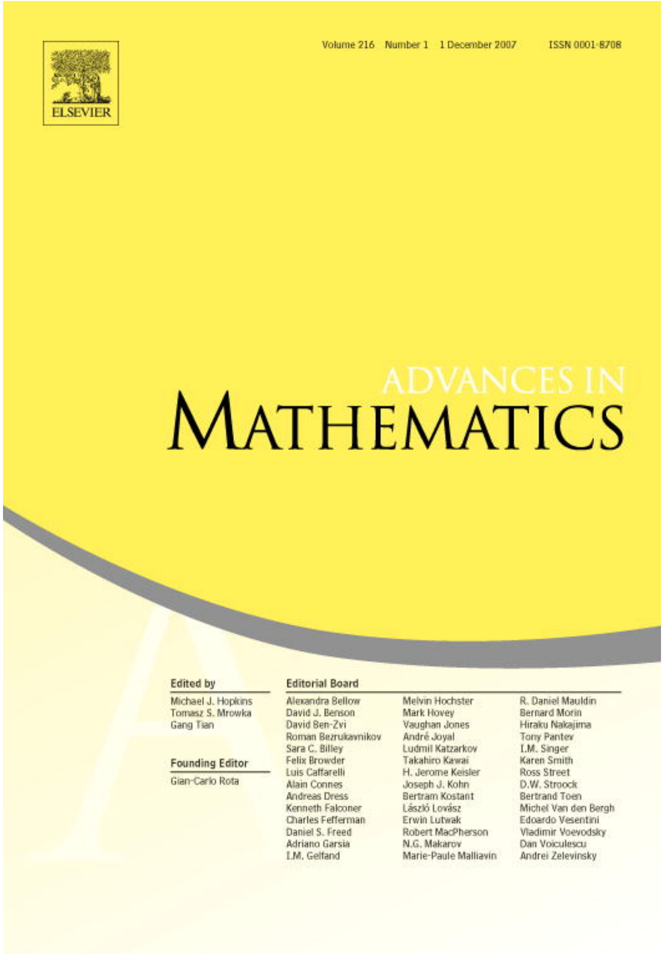
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Coincidences among skew Schur functions ☆

Victor Reiner [a], Kristin M. Shaw [b], Stephanie van Willigenburg [b,*]

[a] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA*
[b] *Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada*

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Abstract

New sufficient conditions and necessary conditions are developed for two skew diagrams to give rise to the same skew Schur function. The sufficient conditions come from a variety of new operations related to *ribbons* (also known as *border strips* or *rim hooks*). The necessary conditions relate to the extent of overlap among the rows or among the columns of the skew diagram.

MSC: primary 05E05, 20C30

Keywords: Symmetric function; Skew Schur function; Ribbon Schur function; Weyl module

Contents

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* Corresponding author.
E-mail addresses: reiner@math.umn.edu (V. Reiner), krishaw@math.ubc.ca (K.M. Shaw), steph@math.ubc.ca (S. van Willigenburg).

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1. Introduction

Symmetric functions play an important role in combinatorics, geometry, and representation theory. Of particular prominence among the symmetric functions are the family of skew Schur functions $s_{\lambda/\mu}$. For example, when they were introduced by Schur [14] over one hundred years ago they were related to the irreducible representations of the symmetric group. Most recently they have been connected to branching rules for classical Lie groups [8,12], and eigenvalues and singular values of sums of Hermitian and of complex matrices [1,5,8] via the study of *inequalities* among products of skew Schur functions.

With this in mind, a natural avenue to pursue is the *equalities* among products of skew Schur functions. As we shall see in Section 6, an equivalent formulation of this question is the study of all *binomial syzygies* among skew Schur functions, which is a more tractable incarnation of a problem that currently seems out of reach: find *all* syzygies among skew Schur functions. Famous non-binomial syzygies include various formulations of the Littlewood–Richardson rule and Eq. (4.1) below, which give some indication of the complexity that any solution would involve.

The study of equalities among skew Schur functions can also be regarded as part of the "calculus of shapes." For an arbitrary subset D of $\mathbb{Z}^2$, there are two polynomial representations $\mathcal{S}^D$ and $\mathcal{W}^D$ of $GL_N(\mathbb{C})$ known as a *Schur module* and *Weyl module* respectively, obtained by row-symmetrizing and column-antisymmetrizing tensors whose tensor positions are indexed by the cells of D. These representations are determined up to isomorphism by their character, namely the symmetric function $s_D(x_1, \ldots, x_N)$, which tells us the trace of any element g in $GL_N(\mathbb{C})$ acting on $\mathcal{S}^D$ and $\mathcal{W}^D$ as a function of the eigenvalues $x_1, \ldots, x_N$ of g. When $D = \lambda/\mu$ is a skew diagram, this symmetric function is the skew Schur function $s_{\lambda/\mu}(x_1, \ldots, x_N)$. Therefore, the question of when two skew Schur or Weyl modules are equivalent, working over $\mathbb{C}$, is precisely the question of equalities among skew Schur functions.

As a consequence of this, the aim of this paper is to study the equivalence relation on skew diagrams D_1, D_2 defined by $D_1 \sim D_2$ if and only if $s_{D_1} = s_{D_2}$, and in particular to use known *skew-equivalences* to generate new ones. Our motivation for this approach is [2] where Billera, Thomas and the third author studied when two elements of the subclass of skew diagrams known as *ribbons* or *border strips* or *rim hooks* were skew-equivalent. They discovered that if ribbons

$\alpha, \beta, \gamma, \delta$ satisfied $\alpha \sim \beta$ and $\gamma \sim \delta$ then the *composition* of ribbons $\alpha \circ \beta$ and $\gamma \circ \delta$ satisfied $\alpha \circ \beta \sim \gamma \circ \delta$.

The paper is structured as follows. In Section 2 we review notation concerning partitions, compositions and skew diagrams. Section 3 recalls the ring of symmetric functions and Section 4 covers various definitions and basic properties of skew Schur functions. Section 5 is our final review section and gives a version of the Littlewood–Richardson rule.

In Section 6 we reduce the question of skew-equivalence to the case of connected skew diagrams. Sections 7 and 8 then build upon this to develop necessary and sufficient conditions for skew-equivalence. Specifically, in Section 7, for ribbons α, β and a skew diagram D we define compositions $\alpha \circ D$ and $D \circ \beta$ that naturally generalize the composition of ribbons, $\circ$, defined in [2] and prove

Theorem *(Theorem 7.6). If one has ribbons α, α' and skew diagrams D, D' satisfying $\alpha \sim \alpha'$ and $D \sim D'$, then*

(i) $\alpha \circ D \sim \alpha' \circ D$,
(ii) $D \circ \alpha \sim D' \circ \alpha$,
(iii) $D \circ \alpha \sim D \circ \alpha'$, *and*
(iv) $\alpha \circ D \sim \alpha \circ D^*$,

where D^* is D rotated by 180 *degrees.*

For certain ribbons ω we also construct an analogous operation to $\circ$ called *amalgamated composition*, $\circ_\omega$, and prove

Theorem *(Theorem 7.22). If α, α' are ribbons with $\alpha \sim \alpha'$, and D, ω satisfy Hypotheses* 7.19*, then one has the following skew-equivalences*:

$$\alpha' \circ_\omega D \sim \alpha \circ_\omega D \sim \alpha \circ_{\omega^*} D^*,$$

where D^* is D rotated by 180 *degrees.*

Additionally, Section 7.3 yields a construction that produces skew diagrams that are skew-equivalent to their conjugate.

