

PROPERTIES OF THE DESCENT ALGEBRAS OF TYPE D

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ABSTRACT. We establish simple combinatorial descriptions of the radical and irreducible representations specifically for the descent algebra of a Coxeter group of type D over any field.

Key Words: Coxeter group, descent algebra, type D .

1. INTRODUCTION

In enumerative combinatorics, quasisymmetric functions arise naturally in a variety of of areas, for example, the chromatic symmetric function of a graph [8], the enumerator of partitions of a poset [13, 16], and the flag vector of a graded poset [9]. Moreover, the universality of the Hopf algebra of quasisymmetric functions \mathcal{Q} has recently been studied by Aguiar and Bergeron (personal communication). The dual of this algebra, formed from the descent algebras of the symmetric groups, also arises in a variety of contexts such as Hochschild homology [7], card shuffling and hyperplane arrangements [10], and planar binary trees [14], in addition to being isomorphic to the Hopf algebra of noncommutative symmetric functions [12]. Since a descent algebra exists for every Coxeter group [15] it is natural to study those descent algebras stemming from other Coxeter groups (*e.g.* [3, 5, 6]) in the hope that a deeper understanding of \mathcal{Q} or the descent algebras of the symmetric groups might be obtained.

In this paper we extend the study initiated in [6] to establish straightforward combinatorial descriptions of some algebraic properties of the descent algebras of (the Coxeter groups of) type D . Throughout we utilise notation first seen in [6] which allows us to state our results in a vein similar to that in [2, 11] for the symmetric groups case and [3, 4, 18] for the hyperoctahedral groups case. In these cases this notation not only allowed results to be stated clearly and concisely but also aided the realisation of the complete algebraic structure in each case. Definitions are given in the remainder of this section, and we state some results concerning our algebras over a field of characteristic zero in Section 2. We then generalise this to a field of finite characteristic in Section 3. Finally we discuss open problems that could be addressed next.

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1.1. The descent algebras of Coxeter groups. Let W be a Coxeter group with generating set S , let J be a subset of S , W_J be the subgroup generated by J , X_J (X_J^{-1}) be the unique set of minimal length left (right) coset representatives of W_J , and \mathcal{X}_J be the formal sum of the elements in X_J . Solomon proved [15, Theorem 1] for J, K, L being subsets of S

$$(1.1) \quad \mathcal{X}_J \mathcal{X}_K = \sum_L a_{JKL} \mathcal{X}_L$$

where a_{JKL} is the number of elements $x \in X_J^{-1} \cap X_K$ such that $x^{-1}Jx \cap K = L$. Moreover the \mathcal{X}_J form a basis for the *descent algebra of W* (over \mathbb{Q}), denoted Σ_W , and the radical of Σ_W is spanned by all differences $\mathcal{X}_J - \mathcal{X}_K$ where J and K are conjugate subsets of S [15, Theorem 3].

If J is a subset of S , let ϕ_J be the permutation character of W acting on the right cosets of W_J , and c_J be a Coxeter element of J , that is, a product of all the elements of J taken in some order. If we choose a set of representatives of the conjugacy classes of subsets of S we find (for example [1]) the columns of the matrix $R = [\phi_J(c_K)]$ with rows and columns indexed by this set can be taken as the irreducible representations of Σ_W .

We delay the discussion of descent algebras over a field of prime characteristic until Section 3.

1.2. The Coxeter groups of type D . From now on we consider D_n , the n -th Coxeter group of type D , which for our purposes will be the group acting on the set

$$\{-n, \dots, -1, 1, \dots, n\}$$

whose Coxeter generators are the set $S = \{s_{1'}, s_1, s_2, \dots, s_{n-1}\}$, where s_i is the product of transpositions $(-i-1, -i)(i, i+1)$ for $i = 1, 2, \dots, n-1$, and $s_{1'}$ is the product of transpositions $(-2, 1)(-1, 2)$. Observe from this definition that if $\sigma \in D_n$ then $\sigma(-i) = -\sigma(i)$ and the parity of σ is even, *i.e.*, the multiplicity of negative numbers in $\{\sigma(1), \dots, \sigma(n)\}$ is even.

