Shifted Quasi-Symmetric Functions and the Hopf algebra of peak functions ⋆

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Abstract

In his work on P-partitions, Stembridge defined the algebra of peak functions Π, which is both a subalgebra and a quotient of the algebra of quasi-symmetric functions. We show that Π is closed under coproduct, and therefore a Hopf algebra, and describe the kernel of the quotient map. Billey and Haiman, in their work on Schubert polynomials, also defined a new class of quasi-symmetric functions — shifted quasi-symmetric functions — and we show that Π is strictly contained in the linear span Ξ of shifted quasi-symmetric functions. We show that Ξ is a coalgebra, and compute the rank of its nth graded component.

Résumé

Dans ses travaux sur les P-partitions, Stembridge définit l'algèbre Π des fonctions de pics. Cette algèbre peut être vue comme une sous-algèbre ou un quotient de l'algèbre des fonctions quasi-symétriques. Nous montrons ici que Π est fermée sous le coproduit, et est donc une algèbre de Hopf. Nous décrivons aussi le noyau du quotient ci-dessus. D'autre part, dans leurs travaux sur les polynômes de Schubert, Billey et Haiman ont défini une nouvelle classe de fonctions quasi-symétriques: les fonctions quasi-symétrique décalé. Nous montrons que Π est strictement contenue dans l'espace linéaire Ξ des fonctions quasi-symétrique décalé. Puis nous montrons que Ξ est une coalgèbre et calculons les dimensions des composantes de degrée n.

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1 Introduction

Schur Q functions first arose in the study of projective representations of S_n [9]. Since then they have appeared in variety of contexts including the representations of Lie superalgebras [10] and cohomology classes dual to Schubert cycles in isotropic Grassmanians [5,8]. While studying the duality between skew Schur P and Q functions and their connection to the Schubert calculus of isotropic flag manifolds, we were led to their quasi-symmetric analogues, the *peak functions* of Stembridge [11]. We show that *the linear span of peak functions is a Hopf algebra* (Theorem 2.2). We also show that these peak functions are contained in the strictly larger set of *shifted quasi-symmetric functions* (Theorem 3.6) introduced by Billey and Haiman [2]. We remark that the quasi-symmetric functions here are not any apparent specialization of the quasi-symmetric q-analogues of Hivert [4].

From extensive calculations, we believe that the linear span Ξ of all shifted quasi-symmetric functions is a Hopf algebra, but at present we can only show that:

The linear span Ξ of all shifted quasi-symmetric functions is a graded coalgebra whose nth graded component has rank π_n, where π_n is given by the recurrence

$$\pi_n = \pi_{n-1} + \pi_{n-2} + \pi_{n-4},$$

with initial conditions $\pi_1 = 1$, $\pi_2 = 1$, $\pi_3 = 2$, $\pi_4 = 4$. [4]

We shall prove this result (Theorems 3.2 and 4.3) and in addition shall establish some other properties of these functions.

A composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ of a positive integer n is an ordered list of positive integers whose sum is n. We denote this by $\alpha \vDash n$. We call the integers α_i the *components* of α, and denote the number of components in α by $k(\alpha)$. There exists a natural one-to-one correspondence between compositions of n and subsets of $[n-1]$. If $A = \{a_1, a_2, \ldots, a_{k-1}\} \subset [n-1]$, where $a_1 < a_2 < \ldots < a_{k-1}$, then A corresponds to the composition, $\alpha = [a_1 - a_0, a_2 - a_1, \ldots, a_k - a_{k-1}]$, where $a_0 = 0$ and $a_k = n$. For ease of notation, we shall denote the set corresponding to a given composition α by $I(\alpha)$. For compositions α and β we say that α is a *refinement* of β if $I(\beta) \subset I(\alpha)$, and denote this by $\alpha \preccurlyeq \beta$.

