# COMPACT SYMMETRIC SOLUTIONS TO THE POSTAGE STAMP PROBLEM

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Abstract. We derive lower and upper bounds on possible growth rates of certain sets of positive integers $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$ such that all integers $n \in \{0, 1, 2, \ldots, ka_k\}$ can be represented as a sum of no more than $k$ elements of $A_k$, with repetition.

## 1. Introduction

The postage stamp problem [2, C 12] is a classic problem in additive number theory and can be described as follows: if $h$ and $k$ are positive integers, $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$, $a_i \in \mathbb{N}$ and

$$S(h, A_k) = \{\sum_{i=1}^{k} x_i a_i | x_i \geq 0, \sum_{i=1}^{k} x_i \leq h\}$$

then what is the smallest positive integer $N(h, A_k) \notin S(h, A_k)$? One focus is to solve the global aspect of this problem, that is, given $h$ and $k$ find $A_k$ such that $N(h, A_k)$ is as large as possible. The case $k = 3$ was solved by Hofmeister [3], and for $k \geq 4$ Rödseth [5] derived the currently best known general upper bound. Another focus is to solve the local aspect, that is, given $h, k$ and $A_k$ determine $N(h, A_k)$. The case $k = 3$ is covered in [6]. Both aspects were solved for the case $k = 2$ in [7].

It is easy to see that $N(h, A_k) \leq ha_k + 1$. In this paper, we focus on integrating the global and local aspects by investigating certain sets generated for which this inequality is actually an equality.

### 1.1. Preliminaries.

From here on we restrict our attention to the situation $h = k$. We say a set $A_k = \{1 = a_1 < a_2 < \ldots < a_k\}$ is *symmetric* if when $k = 2m$ then

$$\begin{aligned} a_1 &= 1 \\ a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

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and when $k = 2m+1$ then

$$\begin{aligned} a_1 &= 1 \\ a_i &> a_{i-1} \text{ for } 2 \le i \le m \\ \hat{a}_m &= a_m + x \text{ for } 0 < x \in \mathbb{N} \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \le i \le m-1 \\ a_{2m} &= \hat{a}_m + a_m, \end{aligned}$$

where the $2m+1$ elements are ordered

$$a_1 < a_2 < \ldots < a_m < \hat{a}_m < a_{m+1} < \ldots < a_{2m}.$$

This labelling of the elements has been chosen to make the enclosed proofs more uniform.

The largest integer that can be represented as the sum of $k$ positive integers chosen from $A_k$, with repetitions allowed, is clearly $ka_k$. If every positive integer $n$, $0 \le n \le ka_k$, can be represented as the sum of at most $k$ positive integers from $A_k$, then we say that $A_k$ is *compact*. We now study the growth rate of the $a_i$ such that $A_k$ is both symmetric and compact. More precisely, if $A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\}$ is symmetric then we derive bounds $\alpha, \beta$ such that if $\frac{a_i}{a_{i-1}} \le \alpha$ for $2 \le i \le m$ then $A_k$ will always be compact, whereas if $\beta \le \frac{a_i}{a_{i-1}}$ for $2 \le i \le m$ then $A_k$ will never be compact. Symmetric compact sets were studied in [8] where the focus was on sets with a stronger symmetry property, known as nested symmetry.

For convenience, we refer to $A_k$ as the *base*, refer to the $a_i$ as *base elements*, and denote the largest base element by $M$.

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## 2. A lower bound

We now describe symmetric sets $A_k$ that are compact. For the remainder of this section, let $A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\}$ be a symmetric base such that

(1) the $a_i$ satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\le 3a_{i-1} \text{ for } 2 \le i \le m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \le i \le m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

or

(2) the $a_i$ satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } 0 < x \leq 2a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

The following theorem on $A_k$ can be proved via [4, Korollar], however, we provide a direct proof, which begins with

**Lemma 2.1.** *Let $1 \leq r \leq m-1$. If $\lfloor \frac{n}{M} \rfloor \leq r$ and $n - \lfloor \frac{n}{M} \rfloor M < a_{r+1}$ then $n$ can be written as a sum of at most $2r$ base elements with repetition.*

*Proof.* We proceed by induction on $r$. When $r = 1$ observe that $n - \lfloor \frac{n}{M} \rfloor M < a_2 \leq 3$, so $n = 0, 1, 2, M, M+1, M+2$. That $n$ can be written as a sum of two base elements is trivial for all cases bar $n = M+2$. This case only arises if $a_2 = 3$, in which case we can write $n = (M-1) + a_2$.

