

COMPACT SYMMETRIC SOLUTIONS TO THE POSTAGE STAMP PROBLEM

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ABSTRACT. We derive lower and upper bounds on possible growth rates of certain sets of positive integers $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$ such that all integers $n \in \{0, 1, 2, \dots, ka_k\}$ can be represented as a sum of no more than k elements of A_k , with repetition.

1. INTRODUCTION

The postage stamp problem [2, C 12] is a classic problem in additive number theory and can be described as follows: if h and k are positive integers, $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$, $a_i \in \mathbb{N}$ and

$$S(h, A_k) = \left\{ \sum_{i=1}^k x_i a_i \mid x_i \geq 0, \sum_{i=1}^k x_i \leq h \right\}$$

then what is the smallest positive integer $N(h, A_k) \notin S(h, A_k)$? One focus is to solve the global aspect of this problem, that is, given h and k find A_k such that $N(h, A_k)$ is as large as possible. The case $k = 3$ was solved by Hofmeister [3], and for $k \geq 4$ Rödseth [5] derived the currently best known general upper bound. Another focus is to solve the local aspect, that is, given h, k and A_k determine $N(h, A_k)$. The case $k = 3$ is covered in [6]. Both aspects were solved for the case $k = 2$ in [7].

It is easy to see that $N(h, A_k) \leq ha_k + 1$. In this paper, we focus on integrating the global and local aspects by investigating certain sets generated for which this inequality is actually an equality.

1.1. Preliminaries. From here on we restrict our attention to the situation $h = k$. We say a set $A_k = \{1 = a_1 < a_2 < \dots < a_k\}$ is *symmetric* if when $k = 2m$ then

$$\begin{aligned} a_1 &= 1 \\ a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= 2a_m \end{aligned}$$

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and when $k = 2m + 1$ then

$$\begin{aligned} a_1 &= 1 \\ a_i &> a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } 0 < x \in \mathbb{N} \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\ a_{2m} &= \hat{a}_m + a_m, \end{aligned}$$

where the $2m + 1$ elements are ordered

$$a_1 < a_2 < \dots < a_m < \hat{a}_m < a_{m+1} < \dots < a_{2m}.$$

This labelling of the elements has been chosen to make the enclosed proofs more uniform.

The largest integer that can be represented as the sum of k positive integers chosen from A_k , with repetitions allowed, is clearly ka_k . If every positive integer n , $0 \leq n \leq ka_k$, can be represented as the sum of at most k positive integers from A_k , then we say that A_k is *compact*. We now study the growth rate of the a_i such that A_k is both symmetric and compact. More precisely, if $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$ is symmetric then we derive bounds α, β such that if $\frac{a_i}{a_{i-1}} \leq \alpha$ for $2 \leq i \leq m$ then A_k will always be compact, whereas if $\beta \leq \frac{a_i}{a_{i-1}}$ for $2 \leq i \leq m$ then A_k will never be compact. Symmetric compact sets were studied in [8] where the focus was on sets with a stronger symmetry property, known as nested symmetry.

For convenience, we refer to A_k as the *base*, refer to the a_i as *base elements*, and denote the largest base element by M .

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2. A LOWER BOUND

We now describe symmetric sets A_k that are compact. For the remainder of this section, let $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$ be a symmetric base such that

(1) the a_i satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\ a_{2m} &= 2a_m \end{aligned}$$

or

(2) the a_i satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\leq 3a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } 0 < x \leq 2a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m-1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

The following theorem on A_k can be proved via [4, Korollar], however, we provide a direct proof, which begins with

Lemma 2.1. *Let $1 \leq r \leq m-1$. If $\lfloor \frac{n}{M} \rfloor \leq r$ and $n - \lfloor \frac{n}{M} \rfloor M < a_{r+1}$ then n can be written as a sum of at most $2r$ base elements with repetition.*

Proof. We proceed by induction on r . When $r = 1$ observe that $n - \lfloor \frac{n}{M} \rfloor M < a_2 \leq 3$, so $n = 0, 1, 2, M, M+1, M+2$. That n can be written as a sum of two base elements is trivial for all cases bar $n = M+2$. This case only arises if $a_2 = 3$, in which case we can write $n = (M-1) + a_2$.

Now let $i = \lfloor \frac{n}{M} \rfloor, j = \lfloor \frac{n-iM}{a_r} \rfloor$. Since $n - iM < a_{r+1} \leq 3a_r$ we know $0 \leq j \leq 2$. If

- (1) $j \leq 1$ let $n' = n - ja_r - \min(i, 1)M$
- (2) $j = 2, i = 0$ let $n' = n - 2a_r$
- (3) $j = 2, i > 0$ let $n' = n - (M - a_r) - a_{r+1}$.

