

EXPRESSIONS FOR CATALAN KRONECKER PRODUCTS

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ABSTRACT. We give some elementary manifestly positive formulae for the Kronecker products $s_{(d,d)} * s_{(d+k,d-k)}$. These formulae demonstrate some fundamental properties of the Kronecker coefficients, and we use them to deduce a number of enumerative and combinatorial results.

1. INTRODUCTION

A classic open problem in algebraic combinatorics is to explain the Kronecker product (or internal product) of two Schur functions in a manifestly positive combinatorial formula. The Kronecker product of two Schur functions is the Frobenius image of the internal tensor product of two irreducible symmetric group modules, or it is alternatively the characters of the induced tensor product of general linear group modules. Although this expression clearly has non-negative coefficients when expanded in terms of Schur functions for representation theoretic reasons, it remains an open problem to provide a satisfying positive combinatorial or algebraic formula for the Kronecker product of two Schur functions.

Many attempts have been made to capture some aspect of these coefficients, for example, special cases [5, 6, 18, 19], asymptotics [1, 2], stability [20], the complexity of calculating them [7], and conditions when they are non-zero [8]. Given that the Littlewood-Richardson rule and many successors have so compactly and cleanly been able to describe the external product of two Schur functions it seems as though some new ideas for capturing the combinatorics of Kronecker coefficients are needed.

The results in this paper were inspired by the symmetric function identity for the Kronecker product of two Schur functions, $s_{(d,d)} * s_{(d,d)}$ that appears as [9, Theorem I.1]. More precisely, for a subset of partitions X of $2d$, if we set $[X] = \sum_{\lambda \in X} s_\lambda$, called a *rug*, then

$$(1.1) \quad s_{(d,d)} * s_{(d,d)} = [4 \text{ parts all even or all odd}] .$$

This particular identity is significantly different from most published results on the Kronecker product because it clearly states exactly which partitions have non-zero coefficients and that all of the coefficients are 0 or 1 instead of giving a combinatorial interpretation or algorithm.

This computation originally arose in the solution to a mathematical physics problem related to resolving the interference of 4 qubits [21] because the sum of these coefficients is

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equal to the dimensions of polynomial invariants of four copies of $SL(2, \mathbb{C})$ acting on \mathbb{C}^8 . Understanding the Kronecker product of $s_{(d,d)}$ with s_λ for partitions λ with 4 parts which are all even or all odd would be useful for calculating the dimensions of invariants of six copies of $SL(2, \mathbb{C})$ acting on \mathbb{C}^{12} which is a measure of entanglement of 6 qubits. Ultimately we would like to be able to compute

$$CT_{a_1, a_2, \dots, a_k} \left(\frac{\prod_{i=1}^k (1 - a_i^2)}{\prod_{S \subseteq \{1, 2, \dots, k\}} (1 - q \prod_{i \in S} a_i / \prod_{j \notin S} a_j)} \right) = \sum_{d \geq 0} \langle s_{(d,d)}^{*k}, s_{(2d)} \rangle q^{2d}$$

(see [10, formulas I.4 and I.5] and [13] for a discussion) where CT represents the operation of taking the constant term and equations of this type are a motivation for understanding the Kronecker product with $s_{(d,d)}$ as completely as possible.

Using $s_{(d,d)} * s_{(d,d)}$ as our inspiration, we were able to show (Corollary 3.6) that

$$s_{(d,d)} * s_{(d+1,d-1)} = [2 \text{ even parts and 2 odd parts }]$$

and with a similar computation we were also able to derive that

$$s_{(d,d)} * s_{(d+2,d-2)} = [4 \text{ parts, all even or all odd, but not 3 the same }] + [4 \text{ distinct parts }] .$$

One interesting feature of this formula is that it says that all of the coefficients in the Schur expansion of $s_{(d,d)} * s_{(d+2,d-2)}$ are either 0, 1 or 2 and the coefficient is 2 with those Schur functions indexed by partitions with 4 distinct parts that are all even or all odd.

These and larger examples suggested that the Schur function expansion of $s_{(d,d)} * s_{(d+k,d-k)}$ has the pattern of a boolean lattice of subsets, in that it can be written as the sum of $\lfloor k/2 \rfloor + 1$ intersecting sums of Schur functions each with coefficient 1. The main result of this article is Theorem 3.1, which states

$$(1.2) \quad s_{(d,d)} * s_{(d+k,d-k)} = \sum_{i=0}^k [(k+i, k, i)P] + \sum_{i=1}^k [(k+i+1, k+1, i)P]$$

where we have used the notation γP to represent the set of partitions λ of $2d$ of length less than or equal to 4 such that $\lambda - \gamma$ (representing a vector difference) is a partition with 4 even parts or 4 odd parts. The disjoint sets of this sum can be grouped so that the sum is of only $\lfloor k/2 \rfloor + 1$ terms, which shows that the coefficients always lie in the range 0 through $\lfloor k/2 \rfloor + 1$. The most interesting aspect of this formula is that we see the lattice of subsets arising in a natural and unexpected way in a representation theoretical setting. This is potentially part of a more general result and the hope is that this particular model will shed light on a general formula for the Kronecker product of two Schur functions, but our main motivation for computing these is to develop computational tools.

There are yet further motivations for restricting our attention to understanding the Kronecker product of $s_{(d,d)}$ with another Schur function. The Schur functions indexed by the partition (d, d) are a special family for several combinatorial reasons and so there is reason to believe that their behavior will be more accessible than the general case for the Kronecker product of two Schur functions. More precisely, Schur functions indexed by partitions with

two parts are notable because they are the difference of two homogeneous symmetric functions, for which a combinatorial formula for the Kronecker product is known. In addition, a partition (d, d) is rectangular and hence falls under a second category of Schur functions that are often combinatorially more straightforward to manipulate than the general case.

From the hook length formula it follows that the number of standard tableau of shape (d, d) is equal to the Catalan number $C_d = \frac{1}{d+1} \binom{2d}{d}$. Therefore, from the perspective of S_{2d} representations, taking the Kronecker product with the Schur function $s_{(d,d)}$ and the Frobenius image of a module explains how the tensor of a representation with a particular irreducible module of dimension C_d decomposes.