For any composition $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ we denote by M_α the *monomial quasi-symmetric function* [3]

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}.$$

We define $M_0 = 1$, where 0 denotes the unique empty composition of 0. We denote

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[4] More recently, we have shown that Ξ is indeed a Hopf algebra [1].

by F_α the *fundamental quasi-symmetric function* [3]

$$F_\alpha = \sum_{\alpha \succcurlyeq \beta} M_\beta .$$

Definition 1.1 *For any subset $A \subset [n-1]$, let $A+1$ be the subset of $\{2, \ldots, n\}$ formed from A by adding 1 to each element of A. Let $\alpha \vDash n$. Then we define*

$$\theta_\alpha = \sum 2^{k(\beta)} M_\beta ,$$

the sum over all compositions β of n where $I(\alpha)$ is a subset of $I(\beta) \cup (I(\beta)+1)$. Thus θ_α includes those M_β occurring in F_α.

This is the natural extension of the definition of peak functions given in [11].

Example 1.2 *We shall often omit the brackets that surround the components of a composition.*

We have $I(21) = \{2\}$, and $\{I(21)+1 = \{2\}+1 = \{3\}$. Hence

$$\begin{aligned} \theta_{21} &= 4M_{21} + 4M_{12} + 8M_{111} \\ F_{21} &= M_{21} + M_{111}. \end{aligned}$$

Let Σ^n be the $\mathbb{Z}$-module of quasi-symmetric functions spanned by $\{M_\alpha\}_{\alpha \vDash n}$ and let $\Sigma = \oplus_{n \geq 0} \Sigma^n$ be the graded $\mathbb{Z}$-algebra of quasi-symmetric functions. This is a Hopf algebra [6] with coproduct given by

$$\Delta(M_\alpha) = \sum_{\alpha = \beta \cdot \gamma} M_\beta \otimes M_\gamma ,$$

where $\beta \cdot \gamma$ is the concatenation of compositions β and γ.

Example 1.3 $\Delta(M_{32}) = 1 \otimes M_{32} + M_3 \otimes M_2 + M_{32} \otimes 1$.

We compute the coproduct of the functions θ_α.

Lemma 1.4 *For any composition α we have that*

$$\Delta(\theta_\alpha) = 1 \otimes \theta_\alpha + \sum \theta_{\epsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)} \tag{1}$$

where the sum is over all ways of writing α as $\varepsilon \cdot (a+b) \cdot \zeta$, that is, the concatenation of compositions ε and ζ, and a component of α written as the sum of numbers $a > 0$, $b \geq 0$. Also $\phi(b \cdot \zeta) = [1 + \zeta_1, \zeta_2, \ldots]$ if $b = 1$ and $b \cdot \zeta$ otherwise.

We shall use this result to show that certain subsets of functions θ_α span coalgebras (Theorems 2.2 and 3.2).

Proof. Definition 1.1 is equivalent to

$$\theta_\alpha = \sum_{\beta^* \preccurlyeq \alpha} 2^{k(\beta)} M_\beta,$$

where β^* is the refinement of β obtained by replacing all components $\beta_i > 1$, for $i > 1$, by $[1, \beta_i - 1]$. Thus the LHS of equation (1) is equal to

$$\sum_{\substack{\beta^* \preccurlyeq \alpha \\ \beta = \gamma \cdot \delta}} 2^{k(\beta)} M_\gamma \otimes M_\delta = \sum_{(\gamma \cdot \delta)^* \preccurlyeq \alpha} 2^{k(\gamma)} M_\gamma \otimes 2^{k(\delta)} M_\delta. \tag{2}$$

