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is odd, so

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. Therefore, the number of negative numbers in $\{1^\sigma, 2^\sigma, \ldots, n^\sigma\}$ is even, and so the parity of σ is even.

11. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}'_n$

Now $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$, $\bigcup_{i>0} \mathcal{J}_i = \{-1, 2, \ldots, n\}$, and $\bigcup_{j>0} \mathcal{K}_{-j} = \{1, -2, \ldots, -n\}$. Since $\sum_{i,j>0} z_{i(-j)}$ is even, it follows that

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is even, so

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd, and so

$$|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. Therefore, the parity of σ is even.

12. $\mathcal{J} \in \mathcal{G}'_n, \mathcal{K} \in \mathcal{G}_{<n}$

Here we have that $\bigcup_{i>0} \mathcal{J}_i = \{-1, 2, \ldots, n\}$. Let $\mathcal{K}_0 = \{-m, \ldots, -1, 1, \ldots, m\}$, and $\bigcup_{j>0} \mathcal{K}_{-j} = \{-m-1, \ldots, -n\}$.

Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

is even. If $|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd, then

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

is odd, hence $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even. If

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$$

is even, then if it is greater than 0, then we can refine σ by choosing $t \in \{-1, 2, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$; otherwise we can refine σ by choosing $t \in \{-1, 2, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = u$, and then setting $t^\sigma = -u$, and $(-t)^\sigma = u$. For every $u \in \mathcal{K}_0$ we know that $-u \in \mathcal{K}_0$, and hence with

σ refined, all z_{ij} remain unchanged, although now $|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ and hence

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

is odd. That is $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even.

Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd. Then

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

is odd. If $|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is even, then

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

is odd, and therefore $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even. However, if

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$$

is odd, then we can refine σ by choosing some $t \in \{-1, 2, \ldots, n\}$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$. Since $-u, u \in \mathcal{K}_0$, it follows that σ refined leaves all z_{ij} unchanged, although now $|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -m\}|$ is even. Hence

$$|\{(-1)^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

is odd, that is $|\{1^\sigma, 2^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even.

Therefore in each case, we can find a σ such that the parity of σ is even.

13. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_1$

Here we have that $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$. Let $\mathcal{J}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$, and $\bigcup_{i>0} \mathcal{J}_i = \{l+1, \ldots, n\}$.

Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is odd then we may refine σ by choosing a $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$. Since $-t, t \in \mathcal{J}_0$, it follows that refining σ leaves all z_{ij} unchanged, although now $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ and hence $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even. Hence we can find a σ such that the parity of σ is even.

Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is odd. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is odd, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. Otherwise, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even, and greater than 0, we can refine σ by choosing $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even, and equal to 0, then choose $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = u$, and set $t^\sigma = -u$, and $(-t)^\sigma = u$. Note that since $-t, t \in \mathcal{J}_0$, σ refined leaves all z_{ij} unchanged, although we now have that

$$|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is odd, that is $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even. Hence we can find a σ such that the parity of σ is even.

14. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_n$

Again, we may use the same argument as the last case, since we again have that $\bigcup_{j>0} \mathcal{K}_{-j} = \{-1, -2, \ldots, -n\}$. Let $\mathcal{J}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$, and $\bigcup_{i>0} \mathcal{J}_i = \{l+1, \ldots, n\}$.

Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is odd then we refine σ by choosing $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$. Since $-t, t \in \mathcal{J}_0$, we have that σ refined leaves all z_{ij} unchanged. However, now $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ and hence $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even.

Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is odd. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is odd, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$$

is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even, then if it is greater than 0, we refine σ by choosing $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$; otherwise, choose $-u \in \{-1, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = u$, and set $t^\sigma = -u$, and $(-t)^\sigma = u$. Since $-t, t \in \mathcal{J}_0$, σ refined leaves all z_{ij} unchanged, although we now have that $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is odd, and so $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even.

Therefore, in each case we can find a suitable σ such that the parity of σ is even.

15. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}'_n$

Now we have that $\bigcup_{j>0} \mathcal{K}_{-j} = \{1, -2, \ldots, -n\}$. Let $\mathcal{J}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$, and $\bigcup_{i>0} \mathcal{J}_i = \{l+1, \ldots, n\}$.

Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is even. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$ is odd, then

$$|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd, and hence $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$ is even, then if it is greater than 0, then σ can be refined by choosing $-u \in \{1, -2, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and setting $t^\sigma = u$, and $(-t)^\sigma = -u$; otherwise, choose $-u \in \{1, -2 \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = u$, and set $t^\sigma = -u$, and $(-t)^\sigma = u$. We know that $-t, t \in \mathcal{J}_0$, and so σ refined leaves all z_{ij} unchanged. However, $|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$ and hence

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd. Therefore, $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even.

Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$ is even, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd, therefore $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even. If

$$|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd, then we can refine σ by choosing $-u \in \{1, -2, \ldots, -n\}$ and $0 < t \in \mathcal{J}_0$ such that $t^\sigma = -u$, and then setting $t^\sigma = u$, and $(-t)^\sigma = -u$. Since $-t, t \in \mathcal{J}_0$, σ refined leaves all z_{ij} unchanged, although we now have that $|\{1^\sigma, \ldots, l^\sigma\} \cap \{1, -2, \ldots, -n\}|$ is even, so

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{1, -2, \ldots, -n\}|$$

is odd, and $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, -2, \ldots, -n\}|$ is even.

Therefore, in either case we can find σ such that the parity of σ is even.

16. $\mathcal{J} \in \mathcal{G}_{<n}, \mathcal{K} \in \mathcal{G}_{<n}$

    Let $\mathcal{J}_0 = \{-l, \ldots, -1, 1, \ldots, l\}$, and $\mathcal{K}_0 = \{-m, \ldots, -1, 1, \ldots, m\}$.

    Suppose that $\sum_{i,j>0} z_{i(-j)}$ is even, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

    is even. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ is even, then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

    is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd, then we may refine σ by choosing some $0 < t \in \mathcal{J}_0$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, and then setting $t^\sigma = u$, and $(-t)^\sigma = -u$. Since $-t, t \in \mathcal{J}_0$ and $-u, u \in \mathcal{K}_0$ it follows that σ refined leaves all z_{ij} unchanged, however, now $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ and hence $|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$ is even.

    Alternatively, suppose that $\sum_{i,j>0} z_{i(-j)}$ is odd, then

$$|\{(l+1)^\sigma, \ldots, n^\sigma\} \cap \{-m-1, \ldots, -n\}|$$

    is odd. If $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd then

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

    is even. However, if $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ is even, then if it is greater than 0, then we can refine σ by choosing some $0 < t \in \mathcal{J}_0$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = -u$, and then setting $t^\sigma = u$, and $(-t)^\sigma = -u$; if not, then choose some $0 < t \in \mathcal{J}_0$ and $0 < u \in \mathcal{K}_0$ such that $t^\sigma = u$, and set $t^\sigma = -u$, and $(-t)^\sigma = u$. Since $-t, t \in \mathcal{J}_0$ and $-u, u \in \mathcal{K}_0$ it follows that the refining of σ leaves all z_{ij} unchanged. However, we now have that $|\{1^\sigma, \ldots, l^\sigma\} \cap \{-1, \ldots, -m\}|$ is odd, and hence

$$|\{1^\sigma, \ldots, n^\sigma\} \cap \{-1, \ldots, -n\}|$$

    is even.

As in previous sections we now require, for each $x \in X_J^{-1} \cap X_K$, that a matrix $(z_{ij}) = \zeta(x)$ should yield some composition η such that $\boldsymbol{D_\eta}$ is isomorphic to $W_{x^{-1}Jx \cap K}$. Since $\eta \in \mathcal{C}(n)$ we need to know

1. What the components of η are.

2. Whether η is plain or primed.

To obtain the first, we observe that reading the non-zero matrix entries by row gives us a list of numbers $\eta_{-l}, \ldots, \eta_0, \ldots, \eta_l$, if $\eta_0 = z_{00} \neq 0$, or $\eta_{-l}, \ldots, \eta_{-1}, \eta_1, \ldots, \eta_l$ if $z_{00} = 0$, where η_1 is the first non-zero entry read by row after z_{00}. Observe since $z_{ij} = z_{(-i)(-j)}$, that $\eta_i = \eta_{-i}$. This list is such that

$$(S_{\eta_{-l}} \times \ldots \times S_{\eta_{-1}} \times D_{\frac{1}{2}\eta_0} \times S_{\eta_1} \times \ldots \times S_{\eta_l})|_D$$

or

$$(S_{\eta_{-l}} \times \ldots \times S_{\eta_{-1}} \times D_0 \times S_{\eta_1} \times \ldots \times S_{\eta_l})|_D$$

if $z_{00} = 0$, is isomorphic to $x^{-1}W_J x \cap W_K$, and so we may use this list to obtain the components of η.

Since we know that

$$(S_{\eta_{-l}} \times \ldots \times S_{\eta_{-1}} \times D_{\frac{1}{2}\eta_0} \times S_{\eta_1} \times \ldots \times S_{\eta_l})|_D \cong D_{\frac{1}{2}\eta_0} \times S_{\eta_1} \times \ldots \times S_{\eta_l} \qquad (2.5)$$

it follows that unless $\frac{1}{2}\eta_0 = 1$, the components of η in order will be $\eta_1, \ldots \eta_l$. However, if $\frac{1}{2}\eta_0 = 1$, then we recall that this corresponds to

$$\mathcal{J}_0 x \cap \mathcal{K}_0 = \{-1, 1\} = \{-1\} \cup \{\ \} \cup \{1\}.$$

Therefore on the right hand side of Equation 2.5, $D_1 = D_0 \times S_1$, and so the components of η in order will be, by definition: $1, \eta_1, \ldots \eta_l$.

Now that we have found the components of η, to obtain whether or not η is plain or primed, we shall classify the graph of $\mathcal{J}x \cap \mathcal{K}$, and hence derive which subset of $\mathcal{C}(n)$ the composition η belongs to.

Since $\mathcal{J}x \cap \mathcal{K}$ is a subgraph of $\mathcal{K}$, it is convenient to consider four separate cases:

1. $\mathcal{K} \in \mathcal{G}_1$. In this case $\eta \in \mathcal{C}_1$.

2. $\mathcal{K} \in \mathcal{G}_n$. In this case $\eta \in \mathcal{C}_n$, unless $\eta_1 = 1$, in which case $\eta \in \mathcal{C}_1$.

3. $\mathcal{K} \in \mathcal{G}'_n$. In this case $\eta \in \mathcal{C}'_n$, unless $\eta_1 = 1$, in which case $\eta \in \mathcal{C}_1$.

4. $\mathcal{K} \in \mathcal{G}_{<n}$.

   Since $-2, -1, 1, 2 \in \mathcal{K}_0$ we have a more complex case to consider. Therefore, to classify η let us study $\mathcal{J}$ as well. Recall from Lemma 6 that an ordered representation of $\mathcal{J}x \cap \mathcal{K}$ is given by

$$(\mathcal{J}_{-r}x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r+1}x \cap \mathcal{K}_{-s}, \ldots, \mathcal{J}_1 x \cap \mathcal{K}_{-s}, \mathcal{J}_{-r}x \cap \mathcal{K}_{-s+1}, \ldots, \mathcal{J}_r x \cap \mathcal{K}_s)$$

with empty sets, apart from $\mathcal{J}_0 x \cap \mathcal{K}_0$ if empty, removed. More precisely, the sets $\mathcal{J}_i x \cap \mathcal{K}_j$ give the node sets of the connected components of $\mathcal{J}x \cap \mathcal{K}$, and so knowledge of these, coupled with the correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$ will yield the classification type of η.

Let us first recall the correspondence: Let $\lambda \in \mathcal{C}(n)$ such that $\lambda \vDash m \leq n$, with components $\{\lambda_i\}_{i>0}$. Then λ corresponds to the graph $\mathcal{J} \in \mathcal{G}(n)$, with node sets $\{\mathcal{J}_i\}_i$, where

i. $|\mathcal{J}_{-i}| = |\mathcal{J}_i| = \lambda_i$ for $i > 0$.

ii. $\frac{1}{2}|\mathcal{J}_0| = \lambda_0 = n - m$.

iii. If $-1, 1 \in \mathcal{J}_0$ then $\lambda \in \mathcal{C}_{<n}$,
If $-1, 2 \in \mathcal{J}_1$ then $\lambda \in \mathcal{C}'_n$,
If $1, 2 \in \mathcal{J}_1$ then $\lambda \in \mathcal{C}_n$,
Otherwise $\lambda \in \mathcal{C}_1$.