The paper is structured as follows. In the next section we review pertinent background information including necessary symmetric function notation and lemmas necessary for later computation. In Section 3 we consider a generalization of formula (1.1) to an expression for $s_{(d,d)} * s_{(d+k,d-k)}$. Finally, Section 4 is devoted to combinatorial and symmetric function consequences of our results. In particular we are able to give generating functions for the partitions that have a particular coefficient in the expression $s_{(d,d)} * s_{(d+k,d-k)}$.

2. BACKGROUND

2.1. Partitions. A *partition* of an integer n is a finite sequence of non-negative integers $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$ whose values sum to n , denoted $\lambda \vdash n$. The *height* or *length* of the partition, denoted $\ell(\lambda)$, is the maximum index for which $\lambda_{\ell(\lambda)} > 0$. We call the λ_i *parts* or *rows* of the partition and if λ_i appears n_i times we abbreviate this subsequence to $\lambda_i^{n_i}$. With this in mind if $\lambda = (k^{n_k}, (k-1)^{n_{k-1}}, \dots, 1^{n_1})$ then we define $z_\lambda = 1^{n_1} 1! 2^{n_2} 2! \dots k^{n_k} k!$. The 0 parts of the partition are optional and we will assume that for $i > \ell(\lambda)$ we have $\lambda_i = 0$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition of n . To form the *diagram* associated with λ , place a cell at each point in matrix notation (j, i) where $1 \leq i \leq \lambda_j$ and $1 \leq j \leq \ell$. We say λ has *transpose* λ' if the diagram for λ' is given by the points (i, j) where $1 \leq i \leq \lambda_j$ and $1 \leq j \leq \ell$.

2.2. Symmetric functions and the Kronecker product. The ring of symmetric functions is the graded subring of $\mathbb{Q}[x_1, x_2, \dots]$ given by

$$\Lambda := \mathbb{Q}[p_1, p_2, \dots]$$

where $p_i = x_1^i + x_2^i + \dots$ are the elementary power sum symmetric functions. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ we define $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_\ell}$. The interested reader should consult a reference such as [14] for more details of the structure of this ring. It is straightforward to see that $\{p_\lambda\}_{\lambda \vdash n \geq 0}$ forms a basis for Λ . This basis is orthogonal to itself with respect to the scalar product on Λ :

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

However, our focus for this paper will be the basis of Λ known as the basis of *Schur functions*, $\{s_\lambda\}_{\lambda \vdash n \geq 0}$, which is the orthonormal basis under this scalar product:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

The *Kronecker product* is the operation on symmetric functions

$$(2.1) \quad \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda}$$

that in terms of the Schur functions becomes

$$s_\mu * s_\nu = \sum_{\lambda \vdash |\mu|} C_{\mu\nu\lambda} s_\lambda.$$

It transpires that the *Kronecker coefficients* $C_{\mu\nu\lambda}$ encode the inner tensor product of symmetric group representations. That is, if we denote the irreducible S_n module indexed by a partition λ by M^λ , and $M^\mu \otimes M^\nu$ represents the tensor of two modules with the diagonal action, then the module decomposes as

$$M^\mu \otimes M^\nu \simeq \bigoplus_{\lambda} (M^\lambda)^{\oplus C_{\mu\nu\lambda}}.$$

The Kronecker coefficients also encode the decomposition of GL_{nm} polynomial representations to $GL_n \otimes GL_m$ representations

$$\text{Res}_{GL_n \otimes GL_m}^{GL_{nm}}(V^\lambda) \simeq \bigoplus_{\mu, \nu} (V^\mu \otimes V^\nu)^{\oplus C_{\mu\nu\lambda}}.$$

It easily follows from (2.1) and the linearity of the product that these coefficients satisfy the symmetries

$$C_{\mu\nu\lambda} = C_{\nu\mu\lambda} = C_{\mu\lambda\nu} = C_{\mu'\nu'\lambda},$$

and

$$C_{\lambda\mu(n)} = C_{\lambda\mu'(1^n)} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

that we will use extensively in what follows.

We will use some symmetric function identities in the remaining sections. Recall that for a symmetric function f , the symmetric function operator f^\perp (read ‘f perp’) is defined to be the operator that is dual to multiplication with respect to the scalar product. That is,

$$(2.2) \quad \langle f^\perp g, h \rangle = \langle g, f \cdot h \rangle.$$

The perp operator can also be defined linearly by

$$s_\lambda^\perp s_\mu = s_{\mu/\lambda} = \sum_{\nu \vdash |\mu| - |\lambda|} c_{\lambda\nu}^\mu s_\nu$$

where the $c_{\lambda\nu}^\mu$ are the Littlewood-Richardson coefficients. We will make use of the following well known relation, which connects the internal and external products,

$$(2.3) \quad \langle s_\lambda f, g * h \rangle = \sum_{\mu, \nu \vdash |\lambda|} C_{\lambda\mu\nu} \langle f, (s_\mu^\perp g) * (s_\nu^\perp h) \rangle.$$

This formula follows because of the relationship between the internal coproduct and the scalar product. It may also be seen to hold on the power sum basis since

$$(2.4) \quad \left\langle p_\lambda p_\mu, \frac{p_\gamma}{z_\gamma} * \frac{p_\nu}{z_\nu} \right\rangle = \left\langle p_\mu, \left(p_\lambda^\perp \frac{p_\gamma}{z_\gamma} \right) * \left(p_\lambda^\perp \frac{p_\nu}{z_\nu} \right) \right\rangle$$

which holds because both the expressions are 1 if and only if $\gamma = \nu$ and are both equal to the union of the parts of λ and μ . This given, equation (2.3) follows by linearity.

From these two identities and use of the Littlewood-Richardson rule we derive the following lemma.