Meanwhile, Section 8 discusses two necessary conditions for skew-equivalence. One comes from the Frobenius rank of a skew diagram studied in [3,17,18]. The other is new, and relates to the sizes of the rows and the columns of a skew diagram, and the sizes of their overlaps. Finally, Section 9 suggests further avenues to pursue.

2. Diagrams

In this section, we review partitions, compositions, Ferrers diagrams, skew diagrams and ribbons. The interested reader may wish to consult [10,13,16] for further details.

A *partition* λ of a positive integer n, denoted $\lambda \vdash n$, is a sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers λ_i such that

$$\lambda_1 \geqslant \cdots \geqslant \lambda_\ell > 0$$

and $\sum_{i=1}^{\ell} \lambda_i = n$. We call n the *weight* or *size* of λ, and denote it $|\lambda| := n$. Each λ_i is called a *part* of λ, and the number of parts ℓ is called the *length* $\ell(\lambda) := \ell$. The unique partition of 0 is denoted by $\emptyset$.

The (*Ferrers* or *Young*) *diagram* of λ consists of *boxes* or *cells* such that there are λ_i cells in each row i, so the top row has λ_1 cells, the second-from-top row has λ_2 cells, etc. In addition, the rows of cells are all left-justified. We abuse notation and also denote the Ferrers diagram of λ by λ.

Two partial orders on partitions that arise frequently are

- the *inclusion order*: $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all i,
- the *dominance* (or *majorization*) *order* on partitions λ, μ having the *same weight*: $\mu \leqslant_{\text{dom}} \lambda$ if

  $$\mu_1 + \mu_2 + \cdots + \mu_i \leqslant \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

  for $i = 1, 2, \ldots, \min(\ell(\mu), \ell(\lambda))$.

Given two partitions λ, μ such that $\mu \subseteq \lambda$ the *skew* (*Ferrers*) *diagram* $D = \lambda/\mu$ is obtained from the Ferrers diagram of λ by removing the cells in the subdiagram of μ from the top left corner. For example, the following is a skew diagram whose cells are indicated by $\times$:

$$\lambda/\mu = (5,4,3,3)/(3,1) = \begin{array}{ccccc} & & & \times & \times \\ & \times & \times & \times & \\ \times & \times & \times & & \\ \times & \times & \times & & \end{array}.$$

Cells in skew diagrams will be referred to by their row and column indices (i, j), where $i \leqslant \ell(\lambda)$ and $j \leqslant \lambda_i$. The *content* or *diagonal* of the cell is the integer $c(i, j) = j - i$.

Given two skew diagrams D_1, D_2, a *disjoint union* $D_1 \oplus D_2$ of them is obtained by placing D_2 strictly to the north and east of D_1 in such a way that D_1, D_2 occupy none of the same rows or columns. For example, if $D_1 = (2,2)$, $D_2 = (3,2)/(1)$ then a possible disjoint union is

$$D_1 \oplus D_2 = \begin{array}{ccccc} & & & \times & \times \\ & & \times & \times & \\ \times & \times & & & \\ \times & \times & & & \end{array}. \tag{2.1}$$

We say that a skew diagram D is *connected* if it cannot be written as $D = D_1 \oplus D_2$ for two proper subdiagrams D_1, D_2. A connected skew diagram D is called a *ribbon* or *border strip* or *rim hook* if it does not contain a subdiagram isomorphic to that of the partition $\lambda = (2,2)$. For example,

$$\lambda/\mu = (5,4,3,1)/(3,2) = \begin{array}{ccccc} & & & \times & \times \\ & & \times & \times & \\ \times & \times & \times & & \\ \times & & & & \end{array} \tag{2.2}$$

is a ribbon. Two skew diagrams D, $\tilde{D}$ will be considered equivalent as subsets of the plane if one can be obtained from the other by vertical or horizontal translations, or by the removal or addition of empty rows or columns. As a consequence of this, given two diagrams D_1, D_2 we can now say their disjoint union $D_1 \oplus D_2$ is obtained by placing D_2 immediately to the north and east of D_1 in such a way that D_1, D_2 occupy none of the same rows or columns, as illustrated by Eq. (2.1).

A *composition* α of a positive integer n, denoted $\alpha \vDash n$, is an ordered sequence $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of positive integers α_i such that $\sum_{i=1}^{\ell} \alpha_i = n$. As with partitions, we call n the *weight* or *size* of α, and denote it by $|\alpha| := n$. Again, the number ℓ is called the *length* $\ell(\alpha) := \ell$.

We end with two bijections regarding compositions. For a positive integer n, let $[n] := \{1, 2, \ldots, n\}$. For the first bijection consider the map sending a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ to the set of partial sums $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$, which gives a bijection between compositions of n and the collection $2^{[n-1]}$ of all subsets of $[n-1]$. For the second bijection consider the map sending α to the unique ribbon having α_i cells in the ith row from the *bottom*, which gives a bijection between compositions of n and ribbons of size n. Note that labelling the rows of a composition from bottom to top is slightly inconsistent with the labelling of rows of Ferrers diagrams from top to bottom in English notation, but it is in keeping with the seminal work [6]. Due to this bijection, we will often refer to ribbons by their composition of row sizes. To illustrate these bijections, observe that the composition $\alpha = (1, 3, 2, 2)$ of $n = 8$ corresponds to the subset $\{1, 4, 6\}$ of $[n-1] = [7]$, and to the ribbon depicted in (2.2).

2.1. *Symmetries of diagrams*

We will have occasion to use several symmetries of partitions and skew diagrams and review two of them here.

Given a partition λ, its *conjugate* or *transpose* partition λ^t is the partition whose Ferrers diagram is obtained from that of λ by reflecting across the northwest-to-southeast diagonal. Equivalently, the parts of λ^t are the column sizes of the Ferrers diagram of λ read from left to right. This extends to skew diagrams in a natural way: if $D = \lambda/\mu$ then $D^t := \lambda^t/\mu^t$.

Given a skew diagram D, one can form its *antipodal rotation* D^* by rotating it 180 degrees in the plane. Note that for a ribbon $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, the antipodal rotation of its skew diagram corresponds to the *reverse* composition $\alpha^* = (\alpha_\ell, \ldots, \alpha_1)$.

2.2. *Operations on ribbons and diagrams*

This subsection reviews some standard operations on ribbons. It also discusses a composition operation $\alpha \circ \beta$ on ribbons α, β that was introduced in [2], and its generalization to operations $\alpha \circ D$ and $D \circ \beta$ for skew diagrams D.