2. THE DESCENT ALGEBRAS OF TYPE D

It transpires that for the Coxeter group D_n , (1.1) can be rewritten in a manner that is open to combinatorial manipulation [6, Theorem 1]. This in turn allows us to derive algebraic properties of the algebra in a manner similar to the symmetric or hyperoctahedral groups cases (see [11, 3] resp.). However, before we can do this we need to relabel our basis elements.

Recall a composition κ of a non-negative integer n is an ordered list $[\kappa_1, \kappa_2, \dots, \kappa_k]$ of positive integers whose sum is n , denoted by $\kappa \vDash n$. We call the integers $\kappa_1, \kappa_2, \dots, \kappa_k$ the *components* of κ . By convention $\kappa = []$ is the unique composition of 0. If κ is a non-increasing list, *e.g.*, $[3, 2, 1, 1]$, we call κ a *partition*, and the κ_i are called *parts*.

There exists a natural correspondence between the subsets of S and the multiset $\mathcal{C}(n)$, consisting of the union of the sets $\mathcal{C}_{<n} = \{\kappa | \kappa \vDash m, m \leq n-2\}$, $\mathcal{C}_1 = \{\kappa | \kappa \vDash n, \kappa_1 = 1\}$ $\mathcal{C}_n = \{\kappa | \kappa \vDash n, \kappa_1 \geq 2\}$ and $\mathcal{C}'_n = \{\kappa | \kappa \vDash n, \kappa_1 \geq 2\}$. Observe \mathcal{C}_n and \mathcal{C}'_n are two copies of the same set. The subset corresponding to $\kappa \in \mathcal{C}(n)$ is

- (1) $\{s_{\kappa_0}, s_{\kappa_0+\kappa_1}, \dots, s_{\kappa_0+\dots+\kappa_{(k-1)}}\}$ if $\kappa \in \mathcal{C}_{<n}$ where $\kappa_0 = n - m$,
- (2) $\{s_{1'}, s_1, s_{1+\kappa_2}, \dots, s_{1+\kappa_2+\dots+\kappa_{(k-1)}}\}$ if $\kappa \in \mathcal{C}_1$,
- (3) $\{s_{1'}, s_{\kappa_1}, \dots, s_{\kappa_1+\dots+\kappa_{(k-1)}}\}$ if $\kappa \in \mathcal{C}_n$,
- (4) $\{s_1, s_{\kappa_1}, \dots, s_{\kappa_1+\dots+\kappa_{(k-1)}}\}$ if $\kappa \in \mathcal{C}'_n$.

Our relabelling is then simply $B_\kappa := \mathcal{X}_{J^c}$ where J^c is the complement of J in S , and κ is the composition in $\mathcal{C}(n)$ corresponding to J by the above bijection.

Definition 2.1. *If $\kappa, \nu \in \mathcal{C}(n)$ then we say that $\kappa \approx \nu$ if the components of κ can be re-ordered to give to components of ν , but κ and ν do not satisfy either*

- (1) $\kappa \in \mathcal{C}_n, \nu \in \mathcal{C}'_n$, all components are even, or
- (2) $\kappa \in \mathcal{C}'_n, \nu \in \mathcal{C}_n$, all components are even.

Example 2.1. In $\mathcal{C}(6)$ let $[2, 1, 2, 1], [4, 2] \in \mathcal{C}_6$ and $[1, 2, 2, 1]^\vee, [2, 4]^\vee \in \mathcal{C}'_6$ where the \vee is simply to distinguish the compositions in \mathcal{C}_6 from those in \mathcal{C}'_6 . Then $[2, 1, 2, 1] \approx [1, 2, 2, 1]^\vee$ but $[4, 2] \not\approx [2, 4]^\vee$.