Lemma 2.1. *If $\ell(\lambda) > 4$, then $C_{(d,d)(a,b)\lambda} = 0$. Otherwise it satisfies the following recurrences. If $\ell(\lambda) = 4$, then*

$$(2.5) \quad \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle = \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{\lambda-(1^4)} \rangle .$$

If $\ell(\lambda) = 3$, then

$$(2.6) \quad \begin{aligned} \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle &= \langle s_{(d-1,d-1)} * s_{(d+k-1,d-k-1)}, s_{(1)} s_{\lambda-(1^3)} \rangle \\ &\quad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)} \rangle . \end{aligned}$$

If $\ell(\lambda) = 2$, then

$$(2.7) \quad \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle = \begin{cases} 1 & \text{if } k \equiv \lambda_2 \pmod{2} \text{ and } \lambda_2 \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For (2.7) we refer the reader to [18, Theorem 3.3] and [19, Corollary 1]. The same result also appears in [10, Theorem 2.2].

Regev [17] proves that the maximum height of an indexing partition of the terms of $s_{(d,d)} * s_{(d+k,d-k)}$ will be at most 4, hence we can conclude that if $\ell(\lambda) > 4$, then $C_{(d,d)(a,b)\lambda} = 0$.

Assume that $\ell(\lambda) = 4$. By the Pieri rule we have that $s_{(1^4)} s_{\lambda-(1^4)} = s_\lambda +$ terms of the form s_γ where $\ell(\gamma) > 4$. By consequence,

$$\begin{aligned} \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle &= \langle s_{(d,d)} * s_{(d+k,d-k)}, s_{(1^4)} s_{\lambda-(1^4)} \rangle \\ &= \sum_{\mu \vdash 4} \langle (s_\mu^\perp s_{(d,d)}) * (s_{\mu'}^\perp s_{(d+k,d-k)}), s_{\lambda-(1^4)} \rangle . \end{aligned}$$

Every term in this sum is 0 unless both μ and μ' have length less than or equal to 2. The only term for which this is true is $\mu = (2, 2)$ and $s_{(2,2)}^\perp(s_{(a,b)}) = s_{(a-2,b-2)}$, hence (2.5) holds.

Assume that $\ell(\lambda) = 3$. Although there are cases to check, it follows again from the Pieri rule that $s_{(1^3)} s_{\lambda-(1^3)} - s_{(1^4)} s_{(1)}^\perp s_{\lambda-(1^3)} = s_\lambda +$ terms involving s_γ where $\ell(\gamma) > 4$. Therefore,

$$\begin{aligned} \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle &= \langle s_{(d,d)} * s_{(d+k,d-k)}, s_{(1^3)} s_{\lambda-(1^3)} - s_{(1^4)} s_{(1)}^\perp s_{\lambda-(1^3)} \rangle \\ &= \sum_{\mu \vdash 3} \langle (s_\mu^\perp s_{(d,d)}) * (s_{\mu'}^\perp s_{(d+k,d-k)}), s_{\lambda-(1^3)} \rangle \\ &\quad - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)} \rangle . \end{aligned}$$

Again, in the sum the only terms that are not equal to 0 are those that have the length of both μ and μ' less than or equal to 2 and in this case the only such partition is $\mu = (2, 1)$. By the Littlewood-Richardson rule we have that $s_{(2,1)}^\perp(s_{(a,b)}) = s_{(1)}^\perp(s_{(a-1,b-1)})$, hence this last expression is equal to

$$\begin{aligned} & \langle (s_{(1)}^\perp s_{(d-1,d-1)}) * (s_{(1)}^\perp s_{(d+k-1,d-k-1)}), s_{\lambda-(1^3)} \rangle - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)} \rangle \\ & = \langle s_{(d-1,d-1)} * s_{(d+k-1,d-k-1)}, s_{(1)} s_{\lambda-(1^3)} \rangle - \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{(1)}^\perp s_{\lambda-(1^3)} \rangle . \end{aligned}$$

□

To express our main results we will use the characteristic of a boolean valued proposition. If R is a proposition then we denote the propositional characteristic (or indicator) function of R by

$$\langle\langle R \rangle\rangle = \begin{cases} 1 & \text{if proposition } R \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

3. THE KRONECKER PRODUCT $s_{(d,d)} * s_{(d+k,d-k)}$

Let P indicate the set of partitions with four even parts or four odd parts and let \overline{P} be the set of partitions with 2 even parts and 2 odd parts (the complement of P in the set of partitions of even size with at most 4 parts).

We let γP represent the set of partitions λ of $2d$ (the value of d will be implicit in the left hand side of the expression) such that $\lambda - \gamma \in P$. We also allow $(\gamma \uplus \alpha)P = \gamma P \cup \alpha P$. Under all cases that we will consider the partitions in γP and αP are disjoint.

Theorem 3.1. *Let λ be a partition of $2d$,*

$$(3.1) \quad \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle = \sum_{i=0}^k \langle\langle \lambda \in (k+i, k, i)P \rangle\rangle + \sum_{i=1}^k \langle\langle \lambda \in (k+i+1, k+1, i)P \rangle\rangle .$$

Because of the notation we have introduced, Theorem 3.1 can easily be restated in the following corollary.

Corollary 3.2. *For $k \geq 0$, Theorem 3.1 is equivalent to: if k is odd then*

$$\begin{aligned} s_{(d,d)} * s_{(d+k,d-k)} &= [((k, k) \uplus (k+1, k, 1) \uplus (k+2, k+1, 1))P] \\ &+ \sum_{i=1}^{(k-1)/2} [((k+2i, k, 2i) \uplus (k+2i+1, k+1, 2i) \uplus \\ &(k+2i+1, k, 2i+1) \uplus (k+2i+2, k+1, 2i+1))P] \end{aligned}$$

and if k is even then

$$s_{(d,d)} * s_{(d+k,d-k)} = [(k, k)P] + \sum_{i=1}^{k/2} [((k + 2i - 1, k, 2i - 1) \uplus (k + 2i, k + 1, 2i - 1) \uplus (k + 2i, k, 2i) \uplus (k + 2i + 1, k + 1, 2i))P] .$$

As a consequence, the coefficients of $s_{(d,d)} * s_{(d+k,d-k)}$ will all be less than or equal to $\lfloor k/2 \rfloor + 1$.