Given two skew diagrams D_1, D_2, aside from their disjoint sum $D_1 \oplus D_2$, there are two closely related important operations called their *concatenation* $D_1 \cdot D_2$ and their *near-concatenation* $D_1 \odot D_2$. The concatenation $D_1 \cdot D_2$ (respectively near-concatenation $D_1 \odot D_2$) is obtained from the disjoint sum $D_1 \oplus D_2$ by moving all cells of D_2 one column west (respectively one row south), so that the same column (respectively row) is occupied by the rightmost column (respectively topmost row) of D_1 and the leftmost column (respectively bottommost row) of D_2. For example, if

$$D_1 = (2, 2),$$

$$D_2 = (3, 2)/(1)$$

then $D_1 \oplus D_2$ was given in Eq. (2.1), while

$$D_1 \cdot D_2 = \begin{array}{cccc} & & 2 & 2 \\ & 2 & 2 & \\ 1 & 1 & & \\ 1 & 1 & & \end{array}, \qquad D_1 \odot D_2 = \begin{array}{ccccc} & & & 2 & 2 \\ 1 & 1 & 2 & 2 & \\ 1 & 1 & & & \end{array}.$$

Observe we have used the numbers 1 and 2 to distinguish the cells in D_1 from the cells in D_2. The reason for the names "concatenation" and "near-concatenation" becomes clearer when we restrict to ribbons. Here if

$$\alpha = (\alpha_1, \dots, \alpha_\ell),$$
$$\beta = (\beta_1, \dots, \beta_m),$$

then

$$\alpha \cdot \beta = (\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m),$$
$$\alpha \odot \beta = (\alpha_1, \dots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \dots, \beta_m),$$

which are the definitions for concatenation and near-concatenation given in [6].

Note that the operations $\cdot$ and $\odot$ are each associative, and associate with each other:

$$\begin{aligned} (D_1 \cdot D_2) \cdot D_3 &= D_1 \cdot (D_2 \cdot D_3), \\ (D_1 \odot D_2) \odot D_3 &= D_1 \odot (D_2 \odot D_3), \\ (D_1 \odot D_2) \cdot D_3 &= D_1 \odot (D_2 \cdot D_3), \\ (D_1 \cdot D_2) \odot D_3 &= D_1 \cdot (D_2 \odot D_3). \end{aligned} \tag{2.3}$$

Consequently a string of operations $D_1 \star_1 D_2 \star_2 \cdots \star_{k-1} D_k$ in which each $\star_i$ is either $\cdot$ or $\odot$ is well-defined without any parenthesization. Also note that ribbons are exactly the skew diagrams that can be written uniquely as a string of the form

$$\alpha = \square \star_1 \square \star_2 \cdots \star_{k-1} \square \tag{2.4}$$

where $\square$ is the diagram with exactly one cell.

Given a composition α and a skew diagram D, define $\alpha \circ D$ to be the result of replacing each cell $\square$ by D in the expression (2.4) for α:

$$\alpha \circ D := D \star_1 D \star_2 \cdots \star_{k-1} D.$$

For example, if

$$\alpha = (2, 3, 1) = \begin{array}{cccc} & & & \times \\ & \times & \times & \times \\ \times & \times & & \end{array} \quad \text{and} \quad D = \begin{array}{cc} \times & \times \\ \times & \times \end{array}$$

then

$$\alpha = \square \odot \square \cdot \square \odot \square \odot \square \cdot \square,$$

$$\alpha \circ D = D \odot D \cdot D \odot D \odot D \cdot D$$

$$= \begin{array}{cccccccccc}
 & & & & & & & & 6 & 6 \\
 & & & & & & & & 6 & 6 \\
 & & & & & & & 5 & 5 & \\
 & & & & & 4 & 4 & 5 & 5 & \\
 & & & 3 & 3 & 4 & 4 & & & \\
 & & & 3 & 3 & & & & & \\
 & & 2 & 2 & & & & & & \\
1 & 1 & 2 & 2 & & & & & & \\
1 & 1 & & & & & & & &
\end{array}$$

where we have used numbers to distinguish between copies of D.

It is easily seen that when $D = \beta$ is a ribbon, then $\alpha \circ \beta$ is also a ribbon, and agrees with the definition in [2].

Similarly, given a skew diagram D and a ribbon β, we can also define $D \circ \beta$ as follows. Create a copy $\beta^{(i)}$ of the ribbon β for each of the cells of D, numbered $i = 1, 2, \ldots, n$ arbitrarily. Then assemble the diagrams $\beta^{(i)}$ into a disjoint decomposition of $D \circ \beta$ by translating them in the plane, in such a way that $\beta^{(i)} \sqcup \beta^{(j)}$ forms a copy of

$$\begin{cases} \beta^{(i)} \odot \beta^{(j)} & \text{if } i \text{ is just left of } j \text{ in some row of } D, \\ \beta^{(i)} \cdot \beta^{(j)} & \text{if } i \text{ is just below } j \text{ in some column of } D. \end{cases} \tag{2.5}$$

For example, if

$$D = \begin{array}{ccc} & 1 & 2 \\ 3 & 4 & 5 \end{array}, \qquad \beta = \begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \end{array}$$

then $D \circ \beta$ is the skew diagram

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & & & 2 & 2 & 2 \\
 & & & & & & & & 1 & 1 & 1 & 2 & 2 & & \\
 & & & & & & & 1 & 1 & 5 & 5 & 5 & & & \\
 & & & & & 4 & 4 & 4 & 5 & 5 & & & & & \\
 & 3 & 3 & 3 & 4 & 4 & & & & & & & & & \\
3 & 3 & & & & & & & & & & & & &
\end{array}$$

where we have used numbers to distinguish between copies of β. One must check that the local constraints defining $D \circ \beta$ given in (2.5) are indeed simultaneously satisfiable globally, and hence that $D \circ \beta$ is well-defined. For this it suffices to check the case $D = \lambda = (2, 2)$, which we leave to the reader as an easy exercise. Again it is clear that when $D = \alpha$ is a ribbon, then $\alpha \circ \beta$ is another ribbon agreeing with that in [2]. The following distributivity properties should also be clear.

Proposition 2.1. *For skew diagrams D, D_1, D_2 and ribbons α and β the operation $\circ$ distributes over $\cdot$ and $\odot$, that is*

$$(\alpha \cdot \beta) \circ D = (\alpha \circ D) \cdot (\beta \circ D),$$
$$(\alpha \odot \beta) \circ D = (\alpha \circ D) \odot (\beta \circ D),$$

and

$$(D_1 \cdot D_2) \circ \beta = (D_1 \circ \beta) \cdot (D_2 \circ \beta),$$
$$(D_1 \odot D_2) \circ \beta = (D_1 \circ \beta) \odot (D_2 \circ \beta).$$

Remark 2.2. Observe that $D_1 \circ D_2$ has not been defined for both D_1 and D_2 being non-ribbons, as certain difficulties arise. We invite the reader to investigate this already in the case where D_1, D_2 are both equal to the smallest non-ribbon, namely the 2×2 rectangular Ferrers diagram $\lambda = (2, 2)$, in order to appreciate these difficulties; see also Remark 7.10 below.

3. The ring of symmetric functions

We now recall the ring of symmetric functions Λ, and some of its polynomial generators and bases. Further details can be found in the excellent texts [10,13,16].