Note in each of these cases, respectively, $n' \geq 0$ since

- (1) if $j = 0$ then $n - iM \geq 0$, whereas if $j = 1$ then $n - iM \geq a_r$
- (2) if $j = 2$ and $i = 0$, then $\lfloor \frac{n}{a_r} \rfloor = 2$, so $n \geq 2a_r$
- (3) if $j = 2$ and $i > 0$, then $\lfloor \frac{n-iM}{a_r} \rfloor = 2$, so $n - iM \geq 2a_r$ and $n - iM + a_r - a_{r+1} \geq 3a_r - a_{r+1} \geq 0$.

Moreover, in each of these cases n' respectively satisfies

- (1) $i' = \lfloor \frac{n'}{M} \rfloor = 0$ if $i = 0$, or $i' = \lfloor \frac{n'}{M} \rfloor = i-1 \leq r-1$ otherwise, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-ja_r}{a_r} \rfloor = 0$ if $i = 0$, or $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n-ja_r-iM}{a_r} \rfloor = 0$ otherwise, so $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
- (2) $i' = \lfloor \frac{n'}{M} \rfloor = 0 < r-1$ since $i = 0$ and $r \geq 2$, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n-2a_r}{a_r} \rfloor = 0$ since $j = 2$, so $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$
- (3) $i' = \lfloor \frac{n'}{M} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-M}{M} \rfloor = i-1 \leq r-1$ since $a_r - a_{r+1} < 0$, and $\lfloor \frac{n'-i'M}{a_r} \rfloor = \lfloor \frac{n'-(i-1)M}{a_r} \rfloor = \lfloor \frac{n+a_r-a_{r+1}-iM}{a_r} \rfloor = 0$ since $n - iM < a_{r+1}$, so $n - iM - a_{r+1} + a_r < a_r$, and hence $n' - \lfloor \frac{n'}{M} \rfloor M < a_r$.

Thus in each case we can apply the induction hypothesis to n' and write n' as a sum of at most $2(r-1)$ base elements with repetition. The result now follows for n . \square

Theorem 2.2. *A_k is compact.*

Proof. Case (1): Suppose first that $n < 2ma_m$. Consider $\lfloor \frac{n}{a_m} \rfloor$. If $\lfloor \frac{n}{a_m} \rfloor$ is even then $\lfloor \frac{n}{a_m} \rfloor = 2l \leq 2(m-1)$ and so $\lfloor \frac{n}{2a_m} \rfloor = \lfloor \frac{n}{M} \rfloor \leq m-1$ and $n - \lfloor \frac{n}{M} \rfloor M = n - \lfloor \frac{n}{2a_m} \rfloor M < a_m$. Thus, by Lemma 2.1 n can be written as a sum of at most $2(m-1)$ base elements. If $\lfloor \frac{n}{a_m} \rfloor$ is odd then $\lfloor \frac{n}{a_m} \rfloor = 2l+1$. Let $n' = n - a_m$ then $\lfloor \frac{n'}{a_m} \rfloor$ is even and by the above argument we can write n' as a sum of at most $2(m-1)$ base elements, and hence in both situations n can be written as a sum of at most $2m-1$ base elements.

If $n = 2ma_m$, it is clear that n can be written as a sum of m base elements, so now suppose that $2ma_m < n \leq 4ma_m$. We have already shown that $4ma_m - n$ can be written as a sum of at most $2m-1$ base elements; by the symmetry of A_k it follows that n can be written as a sum of at most $2m$ base elements.

Case (2): Suppose first that $n \leq (m + \frac{1}{2})(\hat{a}_m + a_m)$. Write

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where $0 \leq r < a_m$ and $|\hat{q} - q|$ is as small as possible. Note that in fact we can always find \hat{q}, q such that $|\hat{q} - q| \leq 2$ by the following. If $|\hat{q} - q| > 2$ then consider $\lfloor \hat{a}_m/a_m \rfloor$. If $\lfloor \hat{a}_m/a_m \rfloor = 1$ and $\hat{q} > q$ then we can rewrite n as