(a) $\mathcal{J} \in \mathcal{G}_1$

We have that $\mathcal{J}_0 = \{\ \}$, and all nodes in $\bigcup_{i>0} \mathcal{J}_i$ are positive.

Suppose first that $\sum_{i,j>0} z_{i(-j)} = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ is even. Then since the parity of x is even it follows that the number of $u \in \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$ such that $u < 0$, if any, must be even. However, since $\{\mathcal{J}_i x \cap \mathcal{K}_0\}_{i>0}$ are node sets of connected components of $\mathcal{J}x \cap \mathcal{K}$ appearing after $\mathcal{J}_0 x \cap \mathcal{K}_0$ in the ordered representation of $\mathcal{J}x \cap \mathcal{K}$, it follows that $u = -1$ is the only possibility for u. However, we need an even number of such u and so it follows that there are no negative numbers in $\bigcup_{i>0, j\geq 0} \mathcal{J}_i x \cap \mathcal{K}_j$. Therefore, by the correspondence, the composition η that corresponds to $\mathcal{J}x \cap \mathcal{K}$ belongs to $\mathcal{C}_n$, or $\mathcal{C}_1$ if $\eta_1 = 1$.

Alternatively, if $\sum_{i,j>0} z_{i(-j)} = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ is odd, then since the parity of x is even, it follows that an odd number of u belong to $\bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$ such that $u < 0$. Since $\{\mathcal{J}_i x \cap \mathcal{K}_0\}_{i>0}$ are node sets of $\mathcal{J}x \cap \mathcal{K}$ appearing after $\mathcal{J}_0 x \cap \mathcal{K}_0$ in the ordered representation of $\mathcal{J}x \cap \mathcal{K}$, it follows that $u = -1$. Hence $-1 \in \bigcup_{i>0, j\geq 0} \mathcal{J}_i x \cap \mathcal{K}_j$. By the correspondence it follows that the composition η corresponding to $\mathcal{J}x \cap \mathcal{K}$ belongs to $\mathcal{C}'_n$, or $\mathcal{C}_1$ if $\eta_1 = 1$.

(b) $\mathcal{J} \in \mathcal{G}_n$

Since $\mathcal{J}_0 = \{\ \}$, and all nodes in $\bigcup_{i>0} \mathcal{J}_i$ are positive, we may immediately deduce that if $\sum_{i,j>0} z_{i(-j)}$ is even then $\eta \in \mathcal{C}_n$, or $\mathcal{C}_1$ if $\eta_1 = 1$ and if $\sum_{i,j>0} z_{i(-j)}$ is odd, then $\eta \in \mathcal{C}'_n$, or $\mathcal{C}_1$ if $\eta_1 = 1$

(c) $\mathcal{J} \in \mathcal{G}'_n$

Here although $\mathcal{J}_0 = \{\ \}$, *not* all nodes in $\bigcup_{i>0} \mathcal{J}_i$ are positive, and so we must use a different argument.

If $\sum_{i,j>0} z_{i(-j)} = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ is even then since $\psi(\mathcal{K}) = \mathcal{K}$, we have that

$$\sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j})| = \sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_{-j}|$$

is even. However, $\psi(\mathcal{J}) \in \mathcal{G}_n$, and in this case, $1 \in \bigcup_{i>0} \psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_0$, and so

$$1^{(-1,1)} = -1 \in \psi(\bigcup_{i>0} \psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_0) = \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0.$$

Hence by the correspondence $\eta \in \mathcal{C}'_n$, or $\mathcal{C}_1$ if $\eta_1 = 1$.

Similarly, if $\sum_{i,j>0} z_{i(-j)} = \sum_{i,j>0} |\mathcal{J}_i x \cap \mathcal{K}_{-j}|$ is odd, then since $\psi(\mathcal{K}) = \mathcal{K}$, we have that

$$\sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_{-j})| = \sum_{i,j>0} |\psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_{-j}|$$

is odd. Now $\psi(\mathcal{J}) \in \mathcal{G}_n$, and so we can deduce from the argument for this case that $-1 \in \bigcup_{i>0} \psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_0$. Therefore,

$$(-1)^{(-1,1)} = 1 \in \psi(\bigcup_{i>0} \psi(\mathcal{J}_i)\psi(x) \cap \mathcal{K}_0) = \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0,$$

and so $\eta \in \mathcal{C}_n$ or $\mathcal{C}_1$ if $\eta_1 = 1$.

(d) $\mathcal{J} \in \mathcal{G}_{<n}$.

If $|\mathcal{J}_0 x \cap \mathcal{K}_0| \neq \{\ \}$, then via the correspondence, it follows that η corresponding to $\mathcal{J}x \cap \mathcal{K}$ belongs to $\mathcal{C}_{<n}$, unless $|\mathcal{J}_0 x \cap \mathcal{K}_0| = 2$, in which case, by the earlier remark on this situation, $\eta \in \mathcal{C}_1$.

If $z_{00} = 0$, it follows that $\bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$ will contain either 1 or -1. If either 1 or -1 is the only member of a node set of $\mathcal{J}x \cap \mathcal{K}$, then it follows that $\eta \in \mathcal{C}_1$. Let us assume this does not happen; then if $\bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$ contains 1, then

$$1^{(-1,1)} = -1 \in \bigcup_{i>0} \psi(\mathcal{J}_i)\psi(x) \cap \psi(\mathcal{K}_0).$$

However, since $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$ it follows that $\psi(\mathcal{J}_i) = \mathcal{J}_i$, $\psi(\mathcal{K}_0) = \mathcal{K}_0$, and $\psi(x) \in X_J^{-1} \cap X_K$. In addition, the matrix $\zeta(x)$, will be identical to $\zeta(\psi(x))$. Therefore, for such a matrix, to know if η will belong to $\mathcal{C}_n$ or $\mathcal{C}'_n$, we must know, in addition, whether 1 or -1 respectively belongs to $\bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$. By the correspondence if $1 \in \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$, then $\eta \in \mathcal{C}_n$, however, if $-1 \in \bigcup_{i>0} \mathcal{J}_i x \cap \mathcal{K}_0$, then $\eta \in \mathcal{C}'_n$.

At this stage we note that each matrix corresponds to an $x \in X_J^{-1} \cap X_K$, unless $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$, and $z_{00} = 0$, then by point 4d above, the matrix corresponds to two double coset representatives in $X_J^{-1} \cap X_K$, x and x', where $x' = \psi(x)$.

If we now recode the basis elements $\mathcal{X}_J$, given in Solomon's Theorem, to B_λ, where λ and $\mathcal{J}$ correspond via the correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$, then we are ready to give a matrix interpretation of Solomon's Theorem for the descent algebra of D_n.

Proposition 2 *For every $\mu \in \mathcal{C}(n)$ let X_μ be the unique set of minimal length left coset representatives of $D_n/\boldsymbol{D_\mu}$. Let*

$$B_\mu = \sum_{\sigma \in X_\mu} \sigma.$$

Let $\lambda, \mu \in \mathcal{C}(n)$. Let λ be a composition of $l \leq n$, with components $\lambda_1, \ldots, \lambda_r$, and let μ be a composition of $m \leq n$, with components $\mu_1, \ldots, \mu_s$, then

$$B_\lambda B_\mu = \sum_{\boldsymbol{z}} B_\eta$$

where the sum is over all matrices, $\boldsymbol{z} = (z_{ij})$, where $i = -s, \ldots, s$ and $j = -r, \ldots, r$, with non-negative integer entries that satisfy

(a) $\sum_{i=-s}^{s} z_{ij} = \lambda_{|j|},\ j \neq 0,$

(b) $\sum_{i=-s}^{s} z_{i0} = 2(n-l),$

(c) $\sum_{j=-r}^{r} z_{ij} = \mu_{|i|},\ i \neq 0,$

(d) $\sum_{j=-r}^{r} z_{0j} = 2(n-m),$

(e) $z_{ij} = z_{(-i)(-j)},$

(f) *If $\lambda, \mu \in \mathcal{C}_1 \cup \mathcal{C}_n \cup \mathcal{C}'_n$, then*

  i. *If $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mu \in \mathcal{C}'_n$, or $\lambda \in \mathcal{C}'_n$ and $\mu \in \mathcal{C}_1 \cup \mathcal{C}_n$ then $\sum_{i,j>0} z_{i(-j)}$ is odd.*

  ii. *Otherwise $\sum_{i,j>0} z_{i(-j)}$ is even.*

For each matrix $\boldsymbol{z}$, η is the composition in $\mathcal{C}(n)$ obtained by omitting the zero components of $[z_{01}, \ldots, z_{0r}, z_{1(-r)}, \ldots, z_{sr}]$, or $[\frac{1}{2}z_{00}, z_{01}, \ldots, z_{0r}, z_{1(-r)}, \ldots, z_{sr}]$ if $z_{00} = 2$, such that

1. *If $\mu \in \mathcal{C}_1$, then η is plain.*

2. *If $\mu \in \mathcal{C}_n$, then η is plain.*

3. *If $\mu \in \mathcal{C}'_n$, then η is primed, unless the first component is 1, in which case, η is plain.*

4. *If $\mu \in \mathcal{C}_{<n}$,*

   (a) *If $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\sum_{i,j>0} z_{i(-j)}$ is odd, or $\lambda \in \mathcal{C}'_n$ and $\sum_{i,j>0} z_{i(-j)}$ is even, then η is primed if $\eta_1 \geq 2$.*

   (b) *If $\lambda \in \mathcal{C}_{<n}$ and $z_{00} = 0$, then η is plain, but B_η is replaced by $B_\eta + B_{\eta'}$, if $\eta_1 \geq 2$, or $2B_\eta$ if $\eta_1 = 1$.*

   (c) *Otherwise η is plain.*

Remark Observe that the classification of η is a summary of the analysis of η that we have just carried out. Note that conditions 1, 2, and 3 correspond to cases 1, 2, and 3 respectively.

Notice that condition 4a corresponds to when $\eta \in \mathcal{C}'_n$ in cases 4a, 4b, and 4c; condition 4b corresponds to when $z_{00} = 0$ in case 4d; and condition 4c deals with when $\eta \notin \mathcal{C}'_n$ in cases 4a, 4b, 4c, and when $z_{00} \neq 0$ in case 4d.

This proposition is rather unwieldy to use in practice, however, it can be further simplified. To do this we need to make three observations. For the first, let λ and μ be compositions of $\mathcal{C}(n)$ such that $\lambda \vDash l \leq n$ has components $\lambda_1, \ldots, \lambda_r$, and $\mu \vDash m \leq n$ has components $\mu_1, \ldots, \mu_s$. In view of Proposition 2, to obtain η we only need to know the matrix entries $z_{01}, \ldots, z_{0r}, z_{1(-r)}, \ldots, z_{sr}$. Using the equation $z_{ij} = z_{(-i)(-j)}$ we can rewrite conditions (a) to (d) of Proposition 2 as

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + z_{i(-j)}) = \lambda_j,\ j \neq 0$,
2. $\frac{1}{2}z_{00} + \sum_{i=1}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + z_{i(-j)}) = \mu_i,\ i \neq 0$
4. $\frac{1}{2}z_{00} + \sum_{j=1}^{r} z_{0j} = n - m.$

Hence instead of using $(2s+1) \times (2r+1)$ matrices such that conditions (a) to (e) of Proposition 2 hold we can, as in the previous section, use partial matrices

$$P = \begin{pmatrix} & & z_{00} & \ldots & z_{0r} \\ z_{1(-r)} & \ldots & z_{10} & \ldots & z_{1r} \\ \vdots & & \vdots & & \vdots \\ z_{s(-r)} & \ldots & z_{s0} & \ldots & z_{sr} \end{pmatrix}$$

where

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + z_{i(-j)}) = \lambda_j,\ j \neq 0$,
2. $\sum_{i=0}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + z_{i(-j)}) = \mu_i,\ i \neq 0$,
4. $\sum_{j=0}^{r} z_{0j} = n - m.$

As in the previous section, P corresponds to a template

$$\boldsymbol{t} = \begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0r} \\ & y_{11} & y_{12} & \cdots & y_{1r} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{s1} & y_{s2} & \cdots & y_{sr} \\ z_{s0} & z_{s1} & z_{s2} & \cdots & z_{sr}. \end{pmatrix}$$

by the correspondence

$$\begin{aligned} z_{ij} &\leftrightarrow z_{ij} \\ z_{i(-j)} &\leftrightarrow y_{ij} \end{aligned}$$

for $i = 0, 1, \ldots, s$, $j = 0, 1, \ldots, r$, when mapping z_{ij}, and $i = 1, \ldots, s$, $j = 0, 1, \ldots, r$ when mapping $z_{i(-j)}$.