The ring Λ is the subalgebra of the formal power series $\mathbb{Z}[\![x_1, x_2, \ldots]\!]$ in countably many variables, consisting of those series f that are of bounded degree in the x_i, and invariant under all permutations of the variables. If Λ^n denotes the symmetric functions that are homogeneous of degree n, then we have an abelian group direct sum decomposition $\Lambda = \bigoplus_{n \geqslant 0} \Lambda^n$. There is a natural $\mathbb{Z}$-basis for Λ^n given by the *monomial symmetric functions* $\{m_\lambda\}_{\lambda \vdash n}$, where m_λ is the formal sum of all monomials that can be permuted to $x^\lambda := x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$.

The fundamental theorem of symmetric functions states that Λ is a polynomial algebra in the *elementary symmetric functions*

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$$

where

$$e_r := \prod_{1 \leqslant i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

It transpires that it is also a polynomial algebra in the *complete homogeneous symmetric functions*

$$h_r := \prod_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r} x_{i_1} x_{i_2} \cdots x_{i_r},$$

and the map $\omega : \Lambda \to \Lambda$ mapping $e_r \mapsto h_r$ is an involution. To obtain $\mathbb{Z}$-bases for Λ, define for partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_\ell},$$
$$h_\lambda := h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

From here a consequence of the fundamental theorem is that Λ^n has as a $\mathbb{Z}$-basis either $\{e_\lambda\}_{\lambda \vdash n}$ or $\{h_\lambda\}_{\lambda \vdash n}$.

4. Schur and skew Schur functions

This section reviews some definitions of Schur functions $\{s_\lambda\}_{\lambda \vdash n, n \geqslant 0}$ and skew Schur functions that will be useful.

4.1. Tableaux

One way to define the (skew) Schur function s_D for a (skew) diagram D involves tableaux. A *column-strict* (or *semistandard*) *tableau* of *shape* D is a *filling* $T : D \to \{1, 2, \ldots\}$ of the cells of D with positive integers such that the numbers

(i) weakly increase left-to-right in each row,
(ii) strictly increase top-to-bottom down each column.

The (*skew*) *Schur function* s_D is then

$$s_D := \sum_T x^T \tag{4.1}$$

where the sum ranges over all column-strict tableaux of shape D, and

$$x^T := \prod_{(i,j) \in D} x_{T(i,j)}.$$

If D is a ribbon we call s_D a *ribbon Schur function*. That (skew) Schur functions are symmetric follows from the definition

$$s_D = \sum_\mu K_{D,\mu} m_\mu. \tag{4.2}$$

Here $K_{D,\mu}$ is the *Kostka number*, which is a number of column-strict tableaux of shape D and *content* μ, that is, having μ_i occurrences of i for each i. From the definition (4.1), one of the most basic syzygies [10, Chapter 1.5, Example 21, part (a)] among skew Schur functions follows immediately.

Proposition 4.1. *If D_1 and D_2 are skew diagrams then*

$$s_{D_1} s_{D_2} = s_{D_1 \cdot D_2} + s_{D_1 \odot D_2}.$$

Proof. Given a pair (T_1, T_2) of column-strict tableaux of shapes (D_1, D_2), let a_1 be the northeasternmost entry of T_1 and a_2 the southwesternmost entry of T_2. Then either

- $a_1 > a_2$, and hence (T_1, T_2) concatenates to make a column-strict tableaux of shape $D_1 \cdot D_2$, or
- $a_2 \geqslant a_1$, and hence (T_1, T_2) near-concatenates to make a column-strict tableaux of shape $D_1 \odot D_2$. □

4.2. The Jacobi–Trudi determinant and the infinite Toeplitz matrix

Skew Schur functions turn out to be the nonzero minor subdeterminants in certain Toeplitz matrices. Consider the sequence $\mathbf{h} := (h_0(=1), h_1, h_2, \ldots)$ and its *Toeplitz matrix*, the infinite matrix

$$T := (t_{ij})_{i,j \geqslant 0} := (h_{j-i})_{i,j \geqslant 0}$$

with the convention that $h_r = 0$ for $r < 0$. The *Jacobi–Trudi determinant* formula for the skew Schur function $s_{\lambda/\mu}$ asserts that

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{\ell(\lambda)}. \tag{4.3}$$

This can be reinterpreted as follows: the square submatrix of the Toeplitz matrix T having row indices $i_1 < \cdots < i_m$ and column indices $j_1 < \cdots < j_m$ has determinant equal to the skew Schur function s_D for $D = \lambda/\mu$ where for $r = 1, 2, \ldots, m$

$$\begin{aligned} \lambda_r &:= j_m - i_r - m + r, \\ \mu_r &:= j_m - j_r - m + r. \end{aligned} \tag{4.4}$$

In particular, if for some r one has $\lambda_r < \mu_r$, then this determinant will be zero.

We remark here that transposing a skew diagram D corresponds to the involution ω on Λ that exchanges e_r and h_r for all r, that is

$$\omega(s_D) = s_{D^t}. \tag{4.5}$$

As a consequence, there is a *dual Jacobi–Trudi determinant* that is obtained by applying ω to (4.3), which expresses s_D as a polynomial in the elementary symmetric functions e_r.

4.3. The Hamel–Goulden determinant

One can view the Jacobi–Trudi determinant (or its dual) as expressing a skew Schur function in terms of skew Schur functions of particular shapes, namely shapes consisting of a single row (respectively a single column), since by the definition (4.1) $h_r = s_r$ (respectively $e_r = s_{1^r}$). There are other such determinantal formulae for Schur and skew Schur functions such as the *Giambelli determinant* involving hook shapes, the *Lascoux–Pragacz determinant* involving ribbons [9], and most generally the *Hamel–Goulden determinant* [7]. We review this last determinant here, using the reformulation involving the notion of a *cutting strip* due to Chen, Yan and Yang [4].

Given a skew diagram D, an *outside* (*border strip*) *decomposition* is an ordered decomposition $\Pi = (\theta_1, \ldots, \theta_m)$ of D, where each θ_k is a ribbon whose southwesternmost (respectively northeasternmost) cell lies either on the left or bottom (respectively right or top) perimeter of D. Having fixed an outside decomposition Π of D, we can determine for each cell x in D, lying in one of the ribbons θ_k, whether x *goes up* or *goes right* in Π:

- It goes up if the cell immediately north of x lies in the same ribbon θ_k, or if x is the northeasternmost cell of θ_k and lies on the top perimeter of D.

- It goes right if the cell immediately east of x lies in the same ribbon θ_k, or if x is the north-easternmost cell of θ_k and lies on the right perimeter of D.

A basic fact about outside decompositions Π is that cells in the same diagonal within D will either all go up or all go right with respect to Π. One can thus define the *cutting strip* $\theta(\Pi)$ for Π to be the unique ribbon occupying the same nonempty diagonals as D, such that the cell in a given diagonal goes up/right exactly as the cells of D all do with respect to Π. Observe that each ribbon θ_k can be identified naturally with a subdiagram of the cutting strip $\theta(\Pi)$, and hence is uniquely determined by the interval of contents $[p(\theta_k), q(\theta_k)]$ that its cells occupy. In this way we can identify intervals $[p, q]$ with subribbons $\theta[p, q]$ of the cutting strip $\theta(\Pi)$, where we adopt the conventions that

- $\theta[q+1, q]$ represents the empty ribbon $\emptyset$, having corresponding skew Schur function $s_\emptyset := 1$, and
- $\theta[p, q]$ is *undefined* when $p > q+1$, and has corresponding skew Schur function $s_{\theta[p,q]} = 0$.