In fact this equivalence relation completely determines the conjugacy classes of subsets of S .

Lemma 2.1. *J and K are conjugate subsets in S if and only if $\kappa \approx \nu$ in $\mathcal{C}(n)$, where κ, ν corresponds to J^c, K^c respectively.*

Proof. Let $J, K \subseteq S$. If J and K are conjugate subsets then it follows that there exists $\sigma \in D_n$ such that for all $s \in J$ we have $s^\sigma = \hat{s}$ for some $\hat{s} \in K$. Since σ is a bijection it follows that if $s = (-b, -a)(a, b), t = (-c, -b)(b, c) \in J$ then if $s^\sigma = (-e, -d)(d, e) \in K$ then t^σ must be of the form $(-f, -e)(e, f) \in K$. Hence the compositions κ and ν that correspond to J^c and K^c via the natural correspondence have the same components.

However, if all the components of κ and ν are even, $\kappa \in \mathcal{C}_n$ and $\nu \in \mathcal{C}'_n$ then no such $\sigma \in D_n$ exists, for σ would be a bijection which maps some $\{s_i, s_{i+1}, \dots, s_{i+2k}\} \subseteq J$ where $s_{i-1}, s_{i+2k+1} \notin J$, onto $\{s'_1, s_2, \dots, s_{2k+1}\} \subseteq K$. From here, it is straightforward to deduce that for this to occur, the parity of σ must be odd, and so does not belong to D_n .

Similarly if $\kappa \in \mathcal{C}'_n$ and $\nu \in \mathcal{C}_n$ then σ would be a bijection which maps $\{s'_1, s_2, \dots, s_{2m+1}\} \subseteq J$ where $s_{2m+2} \notin J$, onto some $\{s_i, s_{i+1}, \dots, s_{i+2m}\} \subseteq K$. The parity of σ must again be odd.

Since this argument is reversible it follows that J and K are conjugate if and only if $\kappa \approx \nu$ and we are done. \square

We are now ready to reinterpret the form of the radical.

Theorem 2.1. *The radical of Σ_{D_n} is spanned by all $B_\kappa - B_\nu$ such that $\kappa \approx \nu$ where $\kappa, \nu \in \mathcal{C}(n)$.*

Proof. This follows immediately from the description of the radical for Σ_W [15, Theorem 3] and Lemma 2.1. \square

Observe how our choice of notation gives a result whose statement has a similar flavour to the analogous result in the symmetric groups case [11, Theorem 1.1] or the hyperoctahedral groups case [4, Corollary 2.13].

The columns of the matrix $R = [\phi_J(c_K)]$ can be taken as the irreducible representations of Σ_W , where the indexing set of rows and columns of R are pairwise non-conjugate subsets of S . Hence by Lemma 2.1 it follows that for the descent algebra Σ_{D_n} subsets whose complements correspond to partitions in $\mathcal{C}(n) \setminus \mathcal{C}'_n$ or partitions in \mathcal{C}'_n with all parts even form a suitable indexing set.

3. THE p -MODULAR DESCENT ALGEBRAS OF TYPE D

Since the structure constants of Σ_W are the integers a_{JKL} , it follows the \mathbb{Z} -module spanned by all \mathcal{X}_K forms a subring \mathcal{Z}_W of Σ_W , and for any prime p , all combinations of the \mathcal{X}_J that are integer multiples of p form an ideal, \mathcal{P}_W , of this subring. We define $\Sigma(W, p) := \mathcal{Z}_W / \mathcal{P}_W$ to be the p -modular descent algebra of W and $\Sigma(W, p)$ is our desired descent algebra over \mathbb{F}_p , the field of characteristic p .