Remark 3.3. Note that the upper bound on the coefficients that appear in the expressions $s_{(d,d)} * s_{(d+k,d-k)}$ are sharp in the sense that for sufficiently large d , there is a coefficient that will be equal to $\lfloor k/2 \rfloor + 1$.

Remark 3.4. We note that this regrouping of the rugs is not unique but it is useful because there are partitions that will fall in the intersection of each of these sets. This set of rugs is also not unique in that it is possible to describe other collections of sets of partitions (e.g. see (3.4)).

Remark 3.5. This case has been considered in more generality by Remmel and Whitehead [18] and Rosas [19] in their study of the Kronecker product of two Schur functions indexed by two-row shapes. In relation to these results, our first (lengthy) derivation of a similar result started with the formula of Rosas, but was later replaced by the current simpler identity. Meanwhile, Orellana [15] has informed us that the journal version of reference [18] has an error in it. Our computer implementation of this formula indicates that Theorem 2.1 of [18] does not agree, for example, with direct computation for $(h, k) = (4, 1)$ and $(l, m) = (3, 2)$ and $\nu = (3, 1, 1)$. Consequently, we desired an independent proof of Theorem 3.1, and as a result our derivation is an elementary proof that uses induction involving symmetric function identities.

We note that for $k = 2$, the expression stated in the introduction does not exactly follow this decomposition but it does follow from some manipulation.

Corollary 3.6. For $d \geq 1$,

$$(3.2) \quad s_{(d,d)} * s_{(d,d)} = [P]$$

$$(3.3) \quad s_{(d,d)} * s_{(d+1,d-1)} = [\overline{P}]$$

and for $d \geq 2$,

$$(3.4) \quad s_{(d,d)} * s_{(d+2,d-2)} = [P \cap \text{no three parts are equal}] + [\text{distinct partitions}] .$$

Proof. Note that (3.2) is just a restatement of Corollary 3.2 in the case that $k = 0$ and is [9, Theorem I.1].

First, by Theorem 3.1 we note that

$$C_{(d,d)(d+1,d-1)\lambda} = ((\lambda \in (1, 1)P)) + ((\lambda \in (2, 1, 1)P)) + ((\lambda \in (3, 2, 1)P)) .$$

If $\lambda \in P$, then $\lambda - (1, 1)$, $\lambda - (2, 1, 1)$, $\lambda - (3, 2, 1)$ are all not in P so each of the terms in that expression are 0. If λ is a partition with two even parts and two odd parts (that is, $\lambda \in \overline{P}$) then either $\lambda_1 \equiv \lambda_2$ and $\lambda_3 \equiv \lambda_4 \pmod{2}$ or $\lambda_1 \equiv \lambda_3$ and $\lambda_2 \equiv \lambda_4 \pmod{2}$ or $\lambda_2 \equiv \lambda_3$ and $\lambda_1 \equiv \lambda_4 \pmod{2}$. In each of these three cases, exactly one of the expressions $(\lambda \in (1, 1)P)$, $(\lambda \in (2, 1, 1)P)$ or $(\lambda \in (3, 2, 1)P)$ will be 1 and the other two will be zero. Therefore,

$$\sum_{\lambda \vdash 2d} C_{(d,d)(d+1,d-1)\lambda} s_\lambda = [\overline{P}].$$

We also have by Theorem 3.1 that

$$\begin{aligned} C_{(d,d)(d+2,d-2)\lambda} = & (\lambda \in (2, 2)P) + (\lambda \in (4, 2, 2)P) - (\lambda \in (6, 4, 2)P) \\ & + (\lambda \in (3, 2, 1)P) + (\lambda \in (4, 3, 1)P) + (\lambda \in (5, 3, 2)P) + (\lambda \in (6, 4, 2)P). \end{aligned}$$

Any distinct partition that is in P is also in $(6, 4, 2)P$. Every distinct partition that is in \overline{P} will have two odd parts and two even parts and will be in one of $(3, 2, 1)P$, $(4, 3, 1)P$, or $(5, 3, 2)P$ depending on which of λ_2, λ_1 or λ_3 is equal to λ_4 modulo 2 (respectively). Therefore, we have

$$(3.5) \quad [\text{distinct partitions}] = [((3, 2, 1) \uplus (4, 3, 1) \uplus (5, 3, 2) \uplus (6, 4, 2))P].$$

If $\lambda \in (2, 2)P \cap (4, 2, 2)P$, then $\lambda_2 \geq \lambda_3 + 2$ because $\lambda \in (2, 2)P$, and $\lambda_1 \geq \lambda_2 + 2$ and $\lambda_3 \geq \lambda_4 + 2$ because $\lambda \in (4, 2, 2)P$, so $\lambda \in (6, 4, 2)P$. Conversely, one verifies that in fact $(2, 2)P \cap (4, 2, 2)P = (6, 4, 2)P$, hence

$$[(2, 2)P \cup (4, 2, 2)P] = [(2, 2)P] + [(4, 2, 2)P] - [(6, 4, 2)P].$$

If $\lambda \in P$ does not have three equal parts, then either $\lambda_2 > \lambda_3$ or $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$. Therefore, $\lambda \in (2, 2)P \cup (4, 2, 2)P$ and hence $(2, 2)P \cup (4, 2, 2)P = P \cap$ no three parts are equal. \square

Proof. (of Theorem 3.1) Our proof proceeds by induction on the value of d and uses the Lemma 2.1. We will consider two base cases because (2.6) and (2.5) give recurrences for two smaller values of d . The exception for this is of course that λ is a partition of length 2 since it is easily verified that the two sides of (3.1) agree since the only term on the right hand side of the equation that can be non-zero is $(\lambda \in (k, k)P)$.