Let $2^{k(\gamma)} M_\gamma \otimes 2^{k(\delta)} M_\delta$ be a term of this sum, with $\gamma \vDash m$. If $m = 0$, then the term is $1 \otimes 2^{k(\delta)} M_\delta$, where $\delta^* \preccurlyeq \alpha$, and it appears in the summand $1 \otimes \theta_\alpha$ on the RHS of equation (1). If $m > 0$, then the term can only appear in one summand on the RHS of equation (1), namely $\theta_{\varepsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)}$ with $\varepsilon \cdot a \vDash$. To show that it does indeed appear, we need to prove that $\gamma^* \preccurlyeq \varepsilon \cdot a$ and $\delta^* \preccurlyeq \phi(b \cdot \zeta)$. Let δ^{**} be the refinement of δ^* obtained by replacing the component δ_1 by $[1, \delta_1 - 1]$, if $\delta_1 > 1$. We have that

$$\gamma^* \cdot \delta^{**} = (\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta,$$

which implies that $\gamma^* \preccurlyeq \varepsilon \cdot a$, and $\delta^{**} \preccurlyeq b \cdot \zeta \preccurlyeq \phi(b \cdot \zeta)$.

If $\delta \vDash 0$ or $\delta_1 = 1$, then $\delta^* = \delta^{**} \preccurlyeq \phi(b \cdot \zeta)$. However, if $\delta_1 > 1$, then there are two possible cases: either $\delta_1 \leq b$, or $b = 1$ and $\delta_1 - 1 \leq \zeta_1$. In the former case $\delta^* \preccurlyeq b \cdot \zeta = \phi(b \cdot \zeta)$, while in the latter, $\delta_1 \leq 1 + \zeta_1$, whence $\delta^* \preccurlyeq [1 + \zeta_1, \zeta_2, \ldots] = \phi(b \cdot \zeta)$.

Conversely, it is easy to see that all terms belonging to the tensor $1 \otimes \theta_\alpha$ on the RHS of equation (1) also appear in equation (2). Now let $2^{k(\gamma)} M_\gamma \otimes 2^{k(\delta)} M_\delta$ be a term belonging to a tensor $\theta_{\varepsilon \cdot a} \otimes \theta_{\phi(b \cdot \zeta)}$ on the RHS of equation (1). To show that it appears in equation (2) we must prove that $(\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta$. We have that $\gamma^* \preccurlyeq \varepsilon \cdot a$ and $\delta^* \preccurlyeq \phi(b \cdot \zeta)$, which imply that

$$(\gamma \cdot \delta)^* = \gamma^* \cdot \delta^{**} \preccurlyeq \gamma^* \cdot \delta^* \preccurlyeq \varepsilon \cdot a \cdot \phi(b \cdot \zeta).$$

If $b = 0$ or $b > 1$, then

$$(\gamma \cdot \delta)^* \preccurlyeq \varepsilon \cdot a \cdot \phi(b \cdot \zeta) = \varepsilon \cdot a \cdot b \cdot \zeta \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta.$$

If $b = 1$, then $\delta^* \preccurlyeq \phi(b \cdot \zeta) = [1 + \zeta_1, \zeta_2, \ldots]$ implies that

$$\delta^{**} = [1, \ldots] \preccurlyeq [1, \zeta_1, \ldots] = b \cdot \zeta.$$

Therefore,

$$(\gamma \cdot \delta)^* = \gamma^* \cdot \delta^{**} \preccurlyeq \varepsilon \cdot a \cdot b \cdot \zeta \preccurlyeq \varepsilon \cdot (a + b) \cdot \zeta$$

as desired.

Finally, we note that no term $2^{k(\gamma)}M_\gamma \otimes 2^{k(\delta)}M_\delta$ appears more than once in equation (2), or more than once in the expansion of the RHS of equation (1). The former is clear, while the latter follows from the fact that if $\theta_{\varepsilon\cdot a} \otimes \theta_{\phi(b\cdot\zeta)}$ and $\theta_{\varepsilon'\cdot a'} \otimes \theta_{\phi(b'\cdot\zeta')}$ are distinct summands on the RHS of equation (1), then $\varepsilon \cdot a \vDash k$ and $\varepsilon' \cdot a' \vDash l$ where $k \neq l$. □

2 The peak Hopf algebra

Definition 2.1 *For any composition* $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ *we say that* θ_α *is a* peak function *if* $\alpha_i = 1 \Rightarrow i = k$.