Using these conventions, define a new ribbon

$$\theta_i \# \theta_j := \theta\big[p(\theta_j), q(\theta_i)\big]$$

inside the cutting strip $\theta(\Pi)$. Then the *Hamel–Goulden determinant* formula asserts that

Theorem 4.2. *(See [7].) For any outside decomposition $\Pi = (\theta_1, \ldots, \theta_m)$ of a skew diagram D*

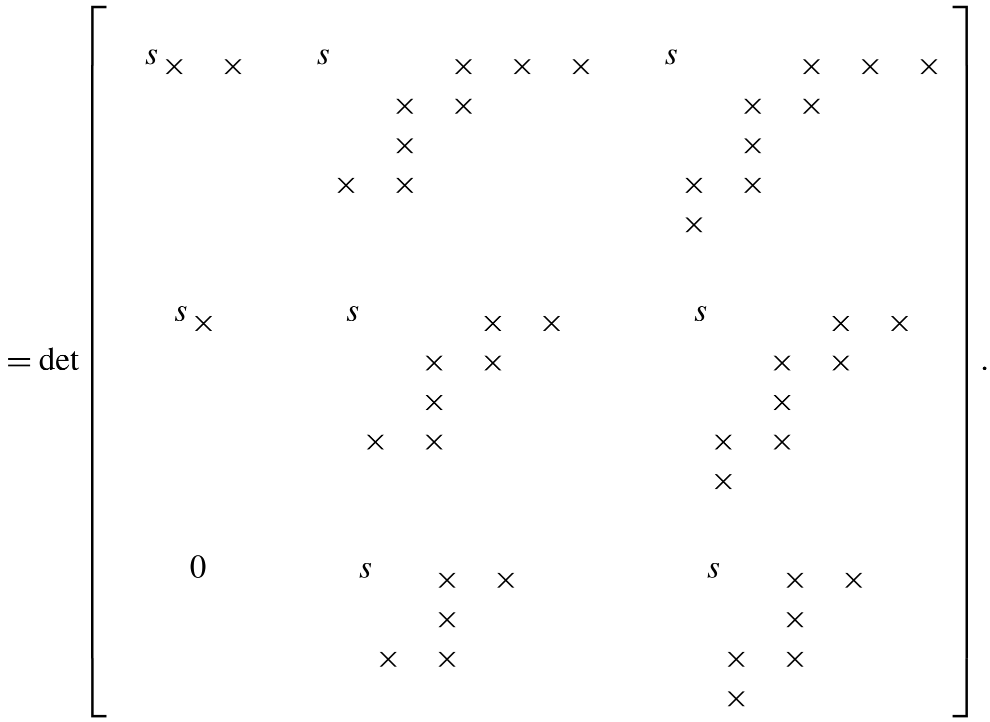
$$s_D = \det(s_{\theta_i \# \theta_j})_{i,j=1}^m.$$

Example 4.3. Consider the following skew diagram D, whose southwesternmost cell is assumed to be (1, 1) with content 0, and outside decomposition $\Pi = (\theta_1, \theta_2, \theta_3)$ where the cells in θ_i are labelled by i. Observe the associated cutting strip $\theta(\Pi)$, and the identification of the ribbons θ_k with intervals of contents within $\theta(\Pi)$:

$$D = \begin{array}{ccccc} & & & 1 & 1 \\ & 3 & 3 & 2 & 2 \\ & 3 & 2 & 2 & \\ 3 & 3 & 2 & & \\ 3 & 2 & 2 & & \end{array}, \qquad \theta(\Pi) = \begin{array}{ccccc} & & \times & \times & \times \\ & \times & \times & & \\ & \times & & & \\ \times & \times & & & \\ \times & & & & \end{array}, \qquad \begin{array}{l} \theta_1 \leftrightarrow \theta[7,8], \\ \theta_2 \leftrightarrow \theta[1,7], \\ \theta_3 \leftrightarrow \theta[0,5]. \end{array}$$

The associated Hamel–Goulden determinant is

$$s_D = \det \begin{bmatrix} s_{\theta[7,8]} & s_{\theta[1,8]} & s_{\theta[0,8]} \\ s_{\theta[7,7]} & s_{\theta[1,7]} & s_{\theta[0,7]} \\ s_{\theta[7,5]} & s_{\theta[1,5]} & s_{\theta[0,5]} \end{bmatrix}$$



There are two particular canonical outside decompositions of a connected skew diagram that will play an important role later.

Definition 4.4. Given a connected skew diagram D, the *southeast decomposition* is the following decomposition into ribbons, which is unique up to reordering. The first ribbon θ starts at the cell on the lower left, traverses the *southeast* border of D, and ends at the cell on the upper right. Now consider D with θ removed, which may decompose into several connected component skew diagrams, and iterate the above procedure on each of these shapes in any order. The *northwest decomposition* is similarly defined, starting with a ribbon θ that traverses the *northwest* border of D.

Example 4.5. For the following skew diagram D, there are four ribbons θ_1, θ_2, θ_3, θ_4 in its southeast decomposition, indicated by the numbers 1, 2, 3, 4, respectively:

$$D = \begin{array}{cccccccccccccccc}
 & & & & & & & & 4 & 4 & 3 & 3 & 3 & 1 & 1 & 1 \\
 & & & & & & & & 3 & 3 & 3 & 1 & 1 & 1 & & \\
 & & & & & & & 3 & 3 & 1 & 1 & 1 & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & & \\
 & & 2 & 2 & 2 & 1 & 1 & 1 & & & & & & & & \\
 & 2 & 2 & 1 & 1 & 1 & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & &
\end{array}.$$

Here the first and largest ribbon $\theta = \theta_1 = (2, 3, 3, 3, 3, 3, 3, 3)$.

Note that for any connected skew diagram D, both the southeast decomposition and the northwest decomposition are outside decompositions of D, and hence give rise to Hamel–Goulden determinants for s_D. In both cases, the associated cutting strip for this outside decomposition coincides with its first and largest ribbon θ.

5. The Littlewood–Richardson rule

The Littlewood–Richardson rule gives the unique expansion of the skew Schur function $s_{\lambda/\mu}$ into Schur functions s_ν for partitions ν, and has many equivalent versions. We will use here a version suited to our purposes, which is known to be equivalent to Zelevinsky's *picture* formulation [19] of the rule.

Definition 5.1. Given a skew diagram D, let its *row filling* $T_{\mathrm{row}}(D)$ be the function from cells of D to the integers which assigns to a cell its row index.

Say that a column-strict tableau T is a *picture* for D if

(i) the content of T is the same as that of $T_{\mathrm{row}}(D)$, and
(ii) the map f from cells of D to cells of T, defined by sending the kth cell from the *right* end of row r of D to the kth occurrence of the entry r from the *left* in T, enjoys this additional property: if a cell x lies lower in the same column of D as some cell x', then $f(x)$ lies in a lower row of T than $f(x')$ (but not necessarily in the same column).

Denote by $\mathrm{Pictures}(D)$ the set of all column-strict tableaux that are pictures for D. Given a column-strict tableau T, let $\lambda(T)$ denote the partition that gives its shape.