Our conditions on P now become the following conditions on $\boldsymbol{t}$

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + y_{ij}) = \lambda_j,\ j \neq 0,$
2. $\sum_{i=0}^{s} z_{i0} = n - l,$
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + y_{ij}) = \mu_i,\ i \neq 0,$
4. $\sum_{j=0}^{r} z_{0j} = n - m.$

The second observation is that if $\lambda, \mu \in \mathcal{C}_1 \cup \mathcal{C}_n \cup \mathcal{C}'_n$, then $\mathcal{J}_0 = \mathcal{K}_0 = \{\ \}$. This implies that each of the matrices (z_{ij}) in Proposition 2 satisfies

$$z_{00} + \sum_{i=-s}^{s} z_{i0} + \sum_{j=-r}^{r} z_{0j} = 0,$$

or in terms of the corresponding templates

$$z_{00} + \sum_{i=1}^{s} z_{i0} + \sum_{j=1}^{r} z_{0j} = 0.$$

Let us define this latter sum to be the *border sum*, $\mathcal{B}(\boldsymbol{t})$, of a template $\boldsymbol{t}$, and $\mathcal{Y}(\boldsymbol{t}) = \sum_{i,j} y_{ij}$ to be the *y-sum*. Then, in terms of templates, condition (f) of Proposition 2 may be re-written as: if $\mathcal{B}(\boldsymbol{t}) = 0$, then $\mathcal{Y}(\boldsymbol{t})$ is odd if

1. $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mu \in \mathcal{C}'_n$, or
2. $\lambda \in \mathcal{C}'_n$ and $\mu \in \mathcal{C}_1 \cup \mathcal{C}_n$.

Otherwise $\mathcal{Y}(\boldsymbol{t})$ is even.

Combining these two observations, we may replace the set of $(2s+1)\times(2r+1)$ matrices with non-negative integer entries such that conditions (a) to (f) of Proposition 2 hold, by the set of templates $Z(\lambda,\mu)$, where $Z(\lambda,\mu)$ is defined as follows.

If λ, and μ are compositions in $\mathcal{C}(n)$ such that $\lambda \vDash l \leq n$ has components $\lambda_1, \lambda_2, \ldots, \lambda_r$, and $\mu \vDash m \leq n$ has components $\mu_1, \mu_2, \ldots, \mu_s$, then the template $\boldsymbol{t}$ belongs to $Z(\lambda,\mu)$ if and only if the following five conditions hold.

1. $z_{0j} + \sum_{i=1}^{s}(z_{ij} + y_{ij}) = \lambda_j,\ j \neq 0$,
2. $\sum_{i=0}^{s} z_{i0} = n - l$,
3. $z_{i0} + \sum_{j=1}^{r}(z_{ij} + y_{ij}) = \mu_i,\ i \neq 0$,
4. $\sum_{j=0}^{r} z_{0j} = n - m$,
5. If $\mathcal{B}(\boldsymbol{t})$=0, $\mathcal{Y}(\boldsymbol{t})$ is odd if
   (a) $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mu \in \mathcal{C}_n'$, or
   (b) $\lambda \in \mathcal{C}_n'$ and $\mu \in \mathcal{C}_1 \cup \mathcal{C}_n$.

   Otherwise $\mathcal{Y}(\boldsymbol{t})$ is even.

The last observation is that the classification of η in Proposition 2 can be realised in terms of $Z(\lambda,\mu)$, and $\mathcal{Y}(\boldsymbol{t})$ as:

Let $\lambda \in \mathcal{C}(n)$. Let $\boldsymbol{t}$ be a template, and the *reading word* of $\boldsymbol{t}$, $r(\boldsymbol{t})$, be

$$[z_{01}, z_{02}, \ldots, z_{0r}, y_{1r}, \ldots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \ldots, z_{1r}, \ldots, z_{s0}, z_{s1}, z_{s2}, \ldots, z_{sr}]$$

with zero entries removed, or

$$[1, z_{01}, z_{02}, \ldots, z_{0r}, y_{1r}, \ldots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \ldots, z_{1r}, \ldots, z_{s0}, z_{s1}, z_{s2}, \ldots, z_{sr}]$$

with zero entries removed, if $z_{00} = 1$. Let $\eta = r(\boldsymbol{t})$. Let $B_{\boldsymbol{t}}$ be the sum of the basis elements yielded by $\boldsymbol{t}$. We note that in most cases we will get one summand, apart from when we are dealing with case 4b in Proposition 2. Then

1. If $\mu \in \mathcal{C}_1$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t}\in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

   where $B_{\boldsymbol{t}} = B_\eta$.

2. If $\mu \in \mathcal{C}_n$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where $B_{\boldsymbol{t}} = B_\eta$.

3. If $\mu \in \mathcal{C}'_n$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where $B_{\boldsymbol{t}} = B_{\eta'}$, or B_η if $\eta_1 = 1$.

4. If $\mu \in \mathcal{C}_{<n}$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where

   (a) If $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mathcal{Y}(\boldsymbol{t})$ is odd, or $\lambda \in \mathcal{C}'_n$ and $\mathcal{Y}(\boldsymbol{t})$ is even. then $B_{\boldsymbol{t}} = B_{\eta'}$, if $\eta_1 \geq 2$,

   (b) If $\lambda \in \mathcal{C}_{<n}$ and $z_{00} = 0$, then $B_{\boldsymbol{t}} = B_\eta + B_{\eta'}$, or $2B_\eta$ if $\eta_1 = 1$.

   (c) Otherwise $B_{\boldsymbol{t}} = B_\eta$.

Hence, with these three observations in mind we can simplify the re-statement of Solomon's Theorem.

2.3.1 The matrix interpretation

Consider "templates" with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0r} \\ & y_{11} & y_{12} & \cdots & y_{1r} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{s1} & y_{s2} & \cdots & y_{sr} \\ z_{s0} & z_{s1} & z_{s2} & \cdots & z_{sr} \end{pmatrix}$$

where

1. All entries in a template are non-negative integers,

2. The y-lines do not have entries in column 0.

Let the above template establish generic names for entries in a template.

Recall that the border-sum, $\mathcal{B}(\boldsymbol{t})$, of the template is the sum

$$z_{00} + \sum_{i=1}^{s} z_{i0} + \sum_{j=1}^{r} z_{0j}$$

and the y-sum, $\mathcal{Y}(\boldsymbol{t})$, is $\sum_{i,j} y_{ij}$. The reading word, $r(\boldsymbol{t})$, of a given template $\boldsymbol{t}$ is given by

$$[z_{01}, z_{02}, \dots, z_{0r}, y_{1r}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1r}, \dots, z_{s0}, z_{s1}, z_{s2}, \dots, z_{sr}]$$

with zero entries omitted, unless $z_{00} = 1$, in which case $r(\boldsymbol{t})$ is given by

$$[1, z_{01}, z_{02}, \dots, z_{0r}, y_{1r}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1r}, \dots, z_{s0}, z_{s1}, z_{s2}, \dots, z_{sr}]$$

with zero entries omitted.

If λ, and μ are compositions in $\mathcal{C}(n)$ such that $\lambda \vDash l \leq n$, and $\mu \vDash m \leq n$, then we recall that $Z(\lambda, \mu)$ is the set of templates, $\boldsymbol{t}$, such that

1. $z_{0j} + \sum_{i \neq 0}(y_{ij} + z_{ij}) = \lambda_j,\ j \neq 0$,
2. $\sum_i z_{i0} = n - l$,
3. $z_{i0} + \sum_{j \neq 0}(y_{ij} + z_{ij}) = \mu_i,\ i \neq 0$,
4. $\sum_j z_{0j} = n - m$,
5. If $\mathcal{B}(\boldsymbol{t})$=0, $\mathcal{Y}(\boldsymbol{t})$ is odd if
   (a) $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mu \in \mathcal{C}'_n$, or
   (b) $\lambda \in \mathcal{C}'_n$ and $\mu \in \mathcal{C}_1 \cup \mathcal{C}_n$.

   Otherwise $\mathcal{Y}(\boldsymbol{t})$ is even.

Theorem 6 *For every $\mu \in \mathcal{C}(n)$, let X_μ be the unique set of minimal length left coset representatives of $D_n/\boldsymbol{D_\mu}$. Let*

$$B_\mu = \sum_{\sigma \in X_\mu} \sigma.$$

Let $\lambda \in \mathcal{C}(n)$. Let $\boldsymbol{t}$ be a template, and $\eta = r(\boldsymbol{t})$. Let $B_{\boldsymbol{t}}$ be the sum of the basis elements yielded by $\boldsymbol{t}$. Then

1. *If $\mu \in \mathcal{C}_1$,*

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

   where $B_{\boldsymbol{t}} = B_\eta$.

2. *If* $\mu \in \mathcal{C}_n$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where $B_{\boldsymbol{t}} = B_\eta$.

3. *If* $\mu \in \mathcal{C}'_n$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where $B_{\boldsymbol{t}} = B_{\eta'}$, *or* B_η *if* $\eta_1 = 1$.

4. *If* $\mu \in \mathcal{C}_{<n}$,

$$B_\lambda B_\mu = \sum_{\boldsymbol{t} \in Z(\lambda,\mu)} B_{\boldsymbol{t}}$$

where

   (a) *If* $\lambda \in \mathcal{C}_1 \cup \mathcal{C}_n$ *and* $\mathcal{Y}(\boldsymbol{t})$ *is odd, or* $\lambda \in \mathcal{C}'_n$ *and* $\mathcal{Y}(\boldsymbol{t})$ *is even. then* $B_{\boldsymbol{t}} = B_{\eta'}$, *if* $\eta_1 \geq 2$,

   (b) *If* $\lambda \in \mathcal{C}_{<n}$ *and* $z_{00} = 0$, *then* $B_{\boldsymbol{t}} = B_\eta + B_{\eta'}$, *or* $2B_\eta$ *if* $\eta_1 = 1$.

   (c) *Otherwise* $B_{\boldsymbol{t}} = B_\eta$.

Some examples of this matrix interpretation in use can be found in Appendix A.

Chapter 3

The p-modular descent algebra

"Generalities are intellectually necessary evils"

– A. Huxley, Brave New World

When group algebras were first investigated for their pivotal role in the representation theory of finite groups, they were studied over a field of zero characteristic, for example [Bur11], [FS06]. Since their structure constants with respect to the basis $\{g\}_{g\in G}$ are integers, it followed that they could also be defined over a field of prime characteristic, p. Furthermore, research into this "p-modular" version of a group algebra was seen to contribute significantly, via modular representation theory, to the subject of group theory, and since then such investigations have played an important role in the study of algebra, for example [Bra39], [Bra56], [BN37], [BN41] (for extensive examples in either case consult the Bibliography in [CR62]).

The descent algebra Σ_W is a subalgebra of the group algebra, and we have seen (Chapter 1) how all properties established for it so far have been over the rationals. Hence it is natural to ask if there might exist a p-modular analogue, since it may reveal more about the subject in a way similar to that seen for group theory. Certainly, we can answer this, for it is not difficult to see that a p-modular analogue does exist for each descent algebra. Recall Solomon's Theorem:

Let W be a Coxeter group, with generating set S. For every subset K of S, let

$$\mathcal{X}_K = \sum_{\sigma\in X_K} \sigma.$$

Then for subsets J and K in S

$$\mathcal{X}_J\mathcal{X}_K = \sum a_{JKL}\mathcal{X}_L$$

where a_{JKL} is the number of elements $x \in X_J^{-1} \cap X_K$ such that $x^{-1}W_Jx \cap W_K = W_L$, with $L = x^{-1}Jx \cap K$.