Observe that if θ_α is a peak function and $\alpha \vDash n$, then $I(\alpha)$ is a subset of $\{2, \ldots, n-1\}$ with no two consecutive elements.

Let Π^n be the $\mathbb{Z}$-module spanned by all peak functions θ_α, $\alpha \vDash n$, and let $\Pi = \oplus_{n\geq 0}\Pi^n$. This was studied by Stembridge [11] who showed that the peak functions are F-positive, are closed under product, and form a basis for Π, and so the rank of Π^n is the nth Fibonacci number. In addition we also know the following about the *algebra of peaks*, Π.

Theorem 2.2 Π *is closed under coproduct.*

Proof. If all components of a composition α, except perhaps the last, are greater than 1, then the same is true for all compositions $\varepsilon \cdot a$ and $\phi(b \cdot \zeta)$ appearing in the RHS of equation (1). □

Let Θ be the $\mathbb{Z}$-linear map from Σ to Π defined by $\Theta(F_\alpha) = \theta_{\Lambda(\alpha)}$, where $\Lambda(\alpha)$ is the composition formed from $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ by adding together adjacent components $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+j}$ where $\alpha_{i+l} = 1$ for $l = 0, \ldots, j-1$, and either $\alpha_{i+j} \neq 1$, or $i + j = k$.

Example 2.3 *If* $\alpha = 31125111$ *then* $\Lambda(\alpha) = 3453$.

Stembridge [11] showed that $\Theta : \Sigma \to \Pi$ is a graded surjective ring homomorphism, and was an analogue of the retraction from the algebra of symmetric functions to Schur Q functions. It is clear from our proof above that this morphism is in fact a Hopf homomorphism. We can describe the kernel of Θ as follows.

Lemma 2.4 *The non-zero differences* $F_\alpha - F_{\Lambda(\alpha)}$ *form a basis of the kernel of* Θ.

Proof. Each difference $F_\alpha - F_{\Lambda(\alpha)}$ is in the kernel of Θ as $\Theta(F_\alpha - F_{\Lambda(\alpha)}) = 0$ since $\Lambda(\Lambda(\alpha)) = \Lambda(\alpha)$. In addition, the non-zero differences are linearly independent as they have different leading terms. Letting f_n denote the nth Fibonacci number, there are $2^{n-1} - f_n$ such differences, and since the dimension of the nth graded component of the kernel of Θ is

$$\dim \Sigma^n - \dim \Pi^n \;=\; 2^{n-1} - f_n,$$

our result follows. □

3 The coalgebra of shifted quasi-symmetric functions

Definition 3.1 *For any composition* $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k] \vDash n$ *we say that* θ_α *is a* shifted quasi-symmetric function *(sqs-function) if* $n \leq 1$ *or* $\alpha_1 > 1$.

Observe that if θ_α is an sqs-function and $\alpha \vDash n$, then $I(\alpha) \subset \{2, \ldots, n-1\}$.

For integers $n \geq 0$, let Ξ^n be the $\mathbb{Z}$-module spanned by all sqs-functions θ_α, $\alpha \vDash n$, and let $\Xi = \oplus_{n \geq 0} \Xi^n$.

Theorem 3.2 Ξ *is closed under coproduct.*

Proof. If α is a composition of 0 or 1, or has first component greater than 1, then the same is true for all compositions $\varepsilon \cdot a$ and $\phi(b \cdot \zeta)$ appearing in the RHS of equation (1). □

Unlike peak functions [11], sqs-functions are not F-positive since

$$\theta_{211} = F_{22} + F_{112} + 2F_{121} + F_{211} - F_{1111}.$$

Definition 3.3 *For any composition,* $\alpha \vDash n$, *we define the complement* α^c *of* α *to be the composition for which* $I(\alpha^c) = (I(\alpha))^c$, *the set complement of* $I(\alpha)$ *in* $[n-1]$. *We define the graph* $G(\alpha)$ *of* α *to be the graph obtained from*

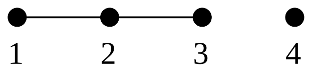
by removing the edge $(i, i+1)$ *if and only if* $i \in I(\alpha)$.

