COMPACT SYMMETRIC SOLUTIONS TO THE POSTAGE STAMP PROBLEM

HUGH THOMAS and STEPHANIE VAN WILLIGENBURG

Department of Mathematics and Statistics
University of New Brunswick
Fredericton NB, E3B 5A3, Canada
e-mail: hugh@math.unb.ca

Department of Mathematics
University of British Columbia
Vancouver BC, V6T 1Z2, Canada
e-mail: steph@math.ubc.ca

Abstract

We derive lower and upper bounds on possible growth rates of certain sets of positive integers $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$ such that all integers $n \in \{0, 1, 2, \ldots, ka_k\}$ can be represented as a sum of no more than $k$ elements of $A_k$, with repetition.

1. Introduction

The postage stamp problem [2, C 12] is a classic problem in additive number theory and can be described as follows: if $h$ and $k$ are positive integers, $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$, $a_i \in \mathbb{N}$ and

$$S(h, A_k) = \left\{ \sum_{i=1}^{k} x_i a_i \mid x_i \geq 0, \sum_{i=1}^{k} x_i \leq h \right\}$$


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then what is the smallest positive integer $N(h, A_k) \in S(h, A_k)$? One focus is to solve the global aspect of this problem, that is, given $h$ and $k$ find $A_k$ such that $N(h, A_k)$ is as large as possible. The case $k = 3$ was solved by Hofmeister [3], and for $k \geq 4$ Rödseth [5] derived the currently best known general upper bound. Another focus is to solve the local aspect, that is, given $h, k$ and $A_k$ determine $N(h, A_k)$. The case $k = 3$ is covered in [6]. Both aspects were solved for the case $k = 2$ in [7].

It is easy to see that $N(h, A_k) \leq ha_k + 1$. In this paper, we focus on integrating the global and local aspects by investigating certain sets generated for which this inequality is actually an equality.

**Preliminaries 1.1.** From here on we restrict our attention to the situation $h = k$. We say a set $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$ is symmetric if when $k = 2m$, then

\[
\begin{align*}
a_1 &= 1 \\
a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\
a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\
a_{2m} &= 2a_m
\end{align*}
\]

and when $k = 2m + 1$, then

\[
\begin{align*}
a_1 &= 1 \\
a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\
\hat{a}_m &= a_m + x \text{ for } 0 < x \in \mathbb{N} \\
a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\
a_{2m} &= \hat{a}_m + a_m,
\end{align*}
\]

where the $2m + 1$ elements are ordered

\[a_1 < a_2 < \ldots < a_m < \hat{a}_m < a_{m+1} < \ldots < a_{2m}.\]

This labelling of the elements has been chosen to make the enclosed proofs more uniform.
The largest integer that can be represented as the sum of \( k \) positive integers chosen from \( A_k \), with repetitions allowed, is clearly \( ka_k \). If every positive integer \( n \), \( 0 \leq n \leq ka_k \), can be represented as the sum of at most \( k \) positive integers from \( A_k \), then we say that \( A_k \) is compact. We now study the growth rate of the \( a_i \) such that \( A_k \) is both symmetric and compact. More precisely, if \( A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\} \) is symmetric, then we derive bounds \( \alpha, \beta \) such that if \( \frac{a_i}{a_{i-1}} \leq \alpha \) for \( 2 \leq i \leq m \), then \( A_k \) will always be compact, whereas if \( \beta \leq \frac{a_i}{a_{i-1}} \) for \( 2 \leq i \leq m \), then \( A_k \) will never be compact. Symmetric compact sets were studied in [8] where the focus was on sets with a stronger symmetry property, known as nested symmetry.

For convenience, we refer to \( A_k \) as the base, refer to the \( a_i \) as base elements, and denote the largest base element by \( M \).