From this we see that since the set of all $\mathcal{X}_K$ is a basis for Σ_W, the structure constants of Σ_W are the integers a_{JKL}. Hence, the Z-module, $\mathcal{Z}_W$, spanned by all $\mathcal{X}_K$ is a subring of Σ_W, and we are able to define descent algebras over a field of characteristic p.

Let p be any prime, and let $\mathcal{P}_W = p\mathcal{Z}_W$, which is an ideal of $\mathcal{Z}_W$. We define $\Sigma(W,p) = \mathcal{Z}_W/\mathcal{P}_W$ to be the *p-modular descent algebra* of W. Then $\Sigma(W,p)$ is our desired descent algebra over $\mathcal{F}_p$, the field of order p.

Now that we have defined the p-modular descent algebra, we must establish some properties of it. Clearly we cannot hope to establish here the equivalent of what is known for Σ_W. After all, this knowledge has taken 20 years to collect. However, we can provide a base, and since Solomon's work in [Sol76] initiated research into Σ_W, it seems fitting to establish an analogy to the results seen there. To do this, we recall one of the main results of [Sol76], the identification of the radical of Σ_W. This Solomon obtained using the following constructs.

If K is any subset of S, let χ_K be the permutation character of W acting on the right cosets of W_K. Let G_W be the Z-module generated by all χ_K, then by the Mackey formula, (Theorem 44.3, [CR62]), G_W is indeed an algebra. Define θ to be the unique linear transformation that satisfies

$$\theta(\mathcal{X}_K) = \chi_K.$$

With this Solomon proved the following theorem:

Theorem 7 *Let $\theta : \Sigma_W \mapsto G_W$, such that $\theta(\mathcal{X}_K) = \chi_K$. Then θ is an algebra homomorphism, and $\ker\theta = \mathrm{rad}(\Sigma_W)$. Moreover, $\mathrm{rad}(\Sigma_W)$ is spanned by all differences $\mathcal{X}_J - \mathcal{X}_K$, where J and K are conjugate subsets of S.*

Our aim, therefore, is to determine $\mathrm{rad}(\Sigma(W,p))$, for a given p. As we shall see, its description can indeed be given in a style similar to Theorem 7.

Let ρ_1 be the natural projection $\mathcal{Z}_W \mapsto \Sigma(W,p)$, and let $\overline{\mathcal{X}}_K = \rho_1(\mathcal{X}_K)$. Then $\{\overline{\mathcal{X}}_J | J \subseteq S\}$ is a basis for $\Sigma(W,p)$ and

$$\overline{\mathcal{X}}_J\overline{\mathcal{X}}_K = \sum \overline{a}_{JKL}\overline{\mathcal{X}}_L$$

where $\overline{a}_{JKL}$ is the image of a_{JKL} in $\mathcal{F}_p$. Since each generalised character in G_W has integer character values, it follows that we may reduce the character values $\mod p$. Let ρ_2 be the map defined on G_W by the reduction of character values $\mod p$, and let the image of G_W under ρ_2 be denoted by $G(W,p)$. Then the map

$$\phi : \Sigma(W,p) \mapsto G(W,p)$$

defined by

$$\phi(\rho_1(x)) = \rho_2(\theta(x)) \text{ for all } x \in \mathcal{Z}_W$$

is clearly well defined and is an algebra homomorphism. Indeed, it is this homomorphism that we shall eventually use to describe $\mathrm{rad}(\Sigma(W,p))$.

3.1 The radical and irreducible representations of $\Sigma(W,p)$

Since we intend to use ϕ to describe $\mathrm{rad}(\Sigma(W,p))$, it follows that we will need to know more about $G(W,p)$. This we can achieve through the permutation characters χ_K, in particular, the study of their character values. To do this we must recall the following.

Let K be a subset of S. Then we say that c_K is a Coxeter element of K if c_K is a product of all the elements of K. By Chapter 5, Section 6, Proposition 1, [Bou68] we know that all such products are conjugate in W. We also partially order the subsets of S by conjugate inclusion: therefore we define $J_1 \prec J_2$ if and only if $J_1 \subset J_2^w$ for some $w \in W$. Let us now fix a total ordering on the subsets of S that extends this partial order.

We are now ready to learn more about $G(W,p)$.

Lemma 9 *If A and B are subgroups of a finite group G then*

$$|\{Ax|AxB = Ax\}| = [N_G(A) : A]|\{A^x|B \subseteq A^x\}|$$

PROOF $AxB = Ax$ if and only if $A^xB = A^x$ if and only if $B \subseteq A^x$ and so

$$\{x|AxB = Ax\} = \{x|B \subseteq A^x\}$$

The set on the left-hand side is a union of cosets of A while, on the right-hand side, every conjugate A^x of A is associated with $|N_G(A)|$ elements x. Therefore

$$|A||\{Ax|AxB = Ax\}| = |N_G(A)||\{A^x|B \subseteq A^x\}|$$

and the result follows. □

Lemma 10 $\chi_J(c_K) = a_{JKK} = [N_W(W_J) : W_J]|\{W_J^w|W_K \subseteq W_J^w\}|$

PROOF The first equality is contained in Theorem 6.2 of [BBHT92]. To prove the second equality we argue from the definitions. By Theorem 1 $a_{JKK} = |\{x \in X_J^{-1} \cap X_K|W_K \subseteq W_J^x\}|$. However, if $x \in X_J^{-1}$ and $W_K \subseteq W_J^x$ then x is a word of shortest length in W_Jx and $xW_K \subseteq W_Jx$; therefore x is a word of shortest length in xW_K and so $x \in X_K$. Hence

$$\begin{aligned} a_{JKK} &= |\{x \in X_J^{-1}|W_K \subseteq W_J^x\}| \\ &= |\{W_Jx|W_K \subseteq W_J^x\}| \\ &= |\{W_Jx|W_JxW_K = W_Jx\}| \end{aligned}$$

and the result now follows from the previous lemma. ☐

Corollary 1 *1.* $\chi_J(c_J) = a_{JJJ} = [N_W(W_J) : W_J] \neq 0$.

2. $\chi_J(c_J) = a_{JJJ}$ *divides* $\chi_J(c_K) = a_{JKK}$ *for all* J *and* K.

It follows from Theorem 7 that $\chi_J(c_K) = \chi_{J'}(c_K)$ for all K, if J and J' are conjugate subsets of S. Also, if K and K' are conjugate subsets of S then c_K and $c_{K'}$ are conjugate and so $\chi_J(c_K) = \chi_J(c_{K'})$ for all J. Motivated by these two observations we now select a fixed representative from every set of conjugate subsets of S and we consider the matrix $R = [\chi_J(c_K)]$ whose rows and columns are indexed by these representatives ordered according to the total ordering on subsets of S defined above.

Let r be the number of rows of R and let s be the number of rows indexed by subsets J with $p|[N_W(W_J) : W_J]$.

Lemma 11 *1.* R *is a lower triangular matrix of rank* $r = \dim G_W$.

2. The p*-rank of* R *(*$= \dim G(W,p)$*) is* $r - s$.

PROOF If the entry $\chi_J(c_K)$ of R is non-zero then, by Lemma 10, there exists $w \in W$ with $W_K \subseteq W_J^w$ and so K equals J, or K precedes J in the conjugate inclusion order on subsets of S. Thus R is indeed lower triangular and since, by Corollary 1, $\chi_J(c_J) \neq 0$, R has full rank $r = \dim G_W$.

If p divides a diagonal entry of R then, by Corollary 1, p divides every entry of that row. Thus the rank of $R \mod p$ (i.e. $\dim G(W,p)$) is the number of non-zero rows in $R \mod p$ and this, by Corollary 1 again, is $r - s$. ☐

Now that we know more about $G(W,p)$, it only remains to prove the following lemma before we can identify $\mathrm{rad}(\Sigma(W,p))$.

Lemma 12 *1.* $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ *is commutative.*

2. Every nilpotent element of $\Sigma(W,p)$ *lies in* $\mathrm{rad}(\Sigma(W,p))$.

PROOF Let θ_1 be the restriction of θ to $\mathcal{Z}_W$. Then θ_1 maps $\mathcal{Z}_W$ onto the commutative ring G_W. By Theorem 7, the kernel of θ_1 is the Z-module $\mathcal{R}_W$ spanned by all $\mathcal{X}_J - \mathcal{X}_K$ where J and K are conjugate subsets of S, and is a nilpotent ideal of $\mathcal{Z}_W$. In particular $\rho_1(\mathcal{R}_W)$ is a nilpotent ideal of $\Sigma(W,p)$, and therefore $\rho_1(\mathcal{R}_W) \subseteq \mathrm{rad}(\Sigma(W,p))$. Hence there exists an ideal $\mathcal{S}_W$ of Σ_W, the pre-image of $\mathrm{rad}(\Sigma(W,p))$, such that $\mathcal{R}_W \subseteq \mathcal{S}_W$ and $\mathcal{S}_W/\mathcal{P}_W \cong \mathrm{rad}(\Sigma(W,p))$. Since $\Sigma(W,p) \cong \mathcal{Z}_W/\mathcal{P}_W$, $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p)) \cong \mathcal{Z}_W/\mathcal{S}_W$ is a homomorphic image of $\mathcal{Z}_W/\mathcal{R}_W \cong G_W$. Since the latter ring is commutative the first part follows.

If x is any nilpotent element of $\Sigma(W,p)$ then the coset $x + \mathrm{rad}(\Sigma(W,p))$ is a nilpotent element in the commutative semi-simple algebra $\Sigma(W,p)/\mathrm{rad}(\Sigma(W,p))$ and so is zero. Therefore $x \in \mathrm{rad}(\Sigma(W,p))$. □

We are now ready to prove the main result of this section.

Theorem 8 $\mathrm{rad}(\Sigma(W,p)) = \ker\phi$. *Moreover,* $\mathrm{rad}(\Sigma(W,p))$ *is spanned by all* $\overline{\mathcal{X}}_J - \overline{\mathcal{X}}_K$ *where* J, K *are conjugate subsets of* S*, together with all* $\overline{\mathcal{X}}_J$ *for which* p *divides* $[N_W(W_J) : W_J]$.

PROOF First we note that $\mathrm{rad}(\Sigma(W,p)) \subseteq \ker\phi$. This follows since the image of ϕ is a space of functions defined over a field and is therefore semi-simple. Consequently the two-sided nilpotent ideal $\phi(\mathrm{rad}(\Sigma(W,p)))$ must be zero.

Now we prove that, if $p|[N_W(W_J) : W_J]$, then $\overline{\mathcal{X}}_J \in \mathrm{rad}(\Sigma(W,p))$. From the definition of a_{JKL} in Theorem 1, $\overline{a}_{JKL} = 0$ unless $L \subseteq K$ and, by Corollary 1, $\overline{a}_{JKK} = 0$ also. Thus $\overline{\mathcal{X}}_J\overline{\mathcal{X}}_K$ is a linear combination of elements $\overline{\mathcal{X}}_L$ with $L \subset K$ (and so $|L| \leq |K| - 1$). Now, by induction, it follows that $\overline{\mathcal{X}}^t_J\overline{\mathcal{X}}_K$ is a linear combination of elements $\overline{\mathcal{X}}_L$ with $|L| \leq |K| - t$ and so $\overline{\mathcal{X}}_J^{|K|+1}\overline{\mathcal{X}}_K$=0 for all K. In particular $\overline{\mathcal{X}}_J$ is nilpotent and so $\overline{\mathcal{X}}_J \in \mathrm{rad}(\Sigma(W,p))$ by Lemma 12.

The elements $\overline{\mathcal{X}}_J - \overline{\mathcal{X}}_K$ where J and K are conjugate subsets of S are all nilpotent and, by Lemma 12, lie in $\mathrm{rad}(\Sigma(W,p))$. They span a space U of dimension $\dim\mathrm{rad}(\Sigma_W) = \dim\Sigma_W - \dim G_W = 2^{n-1} - r$. In addition there are s elements $\overline{\mathcal{X}}_J$ corresponding to those rows of R for which $p|[N_W(W_J) : W_J]$ which also lie in $\mathrm{rad}(\Sigma(W,p))$. These, together with U, span a space of dimension $2^{n-1} - r + s = 2^{n-1} - \dim G(W,p) = \dim\ker\phi$. Hence $\dim\mathrm{rad}(\Sigma(W,p)) \geq \dim\ker\phi$.