Observe that $G(\alpha^c)$ contains the edge $(i, i+1)$ if and only if this edge is not contained in $G(\alpha)$. These graphs will be used later in the proof of Theorem 3.6.

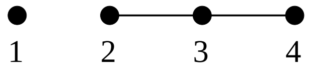
Let a *word* of length n be any n-tuple, $w_1 w_2 \ldots w_n$, and let a *binary word* of length n be a word $w_1 w_2 \ldots w_n$ such that $w_i \in \{0, 1\}$ for all i. For $2 \leq i \leq n-1$, let us denote by $3^{(i)}$ the composition $[1^{i-2}, 3, 1^{n-i-1}]$ of n. For any subset $S \subset \{2, \ldots, n-1\}$, let us denote by $\bigwedge_{i \in S} 3^{(i)}$ the composition of n for which $G(\bigwedge_{i \in S} 3^{(i)})$ has an edge between vertices i and $i+1$ if and only if an edge exists between vertices i and $i+1$ in $G(3^{(i)})$ for some $i \in S$.

Example 3.4 *Let* $S = \{2, 3\} \subset [3]$. *Then*

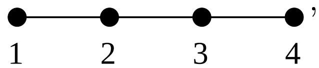
$G(3^{(2)})$ *is*



and $G(3^{(3)})$ *is*



hence $G(\bigwedge_{i \in S} 3^{(i)})$ *is*



so $\bigwedge_{i \in S} 3^{(i)}$ *is the composition* 4.

Definition 3.5 *[2] Let α be a composition of n. Let $\mathcal{A}(\alpha)$ denote the set of weakly increasing sequences $j_1 \leq j_2 \leq \ldots \leq j_n$ of positive integers with no three equal: if $j_i = j_{i+1}$, then $j_{i-1} < j_i$ and $j_{i+1} < j_{i+2}$. The* shifted quasi-symmetric function θ_α^{BH} *is given by*

$$\theta_\alpha^{BH} = \sum_{J=(j_1,\ldots,j_n) \in \mathcal{A}(\alpha)} 2^{\#J} x_{j_1} \ldots x_{j_n},$$

where $\#J$ denotes the number of distinct values j_i in J.

Theorem 3.6 *For any sqs-function θ_α we have that $\theta_\alpha = \theta_\alpha^{BH}$.*

Proof. For each $i \in I(\alpha) \subset [n-1]$, $j_{i-1} = j_i = j_{i+1}$ is forbidden in any monomial

$$x_{j_1} x_{j_2} \ldots x_{j_i} \ldots x_{j_n}$$

appearing as a summand of the function θ_α^{BH}. This is equivalent to saying that M_β is a summand of θ_α^{BH} if and only if $G(3^{(i)}) \not\subset G(\beta)$ for all $i \in I(\alpha)$. Therefore at least one of $i-1$ or i must be the largest label of a vertex in a connected component in $G(\beta)$.

Now when going from compositions of n to subsets of $[n-1]$ we can do so using our graphs, G. All we have to do is list the label of the vertex that is the largest in each connected component, not listing n. We call these vertices the *end-points*. We are now in a position to prove the equivalence of Definitions 1.1 and 3.5 for sqs-functions.