In addition if \mathcal{X}_J is a basis element of Σ_W , and ρ is the natural homomorphism between \mathcal{Z}_W and $\Sigma(W, p)$, then the set of all $\overline{\mathcal{X}}_J := \rho(\mathcal{X}_J)$ forms a basis for $\Sigma(W, p)$. Observe that the multiplicative action in $\Sigma(W, p)$ is the same as that in \mathcal{Z}_W except we now reduce all coefficients modulo p . It was shown in [1, Theorem 3] that the radical of $\Sigma(W, p)$ is spanned by all $\overline{\mathcal{X}}_J - \overline{\mathcal{X}}_K$ where J, K are conjugate subsets of S , together with all $\overline{\mathcal{X}}_J$ for which p divides a_{JJJ} – the coefficient of \mathcal{X}_J in $\mathcal{X}_J \mathcal{X}_J \in \Sigma_W$. In addition it was shown a full set of distinct columns of the matrix $R = [\phi_J(c_K)]$ whose entries have been reduced modulo p can be taken as the irreducible representations of $\Sigma(W, p)$.

Following the conventions laid down so far in a natural way, let the basis of $\Sigma(D_n, p)$ be denoted by $\{\overline{B}_\kappa\}_{\kappa \in \mathcal{C}(n)}$. Then the radical and irreducible modules can be described using the following two notions.

We say a composition has a *component of multiplicity m* if m of its components are of equal value, e.g. $[1, 3, 1]$ has a component of multiplicity 2, and an *m -regular partition* of a positive integer n is a partition with no part divisible by m .

Theorem 3.1. *Let $\kappa, \nu \in \mathcal{C}(n)$. If $p \neq 2$, the radical of $\Sigma(D_n, p)$ is spanned by all $\overline{B}_\kappa - \overline{B}_\nu$ such that $\kappa \approx \nu$, together with all \overline{B}_κ where κ contains a component of multiplicity p or more. However if $p = 2$ and n is even the spanning set consists of*

all \overline{B}_κ , where $\kappa \neq []$; if $p = 2$ and n is odd the spanning set consists of all \overline{B}_κ , where $\kappa \notin \{[], [n], [n]^\vee\}$, and $\overline{B}_{[n]} - \overline{B}_{[n]^\vee}$.

Proof. This result is a straightforward consequence of [1, Theorem 3, Lemma 10] and Lemma 2.1. \square

Again observe the similarity of Theorem 3.1 to [2, Theorem 2] and [18, Theorem 2].

A full set of distinct columns of the matrix $R = [\phi_J(c_K)]$ whose entries are reduced modulo p can be taken as the irreducible representations of $\Sigma(W, p)$, and in fact it was shown in [1] that a suitable set consisted of all columns indexed by a subset J such that p does not divide a_{JJ} . Therefore considering the descent algebra $\Sigma(D_n, p)$, by [1, Lemma 10], if $p \neq 2$ then subsets whose complements correspond to p -regular partitions in $\mathcal{C}(n) \setminus \mathcal{C}'_n$ or p -regular partitions in \mathcal{C}'_n with all parts even form a suitable indexing set. If $p = 2$ and n is even then the subset whose complement corresponds to $[]$ is a suitable indexing set, whereas if $p = 2$ and n is odd, then subsets whose complements correspond to $[]$ or $[n]$ suffice.

4. CONCLUSION

With these results, the next avenue of research would be to establish further properties for Σ_{D_n} or $\Sigma(D_n, p)$, for example a set of minimal idempotents, or the Cartan matrix. For this to be achieved the action of Σ_{D_n} on “ D_n -Lie monomials” would have to be found, as was necessary in the symmetric and hyperoctahedral groups situations [11, 4]. However there has yet to be success in finding such monomials.

We suspect, by observation of the symmetric and hyperoctahedral groups cases, that the action may be derived from multiplication in Σ_{D_n} . More precisely let $A = \{a_1, a_2, \dots, a_M\}$ be an alphabet of M distinct letters. The *free Lie algebra* $L(A)$ is the algebra generated by the letters of A and the bracket operation

$$[P, Q] = PQ - QP.$$

Now consider homogeneous elements $P_i^{(\kappa_i)}$ of degree κ_i in $L(A)$. We call the concatenation product of $P_1^{(\kappa_1)}, \dots, P_k^{(\kappa_k)}$ an S_n -Lie monomial if $\sum_i \kappa_i = n$.