**2. A Lower Bound**

We now describe symmetric sets \( A_k \) that are compact. For the remainder of this section, let \( A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\} \) be a symmetric base such that

1. the \( a_i \) satisfy

\[
\begin{align*}
a_1 &= 1 \\
    a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\
a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\
a_{2m} &= 2a_m
\end{align*}
\]

or

2. the \( a_i \) satisfy

\[
\begin{align*}
a_1 &= 1 \\
    a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m
\end{align*}
\]
\[ \hat{a}_m = a_m + x \text{ for } 0 < x \leq 2a_m \]
\[ a_{m+i} = \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \]
\[ a_{2m} = \hat{a}_m + a_m. \]

The following theorem on \( A_h \) can be proved via [4, Korollar], however, we provide a direct proof, which begins with

**Lemma 2.1.** Let \( 1 \leq r \leq m - 1 \). If \( \left\lfloor \frac{n}{M} \right\rfloor \leq r \) and \( n - \left\lfloor \frac{n}{M} \right\rfloor M < a_{r+1} \), then \( n \) can be written as a sum of at most \( 2r \) base elements with repetition.

**Proof.** We proceed by induction on \( r \). When \( r = 1 \) observe that \( n - \left\lfloor \frac{n}{M} \right\rfloor M < a_2 \leq 3 \), so \( n = 0, 1, 2, M, M + 1, M + 2 \). That \( n \) can be written as a sum of two base elements is trivial for all cases bar \( n = M + 2 \). This case only arises if \( a_2 = 3 \), in which case we can write \( n = (M - 1) + a_2 \).

Now let \( i = \left\lfloor \frac{n}{M} \right\rfloor \), \( j = \left\lfloor \frac{n - iM}{a_r} \right\rfloor \). Since \( n - iM < a_{r+1} \leq 3a_r \), we know \( 0 \leq j \leq 2 \). If

(1) \( j \leq 1 \) let \( n' = n - ja_r - \min(i, 1) M \)

(2) \( j = 2, i = 0 \) let \( n' = n - 2a_r \)

(3) \( j = 2, i > 0 \) let \( n' = n - (M - a_r) - a_{r+1} \).

Note in each of these cases, respectively, \( n' \geq 0 \) since

(1) if \( j = 0 \), then \( n - iM \geq 0 \), whereas if \( j = 1 \), then \( n - iM \geq a_r \)

(2) if \( j = 2 \) and \( i = 0 \), then \( \left\lfloor \frac{n}{a_r} \right\rfloor = 2 \), so \( n \geq 2a_r \)

(3) if \( j = 2 \) and \( i > 0 \), then \( \left\lfloor \frac{n - iM}{a_r} \right\rfloor = 2 \), so \( n - iM \geq 2a_r \) and \( n - iM + a_r - a_{r+1} \geq 3a_r - a_{r+1} \geq 0 \).
Moreover, in each of these cases \( n' \) respectively satisfies

\[
(1) \quad i' = \left\lfloor \frac{n'}{M} \right\rfloor = 0 \text{ if } i = 0, \text{ or } i' = \left\lfloor \frac{n'}{M} \right\rfloor = i - 1 \leq r - 1 \text{ otherwise, and }
\]

\[
\left\lfloor \frac{n' - iM}{a_r} \right\rfloor = \left\lfloor \frac{n - ja_r}{a_r} \right\rfloor = 0 \text{ if } i = 0, \text{ or } \left\lfloor \frac{n' - iM}{a_r} \right\rfloor = \left\lfloor \frac{n' - (i-1)M}{a_r} \right\rfloor = 0 \text{ otherwise, so } n' - \left\lfloor \frac{n'}{M} \right\rfloor M < a_r
\]

\[
(2) \quad i' = \left\lfloor \frac{n'}{M} \right\rfloor = 0 < r - 1 \text{ since } i = 0 \text{ and } r \geq 2, \text{ and } \left\lfloor \frac{n' - i'M}{a_r} \right\rfloor = \left\lfloor \frac{n - 2a_r}{a_r} \right\rfloor = 0 \text{ since } j = 2, \text{ so } n' - \left\lfloor \frac{n'}{M} \right\rfloor M < a_r
\]

\[
(3) \quad i' = \left\lfloor \frac{n'}{M} \right\rfloor = \left\lfloor \frac{n + a_r - a_{r+1} - M}{a_r} \right\rfloor = i - 1 \leq r - 1 \text{ since } a_r - a_{r+1} < 0, \text{ and } \left\lfloor \frac{n' - i'M}{a_r} \right\rfloor = \left\lfloor \frac{n' - (i-1)M}{a_r} \right\rfloor = \left\lfloor \frac{n + a_r - a_{r+1} - iM}{a_r} \right\rfloor = 0 \text{ since } n - iM < a_{r+1}, \text{ so } n - a_{r+1} + a_r < a_r, \text{ and hence } n' - \left\lfloor \frac{n'}{M} \right\rfloor M < a_r.
\]