Theorem 5.2 *(Littlewood–Richardson rule).*

$$s_D = \sum_{T \in \mathrm{Pictures}(D)} s_{\lambda(T)}. \tag{5.1}$$

Example 5.3. Consider the following skew diagram D, and its row filling $T_{\mathrm{row}}(D)$:

$$D = \begin{array}{cccc} & & & \times \\ & & & \times \\ & \times & \times & \\ \times & \times & \times & \\ \times & \times & & \end{array}, \qquad T_{\mathrm{row}}(D) = \begin{array}{cccc} & & & 1 \\ & & & 2 \\ & 3 & 3 & \\ 4 & 4 & 4 & \\ 5 & 5 & & \end{array}.$$

Then one has

$$\mathrm{Pictures}(D) = \left\{ \begin{array}{lll} 1&3&4\\ 2&4&\\ 3&5&\\ 4&&\\ 5&& \end{array},\ \begin{array}{lll} 1&3&4\\ 2&4&5\\ 3&&\\ 4&&\\ 5&& \end{array},\ \begin{array}{lll} 1&3&4\\ 2&4&5\\ 3&5&\\ 4&& \end{array},\ \begin{array}{lll} 1&3&3\\ 2&4&4\\ 4&5&\\ 5&& \end{array},\ \begin{array}{lll} 1&3&3\\ 2&4&4\\ 4&5&5 \end{array}, \right.$$
$$\left. \begin{array}{llll} 1&3&3&4\\ 2&4&&\\ 4&5&&\\ 5&&& \end{array},\ \begin{array}{llll} 1&3&3&4\\ 2&4&5&\\ 4&&&\\ 5&&& \end{array},\ \begin{array}{llll} 1&3&3&4\\ 2&4&5&\\ 4&5&& \end{array},\ \begin{array}{llll} 1&3&3&4\\ 2&4&4&\\ 5&5&& \end{array},\ \begin{array}{llll} 1&3&3&4\\ 2&4&4&5\\ 5&&& \end{array} \right\}.$$

Consequently the Littlewood–Richardson rule says

$$s_D = s_{(3,2,2,1,1)} + s_{(3,3,1,1,1)} + 2s_{(3,3,2,1)} + s_{(3,3,3)} \\ + s_{(4,2,2,1)} + s_{(4,3,1,1)} + 2s_{(4,3,2)} + s_{(4,4,1)}.$$

6. Reduction to connected diagrams

We are now ready to state our key definition.

Definition 6.1. Given two skew diagrams D_1 and D_2, say that they are *skew-equivalent*, denoted $D_1 \sim D_2$, if $s_{D_1} = s_{D_2}$.

The goal of this section is to understand two reductions:

(A) Understanding all binomial syzygies among the skew Schur functions is equivalent to understanding the equivalence relation $\sim$ on all skew diagrams, and
(B) the latter is equivalent to understanding $\sim$ among *connected* skew diagrams.

Both of these reductions will follow from some simple observations about the matrix

$$JT(\lambda/\mu) := (h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{\ell(\lambda)},$$

which appears in the Jacobi–Trudi determinant (4.3) for a skew diagram λ/μ.

Proposition 6.2. *Let λ/μ be a skew diagram with $\ell := \ell(\lambda)$.*

(i) *The largest subscript k occurring on any nonzero entry h_k in the Jacobi–Trudi matrix $JT(\lambda/\mu)$ is*

$$L := \lambda_1 + \ell - 1$$

and this subscript occurs exactly once, on the $(1, \ell)$-entry h_L.

(ii) *The subscripts on the diagonal entries in $JT(\lambda/\mu)$ are exactly the row lengths*

$$(r_1, \dots, r_\ell) := (\lambda_1 - \mu_1, \dots, \lambda_\ell - \mu_\ell)$$

and the monomial $h_{r_1} \cdots h_{r_\ell}$ occurs in the determinant s_D
  (a) *with coefficient $+1$, and*
  (b) *as the monomial whose subscripts rearranged into weakly decreasing order give the smallest partition of $|\lambda/\mu|$ in dominance order among all nonzero monomials.*

(iii) *The subscripts on the nonzero subdiagonal entries in $JT(\lambda/\mu)$ are exactly one less than the adjacent row overlap lengths:*

$$(\lambda_2 - \mu_1, \lambda_3 - \mu_2, \dots, \lambda_\ell - \mu_{\ell-1}).$$

Proof. Assertion (i) follows since the subscripts appearing on nonzero entries in $JT(\lambda/\mu)$ are of the form $\lambda_i - \mu_j - i + j$ with

$$\lambda_i \leqslant \lambda_1, \quad \mu_j \geqslant 0, \quad i \geqslant 1, \ j \leqslant \ell$$

so that

$$\lambda_i - \mu_j - i + j \leqslant \lambda_1 - 0 - 1 + \ell = L.$$

Furthermore, equality can occur only if $i = 1$ and $j = \ell$.

For assertion (ii), expand the determinant of $JT(\lambda/\mu)$ as a signed sum over of permutations in $\mathfrak{S}_\ell$. We claim that only the identity permutation gives rise to the monomial $h_{r_1} \cdots h_{r_\ell}$. This is because any other permutation σ can be obtained from the identity by a sequence of transpositions each increasing the number of inversions, and it is straightforward to check that any such transpositions alters the corresponding monomial so as to make its subscript sequence go *strictly* upwards in the dominance order on partitions of $|\lambda/\mu|$.

Assertion (iii) is straightforward from the definitions, noting that $\lambda_{i+1} - \mu_i$ is indeed the number of columns of overlap between row i and row $i+1$ in the skew diagram. □

Corollary 6.3. *For a disconnected skew diagram $D = D_1 \oplus D_2$, one has the factorization $s_D = s_{D_1} s_{D_2}$. For a connected skew diagram D, the polynomial s_D is irreducible in $\mathbb{Z}[h_1, h_2, \ldots]$.*

Proof. The first assertion of the proposition is well-known, and follows, for example, immediately from the definition (4.1) of s_D using tableaux.

For the second assertion, let $D = \lambda/\mu$ with $\ell := \ell(\lambda)$ and $L := \lambda_1 + \ell - 1$. Then the Jacobi–Trudi determinant (4.3) and Proposition 6.2(i) imply that the expansion of s_D as a polynomial in the h_r is of the form

$$s \cdot h_L + r \tag{6.1}$$

where s, r are polynomials containing no occurrences of h_L. Proposition 6.2(ii) implies that r is not the zero polynomial, as r must contain the monomial $h_{r_1} \cdots h_{r_\ell}$ with coefficient $+1$ where $r_1, \ldots, r_l$ are the lengths of the rows of λ/μ. We wish to show that s is also nonzero, since then Eq. (6.1) would exhibit s_D as a linear polynomial in h_L with nonzero constant term, and hence clearly irreducible in $\mathbb{Z}[h_1, h_2, \ldots]$.