This proves that $\ker\phi = \mathrm{rad}(\Sigma(W,p))$ as required and that it is spanned by the desired set of elements. □

In addition to this, we can also explicitly describe the irreducible representations of $\Sigma(W,p)$. Let E denote the set of subsets of S that index the columns of R. For each $K \in E$ define the map $\xi_K : \Sigma_W \mapsto \mathbb{Q}$ by

$$\xi_K(x) = \theta(x)(c_K) \text{ for all } x \in \Sigma_W$$

Since θ is a homomorphism it follows that

$$\begin{aligned} \xi_K(xy) &= \theta(xy)(c_K) \\ &= \theta(x)\theta(y)(c_K) \\ &= \theta(x)(c_K) \cdot \theta(y)(c_K) \\ &= \xi_K(x) \times \xi_K(y) \end{aligned}$$

as characters are pointwise multiplicative. Therefore, ξ_K is also a homomorphism, and a 1- dimensional representation of Σ_W. Notice that ξ_K is completely determined by its values on basis elements $\mathcal{X}_J$, that $\xi_K(\mathcal{X}_J) = \theta(\mathcal{X}_J)(c_K) = \chi_J(c_K)$, and the values of ξ_K are determined by the appropriate columns of the matrix R. In particular, the set of all $\xi_K|_{\mathcal{Z}_W}$ take integer values and reducing these values modulo p we shall obtain a set of $r - s$ linearly independent columns of R. Moreover, from Theorem 8, we know that $\Sigma(W, p)$ has $r - s$ irreducible representations, and therefore R reduced modulo p has exactly $r - s$ distinct columns. It is precisely these that we can take as the irreducible representations of $\Sigma(W, p)$.

In the next chapter we give, for the descent algebras of the Coxeter groups of types A and B, a rule for choosing an appropriate set of $r - s$ columns of R.

Chapter 4

The p-modular descent algebras of types A and B

"Variety's the very spice of life"

– W. Cowper, The Task

Solomon's paper, [Sol76], established some key results for the descent algebra of any Coxeter group over the rationals. However, as we have seen in Chapter 1, major advances in the subject were made through the study of specific Coxeter group families ([GR89], [BB92a], [Ber92]). In the last chapter, we followed Solomon's lead by deriving some results for the p-modular descent algebras irrespective of their classification, although from the examples we have just cited, we might expect that further progress in this field will now be made through the study of specific families. It is for this reason that in this chapter we restate the results of Chapter 3 in the notation of Chapter 2, for the symmetric and hyperoctahedral groups. This will not only allow a direct comparison of our results with those of [GR89], [BB92a], and [Ber92] to be made, but, as we shall see, also allows more detailed results concerning these descent algebras to be derived. As with Chapter 2, we can adopt a common approach to this study of the descent algebras of the symmetric and hyperoctahedral groups, and it is this that we now give.

We start by recalling the matrix interpretation of our descent algebra over the rationals. Then we define two binary relations, $\approx$ and $\preccurlyeq$, on the set of compositions that we are dealing with as follows. If λ, μ are any two compositions, then we say that $\lambda \approx \mu$ if λ and μ determine the same partition, that is they are the same when their components are re-ordered into numerically non-increasing order. Note that $\approx$ is an equivalence relation. We say that $\lambda \preccurlyeq \mu$ if we can obtain μ from λ by using an algorithm associated with the given descent algebra that involves replacing adjacent components of

λ by their sum. In each case this relation is reflexive and transitive.

From here we define an equivalence on the matrices that appear in the matrix interpretation, and using this and our binary relations, deduce some properties concerning the structure constants of our chosen descent algebras. Then we restate multiplication in the p-modular descent algebra in terms of a matrix interpretation, and note how results concerning the structure constants alter in this case. Combining these p-modular properties with the results of Chapter 3, we obtain an explicit description of the radical of our chosen p-modular descent algebras. With this knowledge and the following lemma, we are then able to derive the respective irreducible representations.

Definition 8 *A* p-singular partition (composition) *of an integer* n *is a partition (composition) with at least one component divisible by* p*. A* p-regular partition (composition) *of an integer* n *is a partition (composition) with no component divisible by* p*.*

Lemma 13 *(p41 [JK81]) Let* p *be a prime and* n *be a non-negative integer, then the number of partitions of* n *with a component of multiplicity* p *or more is equal to the number of* p*-singular partitions of* n*.*

We then introduce a chain of ideals T_i for our p-modular descent algebra, $\Sigma(W,p)$, where

$$\Sigma(W,p) = T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \supseteq T_{n+1} = 0$$

and show that $\mathrm{rad}(\Sigma(W,p))$ raised to successive powers lie in ideals further along the chain. Using this, and the study of a specific element belonging to the radical, we determine the nilpotency index of $\mathrm{rad}(\Sigma(W,p))$.

4.1 The p-modular descent algebra of the symmetric groups

As we saw in Chapter 3, to study the p-modular descent algebra, we must first study the descent algebra over the rationals. Therefore, let us look at Σ_{S_n} a little more. To do this let us recall the matrix interpretation in this case.

Let B_λ, B_μ be basis elements of Σ_{S_n}, where $\lambda = [\lambda_1, \ldots, \lambda_r]$ and $\mu = [\mu_1, \ldots, \mu_s]$ are compositions of n. Let $Z(\lambda,\mu)$ be the set of matrices $\boldsymbol{z} = (z_{ij})$ with non-negative integer entries such that

1. $\sum_i z_{ij} = \lambda_j$, for each $j = 1, 2, \ldots, r$.
2. $\sum_j z_{ij} = \mu_i$, for each $i = 1, 2, \ldots, s$.

Multiplication in Σ_{S_n} is then given by

$$B_\lambda B_\mu = \sum_\eta a_{\lambda\mu\eta} B_\eta$$

where η is the composition of n obtained by omitting zero entries in $[z_{11}, z_{12}, \ldots]$, and $a_{\lambda\mu\eta}$ is the number of matrices in $Z(\lambda, \mu)$ that yield η.

We write $\lambda \preccurlyeq \mu$ if the components of μ can be obtained from the components of λ by summing adjacent components of λ.

Definition 9 *Two matrices are said to be* column equivalent *if one can be obtained from the other by permuting the columns.*

We now use this definition and our binary relations to derive some properties of Σ_{S_n} that can be deduced easily from its matrix interpretation.

Lemma 14 *Let B_λ and B_μ be basis elements of Σ_{S_n} and suppose that, in the composition λ, the number of components equal to i is denoted by t_i. Then*

1. *If the coefficient of B_η in $B_\lambda B_\mu$ is non-zero then $\eta \preceq \mu$.*
2. *The coefficient of B_μ in the product $B_\lambda B_\mu$ is a multiple of $t_1!t_2!\ldots t_n!$ and this coefficient depends on the column equivalence class of λ only.*
3. *If $\lambda \approx \mu$, the coefficient of B_μ in $B_\lambda B_\mu$ is exactly $t_1!t_2!\ldots t_n!$*

PROOF The first statement follows since if B_η is a non-zero summand in the product $B_\lambda B_\mu$, then there exists a matrix whose reading word is η. However, the entries of each row of the matrix sum to give a component of μ. Hence $\eta \preceq \mu$. To prove the remaining statements let $\mu = [\mu_1, \ldots, \mu_s]$. A matrix $\boldsymbol{z} \in Z(\lambda, \mu)$ which contributes to the coefficient of B_λ in $B_\lambda B_\mu$ satisfies

$$\sum_j z_{ij} = \mu_i \text{ and } \sum_i z_{ij} = \lambda_j$$

and the non-zero entries of the rows of $\boldsymbol{z}$, if read in serial order, yield $\mu_1, \ldots, \mu_s$. It follows that the i-th row of $\boldsymbol{z}$ has a single non-zero entry which is equal to μ_i. Note also that, since all $\lambda_j > 0$, every column of $\boldsymbol{z}$ has at least one non-zero entry.

The set of matrices $\mathcal{Q}$ (if any) which satisfy these conditions falls into a number of column equivalence classes. Each of these classes has precisely $t_1!t_2!\ldots t_n!$ members since the set of columns of one of the matrices in $\mathcal{Q}$ with a common sum may be permuted arbitrarily. Thus the coefficient of B_μ in $B_\lambda B_\mu$ is indeed a multiple of $t_1!t_2!\ldots t_n!$ If η is some composition equivalent to λ the set of matrices that is analogous to $\mathcal{Q}$ is related to $\mathcal{Q}$ by permuting columns. This proves the second statement. For the third statement we note that, when $\lambda \approx \mu$, $\mathcal{Q}$ consists of exactly one column equivalence class since then the matrices will have exactly one non-zero entry in each column as well as each row. □

From Chapter 3 we recall that if $\mathcal{X}_J$ is a basis element of Σ_W, and ρ_1 is the natural homomorphism between $\mathcal{Z}_W$ and $\Sigma(W, p)$, then the set of all $\overline{\mathcal{X}}_J$, where $\overline{\mathcal{X}}_J = \rho_1(\mathcal{X}_J)$, forms a basis for $\Sigma(W, p)$. We also recall that the multiplicative action in $\Sigma(W, p)$ is that same as that in $\mathcal{Z}_W$ except all coefficients are now reduced modulo p. It follows that $\{\overline{B}_\lambda | \lambda \vDash n\}$ is a basis for $\Sigma(S_n, p)$ and that Lemma 14 holds also for basis elements $\overline{B}_\lambda, \overline{B}_\mu$ of $\Sigma(S_n, p)$ except that the coefficients in question must be reduced modulo p.

Theorem 9 $\mathrm{rad}(\Sigma(S_n, p))$ *is spanned by all* $\overline{B}_\lambda - \overline{B}_\mu$ *with* $\lambda \approx \mu$ *together with all* $\overline{B}_\lambda$ *where* λ *has a component of multiplicity p or more.*

PROOF This result will follow immediately from Theorem 8 if we can prove that $J \sim K$ if and only if their associated compositions, by condition (2.1), λ and μ determine the same partition, and that p divides $[N_{S_n}(W_J) : W_J]$ if and only if the composition, λ, associated with J has a component of multiplicity p or more.

If J is conjugate to K under some element $x \in S_n$, then it follows by definition that $\mathcal{J}x = \mathcal{K}$. Moreover, each node set $\mathcal{J}_i$ of $\mathcal{J}$ satisfies $\mathcal{J}_i x = \mathcal{K}_j$ for some node set $\mathcal{K}_j$ of $\mathcal{K}$. From this it follows that $|\mathcal{J}_i| = |\mathcal{K}_j|$, and by noting that x is a bijection we can immediately deduce that the compositions λ and μ, determined by J and K respectively, consist of the same components, that is $\lambda \approx \mu$.

If p divides $[N_{S_n}(W_J) : W_J]$, then by Corollary 1, p divides the coefficient of $\mathcal{X}_J$ in the product $\mathcal{X}_J^2$. In terms of our recoded basis elements, by Lemma 14, this is equivalent to saying that p divides $t_1!t_2!\ldots t_n!$ where t_i is the multiplicity of i in the composition, λ, corresponding to J. From this it follows that λ contains a component of multiplicity p or more.

Since both these arguments are reversible, the result follows. □

Now we shall describe the irreducible representations of $\Sigma(S_n, p)$ in even more detail than in Chapter 3.

Let R be the matrix of basis elements of G_{S_n} described in Chapter 3. From this chapter we know that a set of columns of R reduced $\mod p$, R_p, may be chosen to represent the irreducible representations of $\Sigma(S_n, p)$. By Corollary 1 we know that the row rank of R_p is the number of rows indexed by those irreducible characters χ_J for which $p \nmid a_{JJJ}$. For each such character, let λ be the composition of n that corresponds to J via (2.1), and let t_i be the multiplicity of i in λ. Lemma 14 then implies that $p \nmid t_1!t_2!\ldots t_n!$ and therefore that λ contains no component of multiplicity p or more. Thus, the row rank, r_p, of R_p, is the number of partitions with no part of multiplicity p or more. In addition, since r_p is also the column rank of R_p it follows that R_p has at least r_p distinct columns.