Thus in each case we can apply the induction hypothesis to \( n' \) and write \( n' \) as a sum of at most \((r-1)\) base elements with repetition. The result now follows for \( n \).

\[\square\]

**Theorem 2.2.** \( A_k \) is compact.

**Proof.** Case (1). Suppose first that \( n < 2ma_m \). Consider \( \left\lfloor \frac{n}{a_m} \right\rfloor \). If

\[
\left\lfloor \frac{n}{a_m} \right\rfloor \text{ is even, then } \left\lfloor \frac{n}{a_m} \right\rfloor = 2l \leq 2(m - 1) \text{ and so } \left\lfloor \frac{n}{2a_m} \right\rfloor = \left\lfloor \frac{n}{M} \right\rfloor \leq m - 1
\]

and \( n - \left\lfloor \frac{n}{M} \right\rfloor M = n - \left\lfloor \frac{n}{2a_m} \right\rfloor M < a_m \). Thus, by Lemma 2.1 \( n \) can be written as a sum of at most \( 2(m-1) \) base elements. If \( \left\lfloor \frac{n}{a_m} \right\rfloor \) is odd, then

\[
\left\lfloor \frac{n}{a_m} \right\rfloor = 2l + 1. \text{ Let } n' = n - a_m. \text{ Then } \left\lfloor \frac{n'}{a_m} \right\rfloor \text{ is even and by the above}
\]

\[
\left\lfloor \frac{n'}{a_m} \right\rfloor = 2l \leq 2(m - 1)
\]

and

\[
\left\lfloor \frac{n'}{M} \right\rfloor = \left\lfloor \frac{n - a_m}{M} \right\rfloor = \left\lfloor \frac{n - a_m}{2a_m} \right\rfloor = \left\lfloor \frac{n - a_m}{M} \right\rfloor \leq m - 1
\]

and

\[
n - \left\lfloor \frac{n'}{M} \right\rfloor M = n - \left\lfloor \frac{n'}{a_m} \right\rfloor M < a_m
\]

Thus, by Lemma 2.1 \( n \) can be written as a sum of at most \( 2(m-1) \) base elements.
argument we can write $n'$ as a sum of at most $2(m - 1)$ base elements, and hence in both situations $n$ can be written as a sum of at most $2m - 1$ base elements.

If $n = 2ma_m$, it is clear that $n$ can be written as a sum of $m$ base elements, so now suppose that $2ma_m < n \leq 4ma_m$. We have already shown that $4ma_m - n$ can be written as a sum of at most $2m - 1$ base elements; by the symmetry of $A_k$ it follows that $n$ can be written as a sum of at most $2m$ base elements.

Case (2). Suppose first that $n \leq \left( m + \frac{1}{2} \right)(\hat{a}_m + a_m)$. Write

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where $0 \leq r < a_m$ and $|\hat{q} - q|$ is as small as possible. Note that in fact we can always find $\hat{q}, q$ such that $|\hat{q} - q| \leq 2$ by the following. If $|\hat{q} - q| > 2$, then consider $[\hat{a}_m/a_m]$. If $[\hat{a}_m/a_m] = 1$ and $\hat{q} > q$, then we can rewrite $n$ as

$$n = (\hat{q} - 1)\hat{a}_m + (q + 1)a_m + (r + x)$$

if $x + r < a_m$, and

$$n = (\hat{q} - 1)\hat{a}_m + (q + 2)a_m + (r + x - a_m)$$

otherwise. Alternatively if $q > \hat{q}$, then we can rewrite $n$ as

$$n = (\hat{q} + 1)\hat{a}_m + (q - 2)a_m + (r - x + a_m)$$

if $r < x$, or

$$n = (\hat{q} + 1)\hat{a}_m + (q - 1)a_m + (r - x)$$

otherwise. Similarly, if $[\hat{a}_m/a_m] = 2$, then we can rewrite $n$ as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 2)a_m + (r \pm x \mp a_m)$$

or

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + (r \pm x \mp 2a_m).$$
Finally, if \( \frac{\hat{a}_m}{a_m} = 3 \), so \( \hat{a}_m = 3a_m \), then we can rewrite \( n \) as

\[
  n = (\hat{q} + 1)\hat{a}_m + (q + 3)a_m + r.
\]