Now let $i = \lfloor \frac{n}{M} \rfloor, j = \lfloor \frac{n-iM}{a_r} \rfloor$. Since $n - iM < a_{r+1} \leq 3a_r$ we know $0 \leq j \leq 2$. If

(1) $j \leq 1$ let $n' = n - ja_r - \min(i,1)M$
(2) $j = 2, i = 0$ let $n' = n - 2a_r$
(3) $j = 2, i > 0$ let $n' = n - (M - a_r) - a_{r+1}$.

Note in each of these cases, respectively, $n' \geq 0$ since

(1) if $j = 0$ then $n - iM \geq 0$, whereas if $j = 1$ then $n - iM \geq a_r$
(2) if $j = 2$ and $i = 0$, then$\lfloor \frac{n}{a_r} \rfloor = 2$, so $n \geq 2a_r$
(3) if $j = 2$ and $i > 0$, then $\lfloor \frac{n-iM}{a_r} \rfloor = 2$, so $n - iM \geq 2a_r$ and $n - iM + a_r - a_{r+1} \geq 3a_r - a_{r+1} \geq 0$.

Moreover, in each of these cases $n'$ respectively satisfies

(1) $i' = \lfloor \frac{n'}{M} \rfloor = 0$ if $i = 0$, or $i' = \lfloor \frac{n'}{M} \rfloor = i-1 \leq r-1$ otherwise, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-ja_r}{a_r} \rfloor = 0$ if $i = 0$, or $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n-ja_r-iM}{a_r} \rfloor = 0$ otherwise, so $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
(2) $i' = \lfloor \frac{n'}{M} \rfloor = 0 < r-1$ since $i = 0$ and $r \geq 2$, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-2a_r}{a_r} \rfloor = 0$ since $j = 2$, so $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
(3) $i' = \lfloor \frac{n'}{M} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-M}{M} \rfloor = i-1 \leq r-1$ since $a_r - a_{r+1} < 0$, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-iM}{a_r} \rfloor = 0$ since $n - iM < a_{r+1}$, so $n - iM - a_{r+1} + a_r < a_r$, and hence $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$.

Thus in each case we can apply the induction hypothesis to $n'$ and write $n'$ as a sum of at most $2(r-1)$ base elements with repetition. The result now follows for $n$. □

**Theorem 2.2.** *$A_k$ is compact.*

*Proof. Case (1):* Suppose first that $n < 2ma_m$. Consider $\lfloor \frac{n}{a_m} \rfloor$. If $\lfloor \frac{n}{a_m} \rfloor$ is even then $\lfloor \frac{n}{a_m} \rfloor = 2l \leq 2(m-1)$ and so $\lfloor \frac{n}{2a_m} \rfloor = \lfloor \frac{n}{M} \rfloor \leq m-1$ and $n - \lfloor \frac{n}{M} \rfloor M = n - \lfloor \frac{n}{2a_m} \rfloor M < a_m$. Thus, by Lemma 2.1 $n$ can be written as a sum of at most $2(m-1)$ base elements. If $\lfloor \frac{n}{a_m} \rfloor$ is odd then $\lfloor \frac{n}{a_m} \rfloor = 2l+1$. Let $n' = n - a_m$ then $\lfloor \frac{n'}{a_m} \rfloor$ is even and by the above argument we can write $n'$ as a sum of at most $2(m-1)$ base elements, and hence in both situations $n$ can be written as a sum of at most $2m-1$ base elements.

If $n = 2ma_m$, it is clear that $n$ can be written as a sum of $m$ base elements, so now suppose that $2ma_m < n \leq 4ma_m$. We have already shown that $4ma_m - n$ can be written as a sum of at most $2m-1$ base elements; by the symmetry of $A_k$ it follows that $n$ can be written as a sum of at most $2m$ base elements.