Now we need to find a maximal set of distinct columns. Let $K \subseteq S$ and let c_K be a Coxeter element such that K corresponds to a composition $\lambda = \{\lambda_i\}_i$ via (2.1). Then c_K has order $\operatorname{lcm}(\lambda_1, \lambda_2, \ldots)$. Moreover it follows that p does not divide the order of c_K if and only if all λ_i are co-prime to p. By Lemma 40.3 [CR62] we know that for any $x \in S_n$, we can write $x = x_1 x_2$ where x_1 commutes with x_2, the order of x_1 is not divisible by p, and the order of x_2 is a power of p. From Section 82, p587 [CR62], we know that for every character, χ, we have that $\chi(x) = \chi(x_1) \mod p$. Therefore in trying to find a maximal set of distinct columns of R_p, we need only be concerned with those columns headed by K where K corresponds to a p-regular composition via (2.1). However the column rank of R_p is equal to r_p, which by Lemma 13 is equal to the number of p-regular partitions of n. Therefore, it follows that the set of columns of R_p headed by K where K corresponds to a p-regular composition of n via (2.1) are all distinct, and moreover:

Theorem 10 *Let R be the matrix of characters given in Chapter 3, and R_p be this matrix whose entries have been reduced modulo p. Then the irreducible representations of $\Sigma(S_n, p)$ may be taken as those columns of R_p indexed by subsets of S that correspond to p-regular compositions of n via (2.1).*

Let T_m be the ideal of $\Sigma(S_n, p)$ spanned by all $\overline{B}_\mu$ where μ has m or more components. Then

$$\Sigma(S_n, p) = T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \supseteq T_{n+1} = 0$$

Lemma 15 $\operatorname{rad}(\Sigma(S_n, p))T_m \subseteq T_{m+1}$

PROOF Let η be a composition with at least m components (so that $\overline{B}_\eta \in T_m$) and consider the product $X\overline{B}_\eta$ for each of the spanning elements of $\operatorname{rad}(\Sigma(S_n, p))$ given in Theorem 9. Such a product is, by Lemma 14, a linear combination of terms $\overline{B}_\varepsilon$ with $\varepsilon \preceq \eta$ but, as we now prove, the term $\overline{B}_\eta$ itself occurs with coefficient zero. There are two cases to consider:

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. By Lemma 14, the coefficients of $\overline{B}_\eta$ in both $\overline{B}_\lambda \overline{B}_\eta$ and $\overline{B}_\mu \overline{B}_\eta$ are equal; thus, in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$, the coefficient of $\overline{B}_\eta$ is zero.

2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more. Again, by Lemma 14 since $t_1! \ldots t_n!$ is zero in $\mathcal{F}_p$, the coefficient of $\overline{B}_\eta$ in $\overline{B}_\lambda \overline{B}_\eta$ is zero.

It now follows that $XT_m \subseteq T_{m+1}$ for all $X \in \operatorname{rad}(\Sigma(S_n, p))$ and this completes the proof. □

Let $\mathcal{T}$ denote the subspace of $\Sigma(S_n, p)$ generated by all $\overline{B}_\mu - \overline{B}_\lambda$ with $\lambda \approx \mu$. Since $\mathcal{T}$ is the image of $\mathrm{rad}(\Sigma_{S_n})$ under the homomorphism ρ_1, $\mathcal{T}$ is an ideal.

Lemma 16 1. *If n is odd or $p \neq 2$ then* $\mathrm{rad}(\Sigma(S_n, p)) \subseteq T_2 \cap \mathcal{T} + T_3$.

2. *If n is even then* $\mathrm{rad}(\Sigma(S_n, 2)) \subseteq \langle \overline{B}_{[n/2,n/2]} \rangle + T_2 \cap \mathcal{T} + T_3$.

PROOF Consider the spanning set for $\mathrm{rad}(\Sigma(S_n, p))$ given in Theorem 9. An element $\overline{B}_\lambda - \overline{B}_\mu$ with $\lambda \approx \mu$ is non-zero only if λ and μ have at least 2 components and so such an element belongs to $T_2 \cap \mathcal{T}$.

Consider an element $\overline{B}_\lambda$ where the composition λ has a component which occurs p times or more. If n is odd or $p \neq 2$ then λ will have at least 3 components and so $\overline{B}_\lambda \in T_3$. The composition λ can have fewer than 3 components only if $p = 2$ and $\lambda = [n/2, n/2]$. The lemma now follows. ☐

Lemma 17 *If n is even then*

$$\mathrm{rad}(\Sigma(S_n, 2))^2 \subseteq T_3 \cap \mathcal{T} + T_4$$

PROOF By Lemma 15 and Lemma 16

$$\mathrm{rad}(\Sigma(S_n, 2))^2 \subseteq \mathrm{rad}(\Sigma(S_n, 2))\langle \overline{B}_{[n/2,n/2]} \rangle + T_3 \cap \mathcal{T} + T_4$$

and so it is sufficient to prove that all products $X\overline{B}_{[n/2,n/2]}$ lie in $T_3 \cap \mathcal{T} + T_4$ where X runs through the spanning set of $\mathrm{rad}(\Sigma(S_n, p))$ given in Theorem 9.

If $X = \overline{B}_\lambda - \overline{B}_\mu$ then, as $\overline{B}_{[n/2,n/2]} \in T_2$ and $\mathcal{T}$ is a two-sided ideal, $X\overline{B}_{[n/2,n/2]} \in T_3 \cap \mathcal{T}$. Suppose that $X = \overline{B}_\lambda$ where $\lambda = [\lambda_1, \ldots, \lambda_r]$ has a repeated component. Then $X\overline{B}_{[n/2,n/2]}$ is a sum of elements $\overline{B}_\eta$, one for each $2 \times r$ matrix $\boldsymbol{z}$ in $Z(\lambda, [n/2, n/2])$. If such a matrix $\boldsymbol{z}$ has 4 or more non-zero entries then it contributes a summand $\overline{B}_\eta \in T_4$ to the product. If it has 3 non-zero entries then its two rows will not be equal and it may be paired with the matrix $\bar{\boldsymbol{z}}$ obtained from $\boldsymbol{z}$ by interchanging the rows. This pair of matrices contributes a summand $\overline{B}_\gamma + \overline{B}_\delta$ with $\gamma \approx \delta$ which lies in $T_3 \cap \mathcal{T}$. Finally, if $\boldsymbol{z}$ has 2 non-zero entries only it will have one of two possible forms,

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

each of which contributes a summand $\overline{B}_{[n/2,n/2]}$; since $p = 2$ this contribution is zero. ☐

We can now give the nilpotency index of $\mathrm{rad}(\Sigma(S_n, p))$.

Theorem 11 *If $n \geq 3$ the nilpotency index of* $\mathrm{rad}(\Sigma(S_n, p))$ *is* $n - 1$.

PROOF In the proof of Corollary 3.5 of [Atk92] it was proved that, if $w = B_{[1,n-1]} - B_{[n-1,1]}$ and $D(a,b) = B_{[1^a,n-a-b,1^b]}$ then

$$w^r = \sum_{k=0}^{r} (-1)^k \binom{r}{k} D(r-k,k)$$

In particular, $w^{n-2} \notin p\mathcal{Z}_{S_n}$ so that $x = \rho_1(w)$ is an element of $\mathrm{rad}(\Sigma(S_n,p))$ and $x^{n-2} \neq 0$. Therefore the nilpotency index of $\mathrm{rad}(\Sigma(S_n,p))$ is not less than $n-1$.

To prove that the nilpotency index is no more than $n-1$ we consider two cases. First, suppose that either n is odd or $p \neq 2$. Then Lemma 15 and Lemma 16 show that

$$\begin{aligned} \mathrm{rad}(\Sigma(S_n,p))^{n-1} &\subseteq \mathrm{rad}(\Sigma(S_n,p))^{n-2}(T_2 \cap \mathcal{T}) + \mathrm{rad}(\Sigma(S_n,p))^{n-2}T_3 \\ &\subseteq T_n \cap \mathcal{T} + T_{n+1} \end{aligned}$$

On the other hand, if n is even and $p = 2$, Lemma 15 and Lemma 17 show that

$$\begin{aligned} \mathrm{rad}(\Sigma(S_n,p))^{n-1} &= \mathrm{rad}(\Sigma(S_n,p))^{n-3}\mathrm{rad}(\Sigma(S_n,p))^2 \\ &\subseteq \mathrm{rad}(\Sigma(S_n,p))^{n-3}(T_3 \cap \mathcal{T}) + \mathrm{rad}(\Sigma(S_n,p))^{n-3}T_4 \\ &\subseteq T_n \cap \mathcal{T} + T_{n+1} \end{aligned}$$

However, since $T_{n+1} = 0$ and $T_n \cap \mathcal{T} = 0$, the result now follows. ☐

Remark By direct calculation we see that $\mathrm{rad}(\Sigma(S_1,p)) = 0$ and that $\mathrm{rad}(\Sigma((S_2,p)) = \langle \overline{B}_{[1,1]} \rangle$ (so has nilpotency index 2).

4.2 The p-modular descent algebra of the hyperoctahedral groups

In a way similar to the descent algebra of the symmetric groups, we can derive properties of the multiplicative action of Σ_{B_n} using its matrix interpretation, and two binary relations. Recall that for Σ_{B_n} the matrix interpretation for the basis $\{B_\lambda | \lambda \vDash m \leq n\}$ is as follows.

Consider "templates" with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0r} \\ & y_{11} & y_{12} & \cdots & y_{1r} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{s1} & y_{s2} & \cdots & y_{sr} \\ z_{s0} & z_{s1} & z_{s2} & \cdots & z_{sr} \end{pmatrix} \tag{4.1}$$

where

1. $z_{00} = n - N$, where N is the sum of all other entries in the template,

2. All entries in a template are non-negative integers,

3. The y-lines do not have entries in column 0.

Let (4.1) establish generic names for template values.

The reading word of a template is

$$[z_{01}, z_{02}, \ldots, z_{0r}, y_{1r}, \ldots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \ldots, z_{1r}, \ldots, z_{s0}, z_{s1}, z_{s2}, \ldots, z_{sr}]$$

with zero entries omitted.

If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$ is a composition of $k \leq n$, and $\mu = [\mu_1, \mu_2, \ldots, \mu_s]$ is a composition of $m \leq n$, then we define $Z(\lambda, \mu)$ to be the set of templates, $\boldsymbol{t}$, above, such that

1. $z_{0j} + \sum_{i=1}^{s}(y_{ij} + z_{ij}) = \lambda_j$, for each $j = 1, \ldots, r$.

2. $z_{i0} + \sum_{j=1}^{r}(y_{ij} + z_{ij}) = \mu_i$, for each $i = 1, \ldots, s$.

Then

$$B_\lambda B_\mu = \sum_\eta a_{\lambda\mu\eta} B_\eta$$

where η is the reading word of some template in $Z(\lambda, \mu)$, and $a_{\lambda\mu\eta}$ is the number of templates in $Z(\lambda, \mu)$ whose reading word is η.

We also say that if $\lambda \vDash k \leq n$ and $\mu \vDash l$ where $k \leq l \leq n$, then $\mu \preceq \lambda$ if λ can be obtained from μ by deleting components of μ to give some μ_1 and then replacing adjacent components of μ_1 by their sum.

Definition 10 *Two templates with $r + 1$ columns are said to be* column equivalent *if one can be obtained from the other by permuting the columns 1 to r, and* column-row equivalent *if one can be obtained from the other by permuting the columns 1 to r, and exchanging pairs of template entries (z_{ij}, y_{ij}).*

Lemma 18 *Let B_λ, B_μ be basis elements of Σ_{B_n}, and suppose that the composition λ has r components, and the number of components in it equal to i is denoted by t_i. Then*

1. *If the coefficient of B_η in $B_\lambda B_\mu$ is non-zero, $\eta \preccurlyeq \mu$.*

2. *The coefficient of B_μ in $B_\lambda B_\mu$ is a multiple of $2^r t_1! t_2! \ldots t_n!$, and is dependent only on the column equivalence class of λ.*

3. *If $\lambda \approx \mu$, the coefficient of B_μ in $B_\lambda B_\mu$ is precisely $2^r t_1!t_2!\dots t_n!$*

PROOF This proof proceeds in an identical manner to the proof of Lemma 14, except that we are now working with templates instead of matrices.