When $d = k$ and we have that the left hand side of (3.1) is $\langle s_{(k,k)} * s_{(2k)}, s_\lambda \rangle$, which is 1 if $\lambda = (k, k)$ and 0 otherwise. On the right hand side of (3.1) we have that $(\lambda \in (k, k)P)$ is 1 if and only if $\lambda = (k, k)$ and all other terms are 0 and hence the two expressions agree.

If $d = k + 1$, then $s_{(k+1,k+1)} * s_{(2k-1,1)} = s_{(k+1,k,1)} + s_{(k+2,k)}$. Notice that the only partitions λ of $2k + 2$ such that the indicator functions on the right hand side of (3.1) can be satisfied are $(\lambda \in (k, k)P)$ when $\lambda = (k+2, k)$ and $(\lambda \in (k+1, k, 1)P)$ when $\lambda = (k+1, k, 1)$. All others must be 0 because the partitions that are subtracted off are larger than $2k+2$.

Now assume that (3.1) holds for all values strictly smaller than d . If $\ell(\lambda) = 4$, then $\lambda - \gamma \in P$ if and only if $\lambda - \gamma - (1^4) \in P$ for all partitions γ of length less than or equal to

3 so

$$\begin{aligned}
 \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle &= \langle s_{(d-2,d-2)} * s_{(d+k-2,d-k-2)}, s_{\lambda-(1^4)} \rangle \\
 &= \sum_{i=0}^k \left((\lambda-(1^4) \in (k+i, k, i)P) \right) + \sum_{i=1}^k \left((\lambda-(1^4) \in (k+i+1, k+1, i)P) \right) \\
 &= \sum_{i=0}^k \left((\lambda \in (k+i, k, i)P) \right) + \sum_{i=1}^k \left((\lambda \in (k+i+1, k+1, i)P) \right).
 \end{aligned}$$

So we can now assume that $\ell(\lambda) = 3$. By (2.6) we need to consider the coefficients of the form $\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\mu \rangle$ where s_μ appears in the expansion of $s_{(1)}s_{\lambda-(1^3)}$ or $s_{(1)}^\perp s_{\lambda-(1^3)}$. If λ has three distinct parts and $\lambda_3 \geq 2$ then $\mu = \lambda - \delta$ where $\delta \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1, -1), (2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ and we can assume by induction that these expand into terms of the form $\pm((\lambda - \delta - \gamma \in P))$ where γ is a partition. However, if λ is not distinct or $\lambda_3 = 1$, then for some δ in the set, $\lambda - \delta$ will not be a partition and $((\lambda - \delta \in \gamma P))$ will be 0 and we can add these terms to our formulas so that we can treat the argument uniformly and not have to consider different possible λ .

One obvious reduction we can make to treat the expressions more uniformly is to note that $((\lambda - (1, 1, 1, -1) \in \gamma P)) = ((\lambda - (2, 2, 2) \in \gamma P))$.

Let $C_1 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (2, 2, 2)\}$ and $C_2 = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$. By the induction hypothesis and (2.6) we have that $\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle$ is equal to

$$\begin{aligned}
 &\sum_{\delta \in C_1} \left(\sum_{i=0}^k ((\lambda - \delta \in (k+i, k, i)P)) + \sum_{i=1}^k ((\lambda - \delta \in (k+i+1, k+1, i)P)) \right) \\
 &- \sum_{\delta \in C_2} \left(\sum_{i=0}^k ((\lambda - \delta \in (k+i, k, i)P)) + \sum_{i=1}^k ((\lambda - \delta \in (k+i+1, k+1, i)P)) \right).
 \end{aligned}$$

We notice that $\lambda - (2, 2, 2) - (k+i, k, i) = \lambda - (1, 1, 2) - (k+i+1, k+1, i)$ and $\lambda - (2, 1, 1) - (k+i, k, i) = \lambda - (1, 0, 1) - (k+i+1, k+1, i)$ and $\lambda - (1, 2, 1) - (k+i, k, i) = \lambda - (0, 1, 1) - (k+i+1, k+1, i)$ so the corresponding terms always cancel. With this reduction, we are left with the terms

$$\begin{aligned}
 &\sum_{\delta \in C_3} \sum_{i=0}^k ((\lambda - \delta \in (k+i, k, i)P)) + \sum_{\delta \in C_4} \sum_{i=1}^k ((\lambda - \delta \in (k+i+1, k+1, i)P)) \\
 &- \sum_{i=0}^k ((\lambda - (1, 1, 2) \in (k+i, k, i)P)) - \sum_{\delta \in C_5} \sum_{i=1}^k ((\lambda - \delta \in (k+i+1, k+1, i)P)) \\
 &- ((\lambda - (1, 2, 1) \in (k, k)P)) - ((\lambda - (2, 1, 1) \in (k, k)P)) + ((\lambda - (2, 2, 2) \in (k, k)P))
 \end{aligned}$$

where $C_3 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$, $C_4 = \{(1, 1, 0), (2, 2, 2)\}$, $C_5 = \{(2, 1, 1), (1, 2, 1)\}$.

Next we notice that $\lambda - (1, 1, 2) - (k+i, k, i) = \lambda - (0, 1, 1) - (k+i+1, k, i+1)$, $\lambda - (2, 1, 1) - (k+i+1, k+1, i) = \lambda - (1, 1, 0) - (k+i+2, k+1, i+1)$, and $\lambda - (2, 2, 2) - (k+i+1, k+1, i) = \lambda - (1, 2, 1) - (k+i+2, k+1, i+1)$. Then by canceling these terms and joining the compositions that are being subtracted off in the sum, these sums reduce to the following expression.

$$\begin{aligned} & \sum_{i=0}^k \left(\left(\lambda \in (k+i+1, k, i+1)P \right) \right) + \sum_{i=0}^k \left(\left(\lambda \in (k+i+1, k+1, i)P \right) \right) \\ & + \left(\left(\lambda \in (k, k+1, 1)P \right) \right) + \left(\left(\lambda \in (k+3, k+2, 1)P \right) \right) \\ & + \left(\left(\lambda \in (2k+3, k+3, k+2)P \right) \right) + \left(\left(\lambda \in (k+2, k+2, 2)P \right) \right) \\ & - \left(\left(\lambda \in (2k+1, k+1, k+2)P \right) \right) - \left(\left(\lambda \in (2k+3, k+2, k+1)P \right) \right) \\ & - \left(\left(\lambda \in (k+3, k+3, 2)P \right) \right) - \left(\left(\lambda \in (k+1, k+2, 1)P \right) \right) \\ & - \left(\left(\lambda \in (k+2, k+1, 1)P \right) \right) . \end{aligned}$$