*Case (2):* Suppose first that $n \leq (m + \frac{1}{2})(\hat{a}_m + a_m)$. Write

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where $0 \leq r < a_m$ and $|\hat{q} - q|$ is as small as possible. Note that in fact we can always find $\hat{q}, q$ such that $|\hat{q} - q| \leq 2$ by the following. If $|\hat{q} - q| > 2$ then consider $\lfloor \hat{a}_m / a_m \rfloor$. If $\lfloor \hat{a}_m / a_m \rfloor = 1$ and $\hat{q} > q$ then we can rewrite $n$ as

$$n = (\hat{q} - 1)\hat{a}_m + (q+1)a_m + (r + x)$$

if $x + r < a_m$, and

$$n = (\hat{q} - 1)\hat{a}_m + (q+2)a_m + (r + x - a_m)$$

otherwise. Alternatively if $q > \hat{q}$ then we can rewrite $n$ as

$$n = (\hat{q} + 1)\hat{a}_m + (q-2)a_m + (r - x + a_m)$$

if $r < x$, or

$$n = (\hat{q} + 1)\hat{a}_m + (q-1)a_m + (r - x)$$

otherwise. Similarly, if $\lfloor \hat{a}_m / a_m \rfloor = 2$ then we can rewrite $n$ as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 2)a_m + (r \pm x \mp a_m)$$

or

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + (r \pm x \mp 2a_m).$$

Finally, if $\lfloor \hat{a}_m / a_m \rfloor = 3$, so $\hat{a}_m = 3a_m$, then we can rewrite $n$ as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + r.$$

Iterating this procedure we see that we can eventually arrive at coefficients for $\hat{a}_m$ and $a_m$ that differ by at most 2. Thus let

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where $0 \leq r < a_m$ and $|\hat{q} - q| \leq 2$.

Let

$$n' = \begin{cases} n - |\hat{q} - q|\hat{a}_m - \min(q,1)M & \text{if } \hat{q} > q \\ n - |\hat{q} - q|a_m - \min(\hat{q},1)M & \text{if } q > \hat{q} \\ n - \min(\hat{q},1)M & \text{if } \hat{q} = q. \end{cases}$$

Then $\lfloor \frac{n'}{\hat{a}_m+a_m} \rfloor = \lfloor \frac{n'}{M} \rfloor \leq m-1$ and $n' - \lfloor \frac{n'}{M} \rfloor M < a_m$. By Lemma 2.1 we can write $n'$ as a sum of at most $2(m-1)$ base elements, and hence $n$ can be written as a sum of at most $2m+1$ base elements.

If $n > (m+\frac{1}{2})(\hat{a}_m + a_m)$, we apply the symmetry of $A_k$ as in Case (1), and we are done. □

## 3. An upper bound

We now derive upper bounds on the $a_i$ by describing conditions on symmetric sets $A_k$ that force $A_k$ not to be compact. For the remainder of this section let $A_k = \{1 = a_1 < a_2 < \ldots < a_{2m}\}$ be a symmetric base such that

(1) the $a_i$ satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

or

(2) the $a_i$ satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } x \geq 7a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

**Theorem 3.1.** *$A_k$ is not compact.*

*Proof. Case (1):* Suppose that $A_k$ is compact. If $n < a_m$, $n$ must be written as a sum of base elements chosen from $\{a_1, \ldots, a_{m-1}\}$ using at most $2m$ summands. There are $\binom{3m-1}{m-1}$ such sums. Thus, $a_m \leq \binom{3m-1}{m-1}$.

Observe that

$$\binom{3m}{m-1} \leq \binom{3m-1}{i} \text{ for } m-1 \leq i \leq 2m.$$

Thus,

$$\binom{3m-1}{m-1} < \frac{\sum_i \binom{3m-1}{i}}{m+2} = \frac{2^{3m-1}}{m+2} \leq 2^{3m-3}$$

provided $m \geq 2$. Thus, $a_m < 8^{m-1}$, contradicting our choice of $A_k$.

*Case (2):* Similarly, suppose $A_k$ is compact. If $n < a_m$, then $n$ must be written as a sum of base elements from $\{a_1, \ldots, a_{m-1}\}$, using at most $2m+1$ summands, so, by the same argument as before, $a_m \leq \binom{3m}{m-1}$.

Again, using the same argument, we find that

$$\binom{3m}{m-1} < \frac{2^{3m}}{m+3} \leq 2^{3m-3},$$

provided $m \geq 5$, so $a_m < 8^{m-1}$, contradicting our choice of $A_k$. □

*Remark* 3.1. Utilising the above ideas and Stirling's formula it is possible to improve $8 \leq \frac{a_i}{a_{i-1}}$ to $\frac{27}{4} \leq \frac{a_i}{a_{i-1}}$ for $2 \leq i \leq m$, however, we omit the lengthy calculations here.

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