Let $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$ and $\mu = [\mu_1, \mu_2, \dots, \mu_s]$. If B_η occurs in $B_\lambda B_\mu$ with non-zero multiplicity, then there exists a template whose reading word gives the composition η. However, since $z_{i0} + \sum_{j=1}^{r}(y_{ij} + z_{ij}) = \mu_i$, it follows that $\eta \preccurlyeq \mu$.

Now consider the template that contributes to the coefficient of B_μ in $B_\lambda B_\mu$. It satisfies

$$\begin{aligned} \mu_i &= z_{i0} + \sum_{j=1}^{r}(y_{ij} + z_{ij}) \\ \lambda_j &= z_{0j} + \sum_{i=1}^{s}(y_{ij} + z_{ij}) \end{aligned}$$

It follows that $z_{0j} = 0$ for all j, and that amongst the entries $z_{i0}, y_{i1}, z_{i1}, \dots, y_{ir}, z_{ir}$, there will be only one non-zero entry equal to μ_i. The set of templates $\mathcal{Q}$ (if any) that satisfy this fall into a number of column-row equivalence classes. Each of these has precisely $2^r t_1!t_2!\dots t_n!$ members since the set of columns $1, \dots, r$ of a given template in $\mathcal{Q}$ with a common column sum may be permuted arbitrarily. In addition, a non-zero entry equal to μ_i appearing in column j may appear in position z_{ij} or y_{ij} (note that the columns are distinct, so the $t_i!$ permutations of a set of t_i columns give distinct templates). If η is some composition equivalent to λ then the set of matrices analogous to $\mathcal{Q}$ is related to $\mathcal{Q}$ by permuting the columns $1, \dots, r$. This proves the second part.

When $\lambda \approx \mu$ we see that $\mathcal{Q}$ consists of exactly one column-row equivalence class, since each template will have exactly one non-zero entry in each column $1, \dots, r$, and $z_{i0} = 0$ for all i. $\square$

Again we can conclude that a natural basis of $\Sigma(B_n, p)$ can be denoted by $\{\overline{B}_\lambda | \lambda \vDash m \leq n\}$, and this basis is related in the natural way to the basis elements $\{B_\lambda\}_{\lambda \vDash m \leq n}$ of Σ_{B_n}. Thus, we can deduce that the coefficient $\overline{a}_{\lambda\mu\eta}$ of $\overline{B}_\eta$ in the product $\overline{B}_\lambda \overline{B}_\mu$ is $a_{\lambda\mu\eta} \mod p$, where $a_{\lambda\mu\eta}$ is the coefficient of B_η in the product $B_\lambda B_\mu$ in Σ_{B_n}. We note in particular that this implies that Lemma 18 holds for $\overline{B}_\lambda \overline{B}_\mu$ in $\Sigma(B_n, p)$, except that all coefficients are now reduced modulo p.

Theorem 12 *When $p \neq 2$, $\mathrm{rad}(\Sigma(B_n, p))$ is spanned by all $\overline{B}_\lambda - \overline{B}_\mu$ with $\lambda \approx \mu$ together with all $\overline{B}_\lambda$ where λ has a component of multiplicity p or more. However, when $p = 2$, it is spanned by all $\overline{B}_\lambda$ with $\lambda \neq [\,]$.*

PROOF We can use Theorem 8 in this proof in the same way that we exploited it to prove Theorem 9.

Let J and K be subsets of S. If $J \sim K$, then it follows that for some $x \in B_n$ we have that $\mathcal{J}x = \mathcal{K}$. Each node set, $\mathcal{J}_i$ of $\mathcal{J}$ satisfies $\mathcal{J}_i x = \mathcal{K}_j$ for some node set $\mathcal{K}_j$ of $\mathcal{K}$. Since x is a bijection, it follows that the compositions λ and μ, that correspond respectively to J and K by (2.2), have the same components; that is, $\lambda \approx \mu$.

Secondly, if p divides $[N_{B_n}(W_J) : W_J]$, then by Corollary 1, and Lemma 18, it follows that in terms of recoded basis elements, the composition λ, where λ corresponds to J, must have a component of multiplicity p or more if $p \neq 2$, and $\lambda \neq [\,]$ if $p = 2$.

Since these arguments are reversible we are almost done. To complete the proof, we note that for $p = 2$ the set $\{B_\lambda | \lambda \neq [\,]\}$ is in fact sufficient to span $\mathrm{rad}(\Sigma(B_n, 2))$. □

Let us now describe the set of irreducible representations of $\Sigma(B_n, p)$ even more explicitly than in Chapter 3. Let R be the matrix of irreducible characters of Σ_{B_n} as given in Chapter 3, then by Chapter 3 we know that a suitable set of columns of R reduced $\mod p$, R_p, may be taken to represent the irreducible representations of $\Sigma(B_n, p)$.

By Corollary 1 we know that a row in R_p will be non-zero if and only if it is indexed by an irreducible character χ_J such that $p \nmid a_{JJJ}$. For each such character, let λ be the composition of $m \leq n$ with k components that corresponds to J via (2.2), and let t_i be the multiplicity of i in λ. Lemma 18 then implies that $p \nmid 2^k t_1! t_2! \ldots t_n!$, and therefore that for $p \neq 2$, λ contains no part of multiplicity p or more. Hence, when $p \neq 2$, the row rank of R_p is r_p, where r_p is the number of partitions of $m \leq n$ containing no part of multiplicity p or more. Since the row rank and column rank of R_p are equal, when $p \neq 2$ it follows that R_p contains at least r_p distinct columns.

Definition 11 *A* special element *(in B_n) is any Coxeter element c_K such that $K \subseteq S$ corresponds via (2.2) to a composition λ of $m \leq n$, where λ is a p-regular composition of $m \leq n$.*

We will now show that the set of columns of R_p, headed by subsets $K \subseteq S$, such that c_K is a special element, is a full set of irreducible representations of $\Sigma(B_n, p)$ for $p \neq 2$. We shall do this by proving that for every Coxeter element, c, in B_n, there exists a special element, s, such that $\chi(c) = \chi(s) \mod p$ for all characters χ. Once this is established, then it follows that a maximal set of distinct columns of R_p is contained in the set of columns headed by subsets $K \subseteq S$, such that c_K is a special element. However, the number of such columns is equal to the number of p-regular partitions of $m \leq n$, which by Lemma 13 is equal to r_p. Hence we can deduce that all columns headed by a subset $K \subseteq S$, such that c_K is a special element are distinct, and our result follows for $p \neq 2$.

Let $c_K = c_0 c_1$ be a Coxeter element where c_0 is a product of all $s_i \in S$ such that for all j and k switched by s_i, we have that $j, k \in \mathcal{K}_0$. If c_1 has order $p^d e$, where $p \nmid e$, then let $c' = c_0 c_2$, where $c_2 = c_1^{p^d}$. We observe that $c', c_2 \in B_n$, and moreover, p does not divide the order of c_2. From Lemma 40.3 [CR62] we know that for any $x \in B_n$, we can write $x = x_1 x_2$ where x_1 commutes with x_2, the order of x_1 is not divisible by p, and the order of x_2 is a power of p. By construction c_K and c' have the same part whose order is not divisible by p, and so by Section 82 [CR62] it follows that $\chi(c_K) = \chi(c') \mod p$ for all characters χ. Finally, we can always conjugate c' by an element $\alpha \in B_n$, where $\alpha = \prod_{j \neq 0} \alpha_j$ and $\alpha_j \in \mathcal{W}_{\mathcal{K}_j}$, to get a special element, $s = c_0 c_3$, and we are done.

If $p = 2$, then we can immediately deduce that the only non-zero row in R_p will be that indexed by the irreducible character χ_J, where J corresponds to [ ] by (2.2).

We can now deduce the following theorem.

Theorem 13 *Let R be the matrix of characters given in Chapter 3, and R_p be this matrix whose entries have been reduced modulo p. Then if $p \neq 2$, the irreducible representations of $\Sigma(B_n, p)$ may be taken as those columns of R_p indexed by subsets of S that correspond to p-regular compositions of $m \leq n$ via (2.2). If $p = 2$, then the irreducible representation of $\Sigma(B_n, 2)$ may be taken as the column of R_p indexed by the subset of S that corresponds to $\lambda = [\,]$ via (2.2).*

Using Lemma 18, we can also determine the nilpotency index of $\mathrm{rad}(\Sigma(B_n, p))$. Taking the same approach as in the previous section, we shall let T_m be the ideal of $\Sigma(B_n, p)$ spanned by all $\overline{B}_\lambda$ where λ has m or more components. Then it follows

$$\Sigma(B_n, p) = T_0 \supseteq T_1 \supseteq \ldots \supseteq T_n \supseteq T_{n+1} = 0$$

Let $V(i, j)$ be the subspace of $\Sigma(B_n, p)$ generated by all $\overline{B}_\lambda$ where λ has at least $i + 2j$ components, at least j of which are greater than 1.

Lemma 19 *1. If $p = 2$ then $\mathrm{rad}(\Sigma(B_n, 2))T_m \subseteq T_{m+1}$.*

2. If $p \neq 2$ then $\mathrm{rad}(\Sigma(B_n, p))T_m \subseteq T_{m+3} + V(m, 1)$.

PROOF Let η be a composition with m components, so that $\overline{B}_\eta \in T_m$. Consider the product $X\overline{B}_\eta$ for each of the spanning elements of $\mathrm{rad}(\Sigma(B_n, p))$ given in Theorem 12. By Lemma 18, we have that $X\overline{B}_\eta$ is a linear combination of basis elements $\overline{B}_\varepsilon$ with $\varepsilon \preceq \eta$ but the term $\overline{B}_\eta$ itself occurs with coefficient zero, since X can only take one of two forms:

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. By Lemma 18, the coefficients of $\overline{B}_\eta$ in both $\overline{B}_\lambda\overline{B}_\eta$ and $\overline{B}_\mu\overline{B}_\eta$ are equal; thus, in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$, the coefficient of $\overline{B}_\eta$ is zero.

2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more (unless $p = 2$), and r components. By Lemma 18 since $2^r t_1! \ldots t_n!$ is zero in $\mathcal{F}_p$, then irrespective of p, the coefficient of $\overline{B}_\eta$ in $\overline{B}_\eta\overline{B}_\mu$ is zero.

This proves the first part.

To prove the second part, we must do some further analysis.

Consider again $X\overline{B}_\eta$. We have already shown that this is a linear combination of basis elements $\overline{B}_\varepsilon$, with $\varepsilon \preccurlyeq \eta$, such that $\overline{B}_\eta$ occurs with coefficient zero. We shall now show that all such $\overline{B}_\varepsilon$ where $\overline{B}_\varepsilon$ is a contributor to the product $X\overline{B}_\eta$ with exactly $m+1$ components also has coefficient zero.

We note what sort of template would contribute to the coefficient of $\overline{B}_\varepsilon$. Without loss of generality there are two to consider:

TEMPLATE 1: $z_{0j} = 0, z_{i0} = 0$ for all i, j, and there is one non zero entry in each pair of rows contributing to a given η_i, except one pair, which has $y_{i'j'} = \eta_{i'_1}$ and $z_{i'j'} = \eta_{i'_2}$ such that $\eta_{i'_1} + \eta_{i'_2} = \eta_{i'}$.

TEMPLATE 2: $z_{0j} = 0$ for all $j \neq j'$ where $z_{0j'} = \eta_{i'}$ and $z_{i0} = 0$ for all $i \neq i'$ where $z_{i'0} = \eta_{i'}$, and there is one non-zero entry in every other pair of rows contributing to a given η_i.

For each of these types of template we have two cases to consider:

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$.
2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more.

Before we embark on considering each of our four cases, we note that each row of a template of either type, contains at most one non-zero entry. Hence for both template types we can use the following two arguments.

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. It follows that the coefficient of $\overline{B}_\varepsilon$ in $\overline{B}_\lambda\overline{B}_\eta$ will be the same as the coefficient of some $\overline{B}_\delta$ in $\overline{B}_\mu\overline{B}_\eta$, where the templates that yield δ as a reading word are column equivalent to those that yield ε. Since each row in a template consists of at most one non-zero entry, it follows that $\varepsilon = \delta$, and so the coefficient of $\overline{B}_\varepsilon$ in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$ is zero.
2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more. A template, T, that yields ε as a reading word lies in a column equivalence class, $\mathcal{Q}$ containing $t_1!t_2!\ldots t_n!$ templates. Since each row of T contains at most one non-zero entry, it follows that the reading word of each template in $\mathcal{Q}$ is ε, and so in $\mathcal{F}_p$, the coefficient of $\overline{B}_\varepsilon$ in $\overline{B}_\lambda\overline{B}_\eta$ is zero.