$$n = (\hat{q} - 1)\hat{a}_m + (q + 1)a_m + (r + x)$$

if $x + r < a_m$, and

$$n = (\hat{q} - 1)\hat{a}_m + (q + 2)a_m + (r + x - a_m)$$

otherwise. Alternatively if $q > \hat{q}$ then we can rewrite n as

$$n = (\hat{q} + 1)\hat{a}_m + (q - 2)a_m + (r - x + a_m)$$

if $r < x$, or

$$n = (\hat{q} + 1)\hat{a}_m + (q - 1)a_m + (r - x)$$

otherwise. Similarly, if $\lfloor \hat{a}_m/a_m \rfloor = 2$ then we can rewrite n as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 2)a_m + (r \pm x \mp a_m)$$

or

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + (r \pm x \mp 2a_m).$$

Finally, if $\lfloor \hat{a}_m/a_m \rfloor = 3$, so $\hat{a}_m = 3a_m$, then we can rewrite n as

$$n = (\hat{q} \mp 1)\hat{a}_m + (q \pm 3)a_m + r.$$

Iterating this procedure we see that we can eventually arrive at coefficients for \hat{a}_m and a_m that differ by at most 2. Thus let

$$n = \hat{q}\hat{a}_m + qa_m + r$$

where $0 \leq r < a_m$ and $|\hat{q} - q| \leq 2$.

Let

$$n' = \begin{cases} n - |\hat{q} - q|\hat{a}_m - \min(q, 1)M & \text{if } \hat{q} > q \\ n - |\hat{q} - q|\hat{a}_m - \min(\hat{q}, 1)M & \text{if } q > \hat{q} \\ n - \min(\hat{q}, 1)M & \text{if } \hat{q} = q. \end{cases}$$

Then $\lfloor \frac{n'}{\hat{a}_m + a_m} \rfloor = \lfloor \frac{n'}{M} \rfloor \leq m - 1$ and $n' - \lfloor \frac{n'}{M} \rfloor M < a_m$. By Lemma 2.1 we can write n' as a sum of at most $2(m - 1)$ base elements, and hence n can be written as a sum of at most $2m + 1$ base elements.

If $n > (m + \frac{1}{2})(\hat{a}_m + a_m)$, we apply the symmetry of A_k as in Case (1), and we are done. \square

3. AN UPPER BOUND

We now derive upper bounds on the a_i by describing conditions on symmetric sets A_k that force A_k not to be compact. For the remainder of this section let $A_k = \{1 = a_1 < a_2 < \dots < a_{2m}\}$ be a symmetric base such that

(1) the a_i satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ a_{m+i} &= 2a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\ a_{2m} &= 2a_m \end{aligned}$$

or

(2) the a_i satisfy

$$\begin{aligned} a_1 &= 1 \\ a_i &\geq 8a_{i-1} \text{ for } 2 \leq i \leq m \\ \hat{a}_m &= a_m + x \text{ for } x \geq 7a_m \\ a_{m+i} &= \hat{a}_m + a_m - a_{m-i} \text{ for } 1 \leq i \leq m - 1 \\ a_{2m} &= \hat{a}_m + a_m. \end{aligned}$$

Theorem 3.1. A_k is not compact.

Proof. Case (1): Suppose that A_k is compact. If $n < a_m$, n must be written as a sum of base elements chosen from $\{a_1, \dots, a_{m-1}\}$ using at most $2m$ summands. There are $\binom{3m-1}{m-1}$ such sums. Thus, $a_m \leq \binom{3m-1}{m-1}$.

Observe that

$$\binom{3m}{m-1} \leq \binom{3m-1}{i} \text{ for } m-1 \leq i \leq 2m.$$

Thus,

$$\binom{3m-1}{m-1} < \frac{\sum_i \binom{3m-1}{i}}{m+2} = \frac{2^{3m-1}}{m+2} \leq 2^{3m-3}$$

provided $m \geq 2$. Thus, $a_m < 8^{m-1}$, contradicting our choice of A_k .

Case (2): Similarly, suppose A_k is compact. If $n < a_m$, then n must be written as a sum of base elements from $\{a_1, \dots, a_{m-1}\}$, using at most $2m + 1$ summands, so, by the same argument as before, $a_m \leq \binom{3m}{m-1}$.

Again, using the same argument, we find that

$$\binom{3m}{m-1} < \frac{2^{3m}}{m+3} \leq 2^{3m-3},$$

provided $m \geq 5$, so $a_m < 8^{m-1}$, contradicting our choice of A_k . \square

Remark 3.1. Utilising the above ideas and Stirling's formula it is possible to improve $8 \leq \frac{a_i}{a_{i-1}}$ to $\frac{27}{4} \leq \frac{a_i}{a_{i-1}}$ for $2 \leq i \leq m$, however, we omit the lengthy calculations here.

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