Since $\ell(\lambda) = 3$, if $\lambda - (a, b) \in P$, then $\lambda_1 - a \geq \lambda_2 - b \geq \lambda_3 \geq 2$, which is true if and only if $\lambda_1 - a - 2 \geq \lambda_2 - b - 2 \geq \lambda_3 - 2 \geq 0$. In particular, $\left(\left(\lambda \in (k+2, k+2, 2)P \right) \right) = \left(\left(\lambda \in (k, k)P \right) \right)$ and $\left(\left(\lambda \in (k+3, k+3, 2)P \right) \right) = \left(\left(\lambda \in (k+1, k+1)P \right) \right)$.

By verifying a few conditions it is easy to check that $\lambda \in (r, s, s+1)P$ if and only if $\lambda \in (r+2, s+2, s+1)P$ and similarly $\lambda \in (s, s+1, r)P$ if and only if $\lambda \in (s+2, s+1, r)P$. With this relationship, we have the equivalence of $\left(\left(\lambda \in (k, k+1, 1)P \right) \right) = \left(\left(\lambda \in (k+2, k+1, 1)P \right) \right)$, $\left(\left(\lambda \in (k+1, k+2, 1)P \right) \right) = \left(\left(\lambda \in (k+3, k+2, 1)P \right) \right)$, $\left(\left(\lambda \in (2k+1, k+1, k+2)P \right) \right) = \left(\left(\lambda \in (2k+3, k+3, k+2)P \right) \right)$, and $\left(\left(\lambda \in (2k+1, k, k+1)P \right) \right) = \left(\left(\lambda \in (2k+3, k+2, k+1)P \right) \right)$. Each of these appear in the expression above. After we cancel these terms the expression reduces to

$$\sum_{i=0}^{k-1} \left(\left(\lambda \in (k+i+1, k, i+1)P \right) \right) + \sum_{i=1}^k \left(\left(\lambda \in (k+i+1, k+1, i)P \right) \right) + \left(\left(\lambda \in (k, k)P \right) \right) .$$

This concludes the proof by induction on d since we know the identity holds for each partition λ of length 2, 3 or 4. \square

4. COMBINATORIAL AND SYMMETRIC FUNCTION CONSEQUENCES

4.1. Tableaux of height less than or equal to 4. Since every partition of even size and of length less than or equal to 4 lies in either P or \overline{P} , Corollary 3.6 has as a consequence the following corollary.

Corollary 4.1. *For d a positive integer,*

$$(4.1) \quad \sum_{\lambda \vdash 2d, \ell(\lambda) \leq 4} s_\lambda = s_{(d,d)} * (s_{(d,d)} + s_{(d+1, d-1)}) .$$

$$(4.2) \quad \sum_{\lambda \vdash 2d-1, \ell(\lambda) \leq 4} s_\lambda = s_{(d,d-1)} * s_{(d,d-1)} .$$

Proof. For the sum over partitions of $2d$, (1.1) (or (3.1)) says that $s_{(d,d)} * s_{(d,d)}$ is the sum over all s_λ with $\lambda \vdash 2d$ having four even parts or four odd parts and Corollary 3.6 says that $s_{(d,d)} * s_{(d+1,d-1)}$ is the sum over s_λ with $\lambda \vdash 2d$ where λ does not have four odd parts or four even parts. Hence, $s_{(d,d)} * s_{(d,d)} + s_{(d,d)} * s_{(d+1,d-1)}$ is the sum over s_λ where λ runs over all partitions with less than or equal to 4 parts.

For the other identity, we use (2.3) to derive

$$\langle s_{(d,d-1)} * s_{(d,d-1)}, s_\lambda \rangle = \langle s_{(d,d)} * s_{(d,d)}, s_{(1)} s_\lambda \rangle .$$

If λ is a partition of $2d-1$, then the expression is 0 if $\ell(\lambda) > 4$ and if $\ell(\lambda) \leq 4$ then $s_{(1)} s_\lambda$ is a sum of at most 5 terms, $s_{(\lambda_1+1, \lambda_2, \lambda_3, \lambda_4)}$, $s_{(\lambda_1, \lambda_2+1, \lambda_3, \lambda_4)}$, $s_{(\lambda_1, \lambda_2, \lambda_3+1, \lambda_4)}$, $s_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4+1)}$, $s_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, 1)}$. Because λ has exactly 3 or 1 terms that are odd, exactly one of these will have 4 even parts or 4 odd parts. \square

Regev [16], Gouyou-Beauchamps [12], Gessel [11] and subsequently Bergeron, Krob, Favreau, and Gascon [3, 4] studied tableaux of bounded height. If we set $y_k(n)$ = the number of standard tableaux of height less than or equal to k , then Gessel [11] remarks that expressions for $y_k(n)$ exist for $k = 2, 3, 4, 5$ that are simpler than the k -fold sum that one would expect to see. This is perhaps because in all four of these cases have more general formulae in terms of characters.

In the case of $k = 4$, Corollary 4.1 is a statement about characters indexed by partitions of bounded height. In particular, if those characters are evaluated at the identity we see the previously known result,

Corollary 4.2.

$$(4.3) \quad y_4(n) = C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil} .$$

This follows from the hook length formula that says the number of standard tableaux of shape (d, d) is C_d , the number of standard tableaux of $(d, d-1)$ is C_d , and the number of standard tableaux of shape that are either of shape (d, d) or $(d+1, d-1)$ is C_{d+1} .

It is interesting to note that some known expressions for $y_2(n)$, $y_3(n)$ and $y_5(n)$ can also be explained in terms of symmetric function identities using the Pieri rule.