Multiplication in Σ_{S_n} , with basis denoted by $\{B_\kappa\}_{\kappa \models n}$ can be described as follows (say [17, Theorem 2]). If κ, ν are compositions of n , then

$$B_\kappa B_\nu = \sum_z B_{r(z)}$$

where the sum is over all matrices $z = (z_{ij})$ with non-negative integer entries that satisfy

- (1) $\sum_i z_{ij} = \kappa_j$,
- (2) $\sum_j z_{ij} = \nu_i$.

For each matrix, \mathbf{z} , $r(\mathbf{z})$ is the composition obtained by reading the non-zero entries of the matrix \mathbf{z} by row.

Example 4.1. In Σ_{S_4} if calculating $B_{[2,1,1]}B_{[2,2]}$ then the matrices satisfying our conditions are

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence

$$B_{[2,1,1]}B_{[2,2]} = B_{[2,1,1]} + B_{[1,1,2]} + 2B_{[1,1,1,1]}.$$

By comparison of Theorem 2.1 [11] and the above description of multiplication in Σ_{S_n} using matrices we observe that the action of Σ_{S_n} on S_n -Lie monomials can be stated as follows.

Proposition 4.1. *Let $\kappa, \nu \vDash n$ and $P^{(\kappa)} := P_1^{(\kappa_1)} \dots P_k^{(\kappa_k)}$ then*

$$P^{(\kappa)}B_\nu = \sum_{\mathbf{z}} P^{(r(\mathbf{z}))}$$

where the sum is over all matrices $\mathbf{z} = (z_{ij})$ with non-negative integer entries that satisfy

- (1) $\sum_i z_{ij} = \kappa_j$,
- (2) $\sum_j z_{ij} = \nu_i$,
- (3) there is exactly one non-zero entry in each column.

For each matrix, \mathbf{z} , $r(\mathbf{z})$ is the composition obtained by reading the non-zero entries of the matrix \mathbf{z} by row. Moreover $P^{(\kappa)}B_\nu = 0$ unless adjacent components of κ sum to give the components of ν .

An analogous proposition holds if we extend our definition of an S_n -Lie monomial to form a B_n -Lie monomial [4] and compare the action of Σ_{B_n} on these [4, Lemma 1.4] with multiplication in Σ_{B_n} [3, Theorem 1]. Therefore since other results in this paper can be stated in a manner similar to analogous results in the symmetric and hyperoctahedral groups cases it is hoped that when D_n -Lie monomials $P^{(\kappa)}$ can be defined they will have the following properties stemming from multiplication in Σ_{D_n} [6, Theorem 1].

Conjecture 4.1. *Let $\kappa, \nu \in \mathcal{C}(n)$ and $P^{(\kappa)} := P_1^{(\kappa_1)}, \dots, P_k^{(\kappa_k)}$ then*

$$P^{(\kappa)}B_\nu = \sum_{\mathbf{z}} \tilde{P}^{(r(\mathbf{z}))}$$

where the sum is over filled templates $\mathbf{z} \in Z(\kappa, \nu)$ [6, p 701] with exactly one non-zero entry in each column. Also $P^{(\nu)}$ is a summand of $\tilde{P}^{(r(\mathbf{z}))}$ if and only if $B_{(\eta)}$ is a summand of $\tilde{B}_{(r(\mathbf{z}))}$ as determined by [6, Theorem 1].

Moreover $P^{(\kappa)}B_\nu = 0$ unless adjacent components of κ sum to give the components of ν and it is not the case that

- (1) $\kappa \in \mathcal{C}_n$, $\nu \in \mathcal{C}'_n$, all components are even, or
 (2) $\kappa \in \mathcal{C}'_n$, $\nu \in \mathcal{C}_n$, all components are even.

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