Iterating this procedure we see that we can eventually arrive at coefficients for \( \hat{a}_m \) and \( a_m \) that differ by at most 2. Thus let

\[
  n = \hat{q}\hat{a}_m + qa_m + r
\]

where \( 0 \leq r < a_m \) and \( |\hat{q} - q| \leq 2 \).

Let

\[
  n' = \begin{cases} 
    n - |\hat{q} - q|\hat{a}_m - \min(q, 1)M & \text{if } \hat{q} > q \\
    n - |\hat{q} - q|a_m - \min(\hat{q}, 1)M & \text{if } q > \hat{q} \\
    n - \min(\hat{q}, 1)M & \text{if } \hat{q} = q.
  \end{cases}
\]

Then

\[
  \left| \frac{n'}{\hat{a}_m + a_m} \right| = \left| \frac{n'}{M} \right| \leq M - 1 \text{ and } n' = \left| \frac{n'}{M} \right| M < a_m.
\]

By Lemma 2.1 we can write \( n' \) as sum of at most \( 2(m - 1) \) base elements, and hence \( n \) can be written as a sum of at most \( 2m + 1 \) base elements.

If \( n > \left( m + \frac{1}{2} \right)(\hat{a}_m + a_m) \), we apply the symmetry of \( A_k \) as in Case (1), and we are done.

\[\square\]

3. An Upper Bound

We now derive upper bounds on the \( a_i \) by describing conditions on symmetric sets \( A_k \) that force \( A_k \) not to be compact. For the remainder of this section let \( A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\} \) be a symmetric base such that

1. the \( a_i \) satisfy

\[
  a_1 = 1 \\
  a_i \geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\
  a_{m+i} = 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\
  a_{2m} = 2a_m
\]
(2) the $a_i$ satisfy
\[
\begin{align*}
a_1 &= 1 \\
a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\
\hat{a}_m &= a_m + x \text{ for } x \geq 7a_m \\
a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\
a_{2m} &= \hat{a}_m + a_m.
\end{align*}
\]

**Theorem 3.1.** $A_k$ is not compact.

**Proof.** Case (1): Suppose that $A_k$ is compact. If $n < a_m$, $n$ must be written as a sum of base elements chosen from $\{a_1, \ldots, a_{m-1}\}$ using at most $2m$ summands. There are $\binom{3m-1}{m-1}$ such sums. Thus, $a_m \leq \binom{3m-1}{m-1}$.

Observe that
\[
\binom{3m}{m-1} \leq \binom{3m-1}{i} \text{ for } m-1 \leq i \leq 2m.
\]

Thus,
\[
\binom{3m-1}{m-1} < \sum_{i=1}^{2m} \binom{3m-1}{i} = \frac{2^{3m-3}}{m+2} \leq 2^{3m-3}
\]

provided $m \geq 2$. Thus, $a_m < 8^{m-1}$, contradicting our choice of $A_k$.

Case (2). Similarly, suppose $A_k$ is compact. If $n < a_m$, then $n$ must be written as a sum of base elements from $\{a_1, \ldots, a_{m-1}\}$, using at most $2m+1$ summands, so, by the same argument as before, $a_m \leq \binom{3m}{m-1}$.
Again, using the same argument, we find that

\[
\left( \frac{3m}{m-1} \right) < \frac{2^{3m}}{m+3} \leq 2^{3m-m},
\]

provided \( m \geq 5 \), so \( a_m < s^{m-1} \), contradicting our choice of \( A_k \).

\[ \square \]

**Remark 3.1.** Utilising the above ideas and Stirling’s formula it is possible to improve \( 8 \leq \frac{a_i}{a_{i-1}} \) to \( \frac{27}{4} \leq \frac{a_i}{a_{i-1}} \) for \( 2 \leq i \leq m \), however, we omit the lengthy calculations here.

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**References**

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