Note that $s = \det M$ where M is the $(\ell - 1) \times (\ell - 1)$ complementary minor to h_L in $JT(\lambda/\mu)$. Thus M is itself a square submatrix of the Toeplitz matrix, and thus a Jacobi–Trudi determinant for some pair of partitions $\hat{\lambda}$ and $\hat{\mu}$ as defined in Eqs. (4.4). To see that this Toeplitz minor M has nonzero determinant, note that since D is connected, adjacent rows of D have at least one column of overlap, and thus the subscripts on the diagonal entries in M are all nonnegative by Proposition 6.2(iii). However, this implies $\hat{\lambda}_i \geqslant \hat{\mu}_i$ for $i = 1, 2, \ldots, \ell - 1$, so that $\hat{\mu} \subseteq \hat{\lambda}$ and hence

$$s = \det M = s_{\hat{\lambda}/\hat{\mu}} \neq 0. \qquad \square$$

We can now infer reductions (A) and (B) from the beginning of the section. Given a binomial syzygy

$$c s_{D_1} s_{D_2} \cdots s_{D_m} - c' s_{D'_1} s_{D'_2} \cdots s_{D'_m} = 0$$

among the skew Schur functions, with coefficients c, c' in any ring, the first assertion of Corollary 6.3 allows one to rewrite this as $c s_D = c' s_{D'}$, where

$$D := D_1 \oplus D_2 \oplus \cdots \oplus D_m,$$
$$D' := D'_1 \oplus D'_2 \oplus \cdots \oplus D'_m.$$

Proposition 6.2(ii) implies the unitriangular expansion

$$s_D = h_\rho + \sum_{\mu:\ \mu >_{\mathrm{dom}} \rho} c_\mu h_\mu$$

in which ρ is the weakly decreasing rearrangement of the row lengths in D. As $s_{D'}$ has a similar expansion, this forces $c = c'$ above, and hence $s_D = s_{D'}$. That is, $D \sim D'$, achieving reduction (A).

For reduction (B), use the fact that $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ is a unique factorization domain, along with Corollary 6.3.

7. Sufficient conditions

The most basic skew-equivalence is the following well-known fact.

Proposition 7.1. *(See [16, Exercise 7.56(a)].) If D is a skew diagram then $D \sim D^*$, where D^* is the antipodal rotation of D.*

Recently it was also proved that

Theorem 7.2. *(See [2, Theorem 4.1].) Two ribbons β and γ satisfy $\beta \sim \gamma$ if and only if for some k*

$$\beta_1 \circ \cdots \circ \beta_k \sim \gamma_1 \circ \cdots \circ \gamma_k,$$

where for each i either $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$.

It transpires that there are several other constructions and operations on skew diagrams that give rise to more skew-equivalences.

7.1. Composition with ribbons

We now show that the notation for the diagrammatic operations $\alpha \circ D$ and $D \circ \beta$ defined in Section 2.2 are consistent with algebraic operations on skew Schur functions s_D. These operations then lead to nontrivial skew-equivalences.

We begin by reviewing the presentation of the ring Λ of symmetric functions by the generating set of ribbon Schur functions s_α. Let $\mathcal{Q}[z_\alpha]$ denote a polynomial algebra in infinitely many variables z_α indexed by all compositions α.

Proposition 7.3. *(See [2, Proposition 2.2].) The algebra homomorphism*

$$\mathcal{Q}[z_\alpha] \to \Lambda$$
$$z_\alpha \mapsto s_\alpha$$

is a surjection, whose kernel is the ideal generated by the relations

$$z_\alpha z_\beta - (z_{\alpha\cdot\beta} + z_{\alpha\odot\beta}). \tag{7.1}$$

Corollary 7.4. *For a fixed skew diagram D the map*

$$\mathcal{Q}[z_\alpha] \xrightarrow{(-)\circ s_D} \Lambda$$
$$z_\alpha \mapsto s_{\alpha\circ D}$$

descends to a well-defined algebra map $\Lambda \to \Lambda$. In other words, for any symmetric function f, one can arbitrarily write f as a polynomial in ribbon Schur functions $f = p(s_\alpha)$ and then set $f \circ s_D := p(s_{\alpha\circ D})$.

Proof. The relation (7.1) maps under $(-)\circ s_D$ to

$$\begin{aligned}&s_{\alpha\circ D} s_{\beta\circ D} - (s_{(\alpha\cdot\beta)\circ D} + s_{(\alpha\odot\beta)\circ D})\\ &\quad = s_{\alpha\circ D} s_{\beta\circ D} - (s_{(\alpha\circ D)\cdot(\beta\circ D)} + s_{(\alpha\circ D)\odot(\beta\circ D)})\end{aligned}$$

using Proposition 2.1. This last expression is zero by Proposition 4.1. □

We should point out that the notation $f \mapsto f \circ s_D$ has already been used in [2] to denote the *plethysm* or *plethystic composition*, following one of the standard references [10]. We will instead use the notation $f \mapsto f[s_\alpha]$ for plethysm, freeing the symbol $\circ$ for use in the map $f \mapsto f \circ s_D$ defined in Corollary 7.4. Note that we are abusing notation by using $\circ$ both for the map $(-)\circ s_D$ on symmetric functions, as well as the two diagrammatic operations $\alpha \circ D$ and $D \circ \beta$. The previous corollary says that it is well-defined to set

$$s_\alpha \circ s_D = s_{\alpha\circ D} \tag{7.2}$$

so that we are at least consistent with one of the diagrammatic operations. The next result says that we are also consistent with the other.

Proposition 7.5. *For any skew diagram D and ribbon β*

$$s_{D\circ\beta} = s_D \circ s_\beta.$$

Proof. Pick an outside decomposition $\Pi = (\theta_1, \ldots, \theta_m)$ of D, with cutting strip $\theta(\Pi)$, so that Theorem 4.2 asserts $s_D = \det(s_{\theta_i \# \theta_j})_{i,j=1}^m$. It follows from the definition of $D \circ \beta$ and the definition of outside decomposition that

- $\Pi \circ \beta := (\theta_1 \circ \beta, \ldots, \theta_m \circ \beta)$ gives an outside decomposition for $D \circ \beta$ and consequently from the definition of cutting strip
- that the cutting strip satisfies the formula

$$\theta(\Pi \circ \beta) = \theta(\Pi) \circ \beta.$$

- Moreover, the relevant subribbons of this cutting strip satisfy the commutation

$$(\theta_i \circ \beta)\#(\theta_j \circ \beta) = (\theta_i \# \theta_j) \circ \beta.$$

Consequently,

$$\begin{aligned} s_{D\circ\beta} &= \det[s_{(\theta_i\circ\beta)\#(\theta_j\circ\beta)}]_{i,j=1}^m \\ &= \det[s_{(\theta_i\#\theta_j)\circ\beta}]_{i,j=1}^m \\ &= \det[s_{\theta_i\#\theta_j}]_{i,j=1}^m \circ s_\beta \\ &= s_D \circ s_\beta, \end{aligned}$$

where the third equality follows from Corollary 7.4. □

We are now ready to state the first of two main ways to create new skew-equivalences from known ones.

Theorem 7.6. *Assume one has ribbons α, α' and skew diagrams D, D' satisfying $\alpha \sim \alpha'$ and $D \sim D'$. Then*

(i) $\alpha \circ D \sim \alpha' \circ D$,
(ii) $D \circ \alpha \sim D' \circ \alpha$,
(iii) $D \circ \alpha \sim D \circ \alpha'$, *and*
(iv) $\alpha \circ D \sim \alpha \circ D^*$.