4.2. Generating functions for partitions with coefficient r in $s_{(d,d)} * s_{(d+k,d-k)}$. An easy consequence of Theorem 3.1 is a generating function formula for the the sum of the coefficients of the expressions $s_{(d,d)} * s_{(d+k,d-k)}$.

Corollary 4.3. *For a fixed $k \geq 1$,*

$$\begin{aligned} G_k(q) &:= \sum_{d \geq k} \left(\sum_{\lambda \vdash 2d} \langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle \right) q^d \\ &= \frac{q^k + q^{k+1} + q^{2k+1} + \sum_{r=k+2}^{2k} 2q^r}{(1-q)(1-q^2)^2(1-q^3)}. \end{aligned}$$

Remark 4.4. Note that Corollary 4.3 only holds for $k > 0$. In the case that $k = 0$ in the formula in the previous expression, the numerator of the above expression is a bit different and we have from Corollary 3.6,

$$\begin{aligned} G_0(q) &= \sum_{d \geq 0} \left(\sum_{\lambda \vdash 2d} \langle s_{(d,d)} * s_{(d,d)}, s_\lambda \rangle \right) q^d = \sum_{d \geq 0} \langle s_{(d,d)} * s_{(d,d)}, s_{(d,d)} * s_{(d,d)} \rangle q^d \\ &= \sum_{d \geq 0} \langle [P], [P] \rangle q^d = \sum_{d \geq 0} |P| q^d \\ &= \frac{1}{(1-q)(1-q^2)^2(1-q^3)}. \end{aligned}$$

This last equality is the formula given in [10, Corollary 1.2] and it follows because the generating function for partitions with even parts and length less than or equal to 4 is $\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$, and the generating function for the partitions of $2d$ with odd parts of length less than or equal to 4 is $\frac{q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)}$. The sum of these two generating functions will be equal to a generating function for the number of non-zero coefficients of $s_{(d,d)} * s_{(d,d)}$.

Proof. Recall that Theorem 3.1 gives a formula for the expression of $s_{(d,d)} * s_{(d+k,d-k)}$ in terms of rugs of the form $[\gamma P]$. We can calculate that for each of the rugs that appears in this expression

$$\begin{aligned} \sum_{d \geq k} \left(\sum_{\lambda \vdash 2d} \langle [\gamma P], s_\lambda \rangle \right) q^d &= \sum_{d \geq k} \left(\sum_{\lambda \vdash 2d} \langle [P], s_{\lambda-\gamma} \rangle \right) q^d \\ &= \sum_{d \geq k} \left(\sum_{\mu \vdash (2d-|\gamma|)} \langle [P], s_\mu \rangle \right) q^{d+|\gamma|/2} \\ &= \frac{q^{|\gamma|/2}}{(1-q)(1-q^2)^2(1-q^3)}. \end{aligned}$$

Now since $s_{(d,d)} * s_{(d+k,d-k)}$ is the sum of rugs of the form $[\gamma P]$ where each of the $\gamma = (k, k), (k+1, k, 1)$ and $(2k+1, k+1, k)$ each contribute a term to the numerator of the form q^k, q^{k+1} and q^{2k+1} respectively. The rugs with γ equal to $(k+i+1, k+1, i)$ and $(k+i+1, k, i+1)$ for $1 \leq i \leq k-1$ each contribute a term $2q^{k+i+1}$ to the numerator. \square

To allow us to compute other generating functions of Kronecker products we require the following very surprising theorem. It says that the partitions such that $C_{(d,d)(d+k,d-k)\lambda}$ are of coefficient $r > 1$ are exactly the partitions $\gamma + (6, 4, 2)$ where $C_{(d-6,d-6)(d-6+(k-2),d-6-(k-2))\gamma}$ is equal to $r - 1$.

Theorem 4.5. *For $k \geq 2$, assume that $C_{(d,d)(d+k,d-k)\lambda} > 0$, then*

$$C_{(d+6,d+6)(d+k+8,d-k+4)(\lambda+(6,4,2))} = C_{(d,d)(d+k,d-k)\lambda} + 1 .$$

We require the following lemma.

Lemma 4.6. *For γ a partition with $\ell(\gamma) \leq 4$,*

$$\lambda \in \gamma P \Leftrightarrow \lambda + (6, 4, 2) \text{ is in both } (\gamma_1 + 2, \gamma_2 + 2, \gamma_3, \gamma_4)P \text{ and } (\gamma_1 + 4, \gamma_2 + 2, \gamma_3 + 2, \gamma_4)P .$$

Proof. (\Rightarrow) If $\lambda \in \gamma P$, then $\lambda - \gamma$ is a partition with four even parts or four odd parts. Hence, both $\lambda - \gamma + (2, 2) = (\lambda + (6, 4, 2)) - (\gamma + (4, 2, 2))$ and $\lambda - \gamma + (4, 2, 2) = (\lambda + (6, 4, 2)) - (\gamma + (2, 2))$ are elements of P .

(\Leftarrow) Assume that $\lambda + (6, 4, 2)$ is an element of both $(\gamma_1 + 2, \gamma_2 + 2, \gamma_3, \gamma_4)P$ and $(\gamma_1 + 4, \gamma_2 + 2, \gamma_3 + 2, \gamma_4)P$, then $\lambda_1 + 6 - (\gamma_1 + 4) \geq \lambda_2 + 4 - (\gamma_2 + 2)$, $\lambda_2 + 4 - (\gamma_2 + 2) \geq \lambda_3 + 2 - \gamma_3$, and $\lambda_3 + 2 - (\gamma_3 + 2) \geq \lambda_4 - \gamma_4 \geq 0$. This implies that $\lambda - \gamma$ is a partition and since $\lambda - \gamma + (2, 2)$ has four even or four odd parts, then so does $\lambda - \gamma$ and hence $\lambda \in \gamma P$. \square