Hence we have that $\mathrm{rad}(\Sigma(B_n, p))T_m \subseteq T_{m+2}$.

To finish this proof, we must show that if there exists a non-trivial contributor $\overline{B}_\varepsilon$, with $m+2$ components, contributing to the product $X\overline{B}_\eta$, then $\overline{B}_\varepsilon \in V(m,1)$. This is easily done by considering the three templates that could contribute to the coefficient of $\overline{B}_\varepsilon$, which, without loss of generality, can be described as follows:

1. TEMPLATE A: The template resembles TEMPLATE 1, except that there are now two such pairs of rows such that $y_{i'j'} = \eta_{i'_1}$ and $z_{i'j'} = \eta_{i'_2}$ satisfy $\eta_{i'_1} + \eta_{i'_2} = \eta_{i'}$.
2. TEMPLATE B: The template resembles TEMPLATE 2, except that there are now two non-zero entries z_{0j} and $z_{0j'}$ in row 0, and two non-zero entries $z_{i0} = z_{0j}$ and $z_{i'0} = z_{0j'}$ in column 0.
3. TEMPLATE C: $z_{i0} = 0$ for all i, and each pair of rows contributing to a given η_i have one non-zero entry, except two pairs of rows, which can be visualised as follows:

$$\begin{array}{ccccccccc} \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & z_{ij} & 0 & \dots & 0 & z_{ij'} & 0 & \dots \\ \\ \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & z_{i'j} & 0 & \dots & 0 & z_{i'j'} & 0 & \dots \end{array}$$

where $z_{ij}, z_{i'j}, z_{ij'}, z_{i'j'} \neq 0$, $z_{i'j} = z_{ij'}$, $z_{ij} + z_{ij'} = \eta_i$, $z_{i'j} + z_{i'j'} = \eta_{i'}$.

Since X can only take one of two forms, let us check the six cases that arise.

TEMPLATE A

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. Since there is still at most one non-zero entry in each row in each template contributing towards the coefficient of $\overline{B}_\varepsilon$ in $\overline{B}_\lambda \overline{B}_\eta$, the column equivalent templates to these in the product $\overline{B}_\mu \overline{B}_\eta$ also yield ε as a reading word, so the coefficient of $\overline{B}_\varepsilon$ in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$ is zero.
2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more. A template, T, that yields ε as a reading word lies in a column equivalence class with $t_1!t_2!\dots t_n!$ members. Since each row contains at most one non-zero entry, each template in the class also yields ε as a reading word, and so in $\mathcal{F}_p$, the coefficient of $\overline{B}_\varepsilon$ in $\overline{B}_\lambda \overline{B}_\eta$ is zero.

TEMPLATE B

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. If $z_{0j} = z_{0j'}$, then it follows that a template T, contributing to the product $\overline{B}_\lambda \overline{B}_\eta$ and its column equivalent counterpart, T', contributing to the product $\overline{B}_\mu \overline{B}_\eta$ have the same reading word, ε, since all rows $1, \dots$ have at most one non-zero entry. Hence the coefficient of $\overline{B}_\varepsilon$ in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$ will be zero. However, if $z_{0j} \neq z_{0j'}$, then the respective reading words, ε and δ, of T and T' may be distinct, but $z_{0j} + z_{0j'} \geq 3$, and so $\overline{B}_\varepsilon, \overline{B}_\delta \in V(m, 1)$.

2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more. Again we have that the coefficient of $\overline{B}_\varepsilon$ in $\overline{B}_\lambda \overline{B}_\eta$ may not be zero if the column equivalence class of some template with reading word ε contains a template that does not yield ε. In this case, since all but row 0 contains at most one non-zero entry, it follows that we must have $z_{0j} \neq z_{0j'}$ and so we have that $\overline{B}_\varepsilon \in V(m, 1)$.

TEMPLATE C

1. $X = \overline{B}_\lambda - \overline{B}_\mu, \lambda \approx \mu$. If T is a template of type C, with reading word ε, contributing to the product $\overline{B}_\lambda \overline{B}_\eta$, and T' is its column equivalent counterpart contributing to $\overline{B}_\mu \overline{B}_\eta$, then their associated basis elements may not cancel if $z_{ij} \neq z_{ij'}$. However in this case we can achieve cancellation if instead we consider T'', where T'' is identical to T' except that template entries z_{ik} and $z_{ik'}$ that prevent cancellation have been swapped with y_{ik} and $y_{ik'}$. Observing that this cancellation can always happen, it follows that the coefficient of $\overline{B}_\varepsilon$ in $(\overline{B}_\lambda - \overline{B}_\mu)\overline{B}_\eta$ is zero.

2. $X = \overline{B}_\lambda$ where λ has t_i components equal to i with at least one t_i being p or more. Again the only problem that may arise is if $z_{ij} \neq z_{ij'}$ in a template, T, contributing to the coefficient of $\overline{B}_\eta$. However, if there is a template, T', in the column equivalence class of T that does not yield ε, we can instead count T'' related to T' as above. Thus, the number of templates contributing to the coefficient of $\overline{B}_\varepsilon$ will be a multiple of $t_1!t_2!\ldots t_n!$, and the coefficient of $\overline{B}_\eta$ in the product $\overline{B}_\lambda \overline{B}_\eta$ will be zero.

This completes the proof of the lemma. □

As in the previous section, we let $\mathcal{T}$ denote the subspace of $\mathrm{rad}(\Sigma(B_n, p))$ generated by all $\overline{B}_\lambda - \overline{B}_\mu$ with $\lambda \approx \mu$. Again $\mathcal{T}$ is the image of $\rho_1(\mathrm{rad}(\Sigma_{B_n}))$ and hence an ideal.

Lemma 20 *1. If $p = 2$ then $\mathrm{rad}(\Sigma(B_n, 2)) \subseteq T_1$.*

2. If $p \neq 2$ then $\mathrm{rad}(\Sigma(B_n, p)) \subseteq T_2 \cap \mathcal{T} + T_3$.

PROOF Clearly the first part of the lemma holds since $\overline{B}_{[\,]} \notin \mathrm{rad}(\Sigma(B_n, 2))$. For the second part consider the spanning set for $\mathrm{rad}(\Sigma(B_n, p))$ given in Theorem 12. We observe that $\overline{B}_\lambda - \overline{B}_\mu$ with $\lambda \approx \mu$ is non-zero only if λ and μ have at least 2 (distinct) components and that such an element belongs to $T_2 \cap \mathcal{T}$ by definition.

However, an element $\overline{B}_\lambda$ where the composition λ has a component which occurs p or more times will have at least 3 components, and hence belong to T_3. The lemma now follows. □

With these lemmas, we can now prove the following theorem.

Theorem 14 *1. If $p = 2$, the nilpotency index of* $\mathrm{rad}(\Sigma(B_n, 2))$ *is* $n + 1$.

2. If $p \neq 2$, the nilpotency index of $\mathrm{rad}(\Sigma(B_n, p))$ *is* $[\frac{n}{3}] + 1$.

PROOF The following simple induction shows that in $\Sigma(B_n, 2)$, $\overline{B}_{[1]}^{k} = \overline{B}_{[1^k]}$.

Our hypothesis trivially holds true for $k = 1$. Assuming that this hypothesis is true for a given k, we get that

$$\begin{aligned} \overline{B}_{[1]}^{k+1} &= \overline{B}_{[1]}\overline{B}_{[1]}^{k} \\ &= \overline{B}_{[1]}\overline{B}_{[1^k]} \end{aligned}$$

It follows that every template in $Z([1], [1^k])$ with a non-zero entry $z_{i1} = 1,\ i \neq 0$ can be paired with another in $Z([1], [1^k])$, which is identical except that $y_{i1} = 1$, and $z_{i1} = 0$. All templates in $Z([1], [1^k])$ can be paired in this way apart from the one in which $z_{01} = 1$, and $z_{i0} = 1$ for $i = 1, \ldots, k$. This will give us our only summand in the product $\overline{B}_{[1]}^{k+1}$, and its reading word is $[1^{k+1}]$.

Hence $\overline{B}_{[1]}^{n} \neq 0$, and so the nilpotency index of $\mathrm{rad}(\Sigma(B_n, 2))$ is not less than $n + 1$.

To prove it is no more than $n + 1$, we see that from Lemma 19 and Lemma 20

$$\mathrm{rad}(\Sigma(B_n, 2))^{n+1} \subseteq T_{n+1}$$

and since $T_{n+1} = 0$ the first part follows.

When $p \neq 2$, consider the strings 21 and 12, and the product $(21 - 12)^k$, where multiplication is given by concatenation of strings. Since concatenation of strings is non-commutative, it follows that $(21 - 12)^k$ when multiplied out will be a sum $\sum_{i=1}^{2^k} z_i(k)x_i(k)$, where $z_i(k) = \pm$, and $x_i(k)$ is a string containing k 1's and k 2's.

Example 7

$$\begin{aligned} (21 - 12)^3 &= 212121 + 211212 + 122112 + 121221 \\ &\quad -212112 - 211221 - 122121 - 121212. \end{aligned}$$

We shall now prove, by induction, that

$$(\overline{B}_{[2,1]} - \overline{B}_{[1,2]})^k = \sum_i z_i(k)\overline{B}_{[x_i(k)]}.$$

The base case, $k = 1$, follows by definition. Let us assume the hypothesis holds true for $k > 1$. Then

$$\begin{aligned} (\overline{B}_{[2,1]} - \overline{B}_{[1,2]})^{k+1} &= (\overline{B}_{[2,1]} - \overline{B}_{[1,2]})(\overline{B}_{[2,1]} - \overline{B}_{[1,2]})^k \\ &= \overline{B}_{[2,1]}\sum_i z_i(k)\overline{B}_{[x_i(k)]} - \overline{B}_{[1,2]}\sum_i z_i(k)\overline{B}_{[x_i(k)]} \end{aligned}$$

Note that a template T_1 that contributes to $\overline{B}_{[2,1]}\overline{B}_{[x_i(k)]}$ for some $x_i(k)$ can be put in one-to-one correspondence with a template T_2 contributing to the product $\overline{B}_{[1,2]}\overline{B}_{[x_i(k)]}$ by exchanging the first and second columns. Since most of the templates in $Z([2,1],[x_i(k)])$ contain rows, each of which has at most one non-zero entry, or a pair of equal entries, it follows that their counterpart in $Z([1,2],[x_i(k)])$ will give the same reading word, and so the summands they contribute will cancel. The only occasion when this will not happen is if $z_{01} = 2, z_{02} = 1$ in T_1, and $z_{01} = 1, z_{02} = 2$ in T_2. It follows that in this case all $z_{i0} \neq 0$, and that the reading word of T_1, T_2 is of length $2(k+1)$. Clearly all strings in the product $(21 - 12)^{k+1}$ will appear in this way alone, and appear only once as the reading word of a template in the set of all templates contributing to the product $(\overline{B}_{[2,1]} - \overline{B}_{[1,2]})(\overline{B}_{[2,1]} - \overline{B}_{[1,2]})^k$. We also observe that the coefficients will clearly be as stated. Our inductive proof is now done.

Hence $(\overline{B}_{[2,1]} - \overline{B}_{[1,2]})^{[\frac{n}{3}]} \neq 0$, and so the nilpotency index of $\mathrm{rad}(\Sigma(B_n, p))$ is no less than $[\frac{n}{3}] + 1$. Again by Lemma 19 and Lemma 20, and observing that $T_2 \cap \mathcal{T} \subseteq V(0,1)$, we have that

$$\begin{aligned}\mathrm{rad}(\Sigma(B_n,p))^{[\frac{n}{3}]+1} \quad \subseteq \quad & T_{3+3([\frac{n}{3}])} + T_{2+3([\frac{n}{3}])} \cap \mathcal{T} \\ & + V(3, \left[\frac{n}{3}\right]) + V(0, \left[\frac{n}{3}\right] + 1)\end{aligned}$$

and since each of the terms on the right hand side is equal to zero, the theorem is proved. $\square$

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