The powers of 2 agree so we need only show that the indices of summation do too. To see this, take any sqs-function θ_α and let $i \in I(\alpha)$. If M_β is a summand in θ_α^{BH} then at least one of $i-1$ or i is an end-point in $G(\beta)$. Therefore i or $i-1$ belongs to $I(\beta)$, and M_β is a summand of θ_α. Conversely, if M_β is a summand of θ_α, then this implies that for each $i \in I(\alpha)$, we have that $i-1$ or i belongs to $I(\beta)$, so one of $i-1$ or i is an end-point in $G(\beta)$, so M_β is a summand of θ_α^{BH}. □

4 A basis for Ξ

Definition 4.1 *Let θ_α be an sqs-function and $\alpha \models n$. We define an* internal peak *$i \in I(\alpha)$ such that $i-1, i+1 \notin I(\alpha)$, and $i \in \{3, \ldots n-2\}$.*

Remark Observe that the occurrence of an internal peak in the ith position in $I(\alpha) = \{w_1, w_2, \ldots\}$, where $w_1 < w_2 < \ldots$, is equivalent to having two components of α, say α_i, α_{i+1} such that $\alpha_{i+1} \geq 2$, and $\alpha_i \geq 2$ if $i \neq 1$, or $\alpha_i \geq 3$ if $i = 1$.

We can now describe the basis of Ξ as follows.

Theorem 4.2 *The coalgebra Ξ has a basis consisting of all sqs-functions θ_α where $I(\alpha)$ contains no internal peak.*

We sketch the proof of Theorem 4.2 later.

Theorem 4.3 *The rank of Ξ^n is given by the recurrence*

$$\pi_n = \pi_{n-1} + \pi_{n-2} + \pi_{n-4},$$

with initial conditions $\pi_1 = 1$, $\pi_2 = 1$, $\pi_3 = 2$, $\pi_4 = 4$.

This recurrence was suggested by a superseeker query [7].

Proof. By direct calculation we obtain that $\pi_1 = 1$, $\pi_2 = 1$, $\pi_3 = 2$, and $\pi_4 = 4$.

To obtain our recurrence, we observe that for each sqs-function, θ_α where $\alpha \vDash n$, we can encode $I(\alpha)$ as a binary word of length $n-2$, by placing a 1 in position $i-1$ if i is contained in $I(\alpha)$, and 0 otherwise. By this one-to-one correspondence we see that $I(\alpha)$ contains no internal peak if its corresponding binary word does not contain 010 as a subword.

We therefore count binary words of length n that avoid the subword 010. Appending either 1 or 0 to such a binary word of length $n-1$ gives one of length n, provided that we have not created the subword 010 in the last three positions. Let a_n, b_n, c_n, and d_n enumerate those binary words of length $n-2$ that avoid the subword 010 and end in, respectively 00, 01, 10, and 11. We then obtain the following 4 simultaneous recursions.

$$a_n = a_{n-1} + c_{n-1}, \quad b_n = a_{n-1} + c_{n-1}, \quad c_n = d_{n-1}, \quad d_n = b_{n-1} + d_{n-1}.$$

Clearly the number of $I(\alpha)$s in $[n-1]$ with no internal peaks is given by

$$\pi_n = a_n + b_n + c_n + d_n,$$

However by substituting in our recurrences we obtain

$$
\begin{aligned}
\pi_n &= a_n + b_n + c_n + d_n \\
&= 2a_{n-1} + b_{n-1} + 2c_{n-1} + 2d_{n-1} \\
&= \pi_{n-1} + a_{n-1} + c_{n-1} + d_{n-1} \\
&= \pi_{n-1} + a_{n-2} + b_{n-2} + c_{n-2} + 2d_{n-2} \\
&= \pi_{n-1} + \pi_{n-2} + d_{n-2} \\
&= \pi_{n-1} + \pi_{n-2} + b_{n-3} + d_{n-3} \\
&= \pi_{n-1} + \pi_{n-2} + a_{n-4} + b_{n-4} + c_{n-4} + d_{n-4} \\
&= \pi_{n-1} + \pi_{n-2} + \pi_{n-4}.
\end{aligned}
$$

□

We say that M_β is a maximal term of θ_α if for any γ higher in the partial order of compositions M_γ is not a summand of θ_α. The following lemma is stated without proof.