Proof. (of Theorem 4.5) Consider the case where λ is a partition of $2d$ with $\lambda_2 - k \equiv \lambda_4 \pmod{2}$ since the case where $\lambda_2 - k \not\equiv \lambda_4 \pmod{2}$ is analogous and just uses a different non-zero terms in the sum below. From Theorem 3.1, we have

$$(4.4) \quad C_{(d,d)(d+k,d-k)\lambda} = \sum_{i=0}^k \left(\lambda \in (k+i, k, i)P \right) ,$$

since the other terms are clearly zero in this case. If $\lambda_2 - k \geq \lambda_3$, then the terms in this sum will be non-zero as long as $0 \leq i \leq \lambda_3 - \lambda_4$ and $0 \leq k+i \leq \lambda_1 - \lambda_2$ and $\lambda_3 - i \equiv \lambda_4 \pmod{2}$. Consider the case where $\lambda_3 \equiv \lambda_4 \pmod{2}$, then (4.4) is equal to $a + 1$ where $a = \lfloor \min(\lambda_3 - \lambda_4, \lambda_1 - \lambda_2 - k, k) / 2 \rfloor$ since the terms that are non-zero in this sum are $(\lambda \in (k+2j, k, 2j)P)$ where $0 \leq j \leq a$. By Lemma 4.6, these terms are true if and only if $(\lambda + (6, 4, 2) \in (k+2+2j, k+2, 2j)P)$ are true for all $0 \leq j \leq a+1$. But again by Theorem 3.1, in this case we also have

$$C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))} = a + 2 = C_{(d,d)(d+k,d-k)\lambda} + 1 .$$

The case where $\lambda_3 \equiv \lambda_4 + 1 \pmod{2}$ is similar, but the terms of the form $(\lambda \in (k+2j+1, k, 2j+1)P)$ in (4.4) are non-zero if and only if the terms $(\lambda + (6, 4, 2) \in (k+2+2j+1, k+2, 2j+1)P)$ contribute to the expression for $C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))}$ and there is exactly one more non-zero term hence $C_{(d+6,d+6)(d+k+8,d-k-4)(\lambda+(6,4,2))} = C_{(d,d)(d+k,d-k)\lambda} + 1$. \square

Now for computational purposes it is useful to have a way of determining exactly the number of partitions of $2d$ that have a given coefficient. For integers $d, k, r > 0$, we let $L_{d,k,r}$ be the number of partitions λ of $2d$ with $\langle s_{(d,d)} * s_{(d+k,d-k)}, s_\lambda \rangle = r$. Our previous theorems have shown that $L_{d,k,r} = 0$ for $r > \lfloor k/2 \rfloor + 1$ (see Theorem 3.1) and $L_{d,k,r} = L_{d-6,k-2,r-1} + 1$ for $r > 1$ (see Theorem 4.5). These recurrences will allow us to completely determine the generating functions for the coefficients $L_{d,k,r}$. For this purpose, we set

$$L_{k,r}(q) = \sum_{d \geq 0} L_{d,k,r} q^d .$$

Corollary 4.7. *With the convention that $G_k(q) = 0$ for $k < 0$, then $L_{k,r}(q) = 0$ for $r > \lfloor k/2 \rfloor + 1$, and*

$$(4.5) \quad L_{k,1}(q) = G_k(q) - 2q^6 G_{k-2}(q) + q^{12} G_{k-4}(q)$$

$$(4.6) \quad L_{k,r}(q) = q^{6r-6} L_{k-2r+2,1}(q) .$$

Proof. Theorem 4.5 explains (4.6) because

$$\begin{aligned} L_{k,r}(q) &= \sum_{d \geq 0} \#\{\lambda : C_{(d,d)(d+k,d-k)} \lambda = r\} q^d \\ &= \sum_{d \geq 0} \#\{\lambda : C_{(d-6,d-6)(d+k-8,d-k+4)} (\lambda + (6,4,2)) = r - 1\} q^d \\ &= \sum_{d \geq 0} \#\{\lambda : C_{(d-6r+6,d-6r+6)(d+k-8r+8,d-k+4r-4)} (\lambda + (6r-6,4r-4,2r-2)) = 1\} q^d \\ &= q^{6r-6} \sum_{d \geq 0} \#\{\lambda : C_{(d-6r+6,d-6r+6)(d+k-8r+8,d-k+4r-4)} (\lambda + (6r-6,4r-4,2r-2)) = 1\} q^{d-6r+6} \\ &= q^{6r-6} L_{k-2r+2,1}(q) . \end{aligned}$$

Now we also have by definition and Theorem 3.1 that

$$(4.7) \quad G_k(q) = \sum_{r=1}^{\lfloor k/2 \rfloor + 1} r L_{k,r}(q) .$$

Hence, we can use this formula and (4.6) to define $L_{k,r}(q)$ recursively. It remains to show that the formula for $L_{k,1}(q)$ stated in (4.5) satisfies this formula, which we do by induction. Given that $L_{0,1}(q) = G_0(q)$ and $L_{1,1}(q) = G_1(q)$ and $L_{k,1}(q) = 0$ for $k < 0$, then assuming

that the formula holds for values smaller than $k > 1$ then (4.7) yields

$$\begin{aligned}
 L_{k,1}(q) &= G_k(q) - \sum_{r=2}^{\lfloor k/2 \rfloor + 1} r L_{k,r}(q) \\
 &= G_k(q) - \sum_{r \geq 2} r q^{6r-6} L_{k-2r+2,1}(q) \\
 &= G_k(q) - \sum_{r \geq 2} r q^{6r-6} (G_{k-2r+2}(q) - 2q^6 G_{k-2r}(q) + q^{12} G_{k-2r-2}(q)) \\
 &= G_k(q) - \sum_{r \geq 1} (r+1) q^{6r} G_{k-2r}(q) + \sum_{r \geq 2} 2r q^{6r} G_{k-2r}(q) - \sum_{r \geq 3} (r-1) q^{6r} G_{k-2r}(q) \\
 &= G_k(q) - 2q^6 G_{k-2}(q) + q^{12} G_{k-4}(q) .
 \end{aligned}$$

Therefore, by induction we have that (4.5) holds for all $k > 0$. □

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