Lemma 4.4 *Let θ_α be an sqs-function. Consider the collection S of all possible sets derived from $I(\alpha)$ by adding either $i-1$ or $i+1$ to $I(\alpha)$ for all internal peaks $i \in I(\alpha)$. If M_β is a maximal term of θ_α, then β is derived from*

$$\bigwedge_{\substack{i \in (I(\tilde{\alpha}))^c \\ I(\tilde{\alpha}) \in S}} 3^{(i)}$$

by adding adjacent components equal to 1 together to give a component equal to 2 as often as possible.

Lemma 4.5 *Suppose θ_α is an sqs-function such that $I(\alpha)$ has an internal peak in the jth position. Then we have the following linear relation*

$$
\begin{aligned}
\theta_\alpha &= \theta_{[\alpha_1,\ldots,\alpha_j-1,1,\alpha_{j+1},\ldots,\alpha_k]} + \theta_{[\alpha_1,\ldots,\alpha_j,1,\alpha_{j+1}-1,\ldots,\alpha_k]} \\
&\quad -\theta_{[\alpha_1,\ldots,\alpha_j-1,1,1,\alpha_{j+1}-1,\ldots,\alpha_k]}.
\end{aligned}
$$

Proof. By Definition 3.5 we have that the leading terms of θ_α determine the other summands that belong to θ_α. Hence by Lemma 4.4 it follows that the summands of θ_α will be the union of the summands of $\theta_{[\alpha_1,\ldots,\alpha_j-1,1,\alpha_{j+1},\ldots,\alpha_k]}$ and $\theta_{[\alpha_1,\ldots,\alpha_j,1,\alpha_{j+1}-1,\ldots,\alpha_k]}$. However, those summands that appear in both will be duplicated. By definition these will be the summands of $\theta_{[\alpha_1,\ldots,\alpha_j-1,1,1,\alpha_{j+1}-1,\ldots,\alpha_k]}$, and the result follows. □

Sketch of proof of Theorem 4.2. From our relation in Lemma 4.5, it follows that any θ_α can be rewritten as a linear combination of functions $\theta_{\tilde{\alpha}}$, where $I(\tilde{\alpha})$ contains no internal peaks. In addition, by Lemma 4.4 and Definition 3.5, the set of all sqs-functions θ_α where $I(\alpha)$ contains no internal peaks is linearly independent and thus forms a basis for Ξ. □

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References

[1] N. Bergeron, S. Mykytiuk, F. Sottile, and S. van Willigenburg, *Pieri Operations on Posets*, J. of Comb. Theory, Series A, **91**, 84–110 (2000).

[2] S. Billey, and M. Haiman, *Schubert polynomials for the classical groups*, Journal of AMS, **8** (1995) 443–482.

[3] I. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Contemporary Mathematics, **34** (1984) 289–301.

[4] F. Hivert, *Hecke algebras, difference operators, and quasi-symmetric functions*, Adv. Math., 155 (2000), pp. 181–238.

[5] T. Józefiak, *Schur Q functions and cohomology of isotropic Grassmanians*, Math. Proc. Camb. Phil. Soc., **109** (1991) 471–478.

[6] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, Journal of Algebra, **177** (1995) 967–982.

[7] N.J.A. Sloane, *An on-line version of the Encyclopedia of integer sequences*, Elect. J. Combin., **1** No.1, (1994) F1.

`http://akpublic.research.att.com/~njas/sequences/ol.html`.

[8] P. Pragacz, *Algebro-geometric applications of Schur S- and Q- polynomials*, Topics in Invariant Thoery, Séminaire d'Algèbre Dubreil-Malliavin 1989-90, Springer-Verlag, (1991) 130–191.

[9] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. , **139** (1911) 155–250.

[10] A.N. Sergeev, *The tensor algebra of the identity representation as a module over the Lie superalgebras $gl(n,m)$ and $Q(n)$*, Math. USSR Sbornik, **51** (1985) 419–427.

[11] J. Stembridge, *Enriched P-partitions*, Trans. AMS, **349** (1997) 763–788.