Proof. Assertions (i) and (ii) both follow from the fact that if E is any skew diagram, then $D \sim D'$ means $s_D = s_{D'}$, and hence

$$s_D \circ s_E = s_{D'} \circ s_E. \tag{7.3}$$

Note that if D, D' happen to be ribbons α, α', then this gives the middle equality in

$$s_{\alpha\circ E} = s_\alpha \circ s_E = s_{\alpha'} \circ s_E = s_{\alpha'\circ E},$$

while (7.2) gives the outside equalities. This proves $\alpha \circ E \sim \alpha' \circ E$, and hence assertion (i). Similarly, if E happens to be a ribbon α, then (7.3) again gives the middle equality in

$$s_{D\circ\alpha} = s_D \circ s_\alpha = s_{D'} \circ s_\alpha = s_{D'\circ\alpha},$$

and Proposition 7.5 gives the outside equalities. This proves $D \circ \alpha \sim D' \circ \alpha$, and hence assertion (ii).

For assertion (iii), we deduce it first in the special case where the skew diagram D is a ribbon β. It follows then from Theorem 7.2. This characterization asserts that $\alpha \sim \alpha'$ for two ribbons α, α' if and only if there are expressions

$$\begin{aligned} \alpha &= \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_r, \\ \alpha' &= \delta_1 \circ \delta_2 \circ \cdots \circ \delta_r \end{aligned} \tag{7.4}$$

in which for each i one has that γ_i, δ_i are ribbons with either $\gamma_i = \delta_i$ or $\gamma_i = \delta_i^*$. Composing the expressions in (7.4) with β leads to similar such expressions for $\beta \circ \alpha, \beta \circ \alpha'$, and hence $\beta \circ \alpha \sim \beta \circ \alpha'$.

With this in hand, assertion (iii) for an arbitrary skew diagram D is deduced as follows. Arbitrarily express $s_D = p(s_\beta)$ as a polynomial in various ribbon Schur functions s_β. One then has the following string of equalities:

$$s_{D\circ\alpha} \stackrel{1}{=} s_D \circ s_\alpha \stackrel{2}{=} p(s_\beta) \circ s_\alpha \stackrel{3}{=} p(s_{\beta\circ\alpha}) \stackrel{4}{=} p(s_{\beta\circ\alpha'}) \stackrel{5}{=} p(s_\beta) \circ s_{\alpha'} \stackrel{6}{=} s_D \circ s_{\alpha'} \stackrel{7}{=} s_{D\circ\alpha'}.$$

Here the equalities $\stackrel{1}{=}$ and $\stackrel{7}{=}$ use Proposition 7.5, the equalities $\stackrel{2}{=}$ and $\stackrel{6}{=}$ use the expression

$$s_D = p(s_\beta),$$

the equalities $\stackrel{3}{=}$ and $\stackrel{5}{=}$ use Corollary 7.4, and the equality $\stackrel{4}{=}$ uses the special case of (iii) proven in the previous paragraph. Hence $D \circ \alpha \sim D \circ \alpha'$.

Assertion (iv) follows from assertion (i) and Proposition 7.1:

$$\alpha \circ D \sim (\alpha \circ D)^* = \alpha^* \circ D^* \sim \alpha \circ D^*. \qquad \square$$

Remark 7.7. Observe that Theorem 7.6 generalizes [2, Theorem 4.4, parts 1 and 2].

Example 7.8. In general it is not true that $D \sim D'$ implies $\alpha \circ D \sim \alpha \circ D'$. For example, let $D = (4,3,2,1)/(1,1)$ and $D' = (4,3,2,1)/(2)$ and $\alpha = (2)$. Then $D \sim D'$ by Corollary 7.32. However, $(8,7,6,5,3,2,1)/(5,5,4,1,1) = \alpha \circ D \nsim \alpha \circ D' = (8,7,6,5,3,2,1)/(6,4,4,2)$ by Corollary 8.11 below.

Remark 7.9. It was observed in [2, Proposition 3.4] that even though the $\circ$-composition and plethystic composition operations

$$s_\alpha \mapsto s_{\alpha\circ\beta} (= s_\alpha \circ s_\beta),$$
$$s_\alpha \mapsto s_\alpha[s_\beta]$$

are *not* the same, they *do* coincide when one *sums/averages* over all compositions α of a fixed size n:

$$\left(\sum_{\alpha\vDash n} s_\alpha\right) \circ s_\beta = \sum_{\alpha\vDash n} s_{\alpha\circ\beta} = (s_\beta)^n = (s_1^n)[s_\beta] = \left(\sum_{\alpha\vDash n} s_\alpha\right)[s_\beta]$$

in which the second and fourth equalities come from iterating Proposition 4.1. The same holds replacing s_β by s_D for any skew diagram D, with the same proof:

$$\left(\sum_{\alpha\vDash n} s_\alpha\right) \circ s_D = \sum_{\alpha\vDash n} s_{\alpha\circ D} = (s_D)^n = \left(s_1^n\right)[s_D] = \left(\sum_{\alpha\vDash n} s_\alpha\right)[s_D].$$

Remark 7.10. Even though we have not defined a skew diagram $D_1 \circ D_2$ when D_1, D_2 are *both* non-ribbon skew diagrams, the symmetric function $s_{D_1} \circ s_{D_2}$ is still well-defined, via Corollary 7.4. One might ask whether there exists a skew diagram D playing the role of $D_1 \circ D_2$, that is, with $s_D = s_{D_1} \circ s_{D_2}$. Curiously and suggestively, computer calculations show that this seems to be the case in the smallest example, in which D_1, D_2 are both the 2×2 rectangular Ferrers diagram $\lambda = (2, 2)$:

$$s_{(2,2)} \circ s_{(2,2)} = s_{(1)} s_{(5,5,4,4,2)/(3,1,1)}.$$

In other words, $(2, 2) \circ (2, 2)$ cannot be chosen to be a connected skew diagram, but rather should be defined as the direct sum of a single cell with

$$\begin{array}{ccccc} & & & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \\ \times & \times & \times & \times & \\ \times & \times & & & \end{array}.$$

This is somewhat remarkable, and suggests a further avenue of investigation for skew-equivalences, see Section 9 below.

7.2. Amalgamation and amalgamated composition of ribbons

In this section we introduce an operation $\alpha \circ_\omega D$ for certain skew diagrams D and ribbons ω, which we will call the *amalgamated composition* of α and D with respect to ω. It is analogous to the operation $\alpha \circ \beta$ on ribbons α, β and allows us to identify more skew diagrams that are skew-equivalent.

Definition 7.11. Given a skew diagram D and a nonempty ribbon ω, say that ω *protrudes from the top* (*respectively bottom*) *of* D if the restriction of D to its $|\omega|$ northeasternmost (respectively southwesternmost) diagonals is the ribbon ω and the restriction of D to its $|\omega| + 1$ northeasternmost (respectively southwesternmost) diagonals is also a ribbon.

Given two skew diagrams D_1, D_2 and a nonempty ribbon ω protruding from the top of D_1 and the bottom of D_2, the *amalgamation of* D_1 *and* D_2 *along* ω, denoted $D_1 \amalg_\omega D_2$, is the new skew diagram obtained from the disjoint union $D_1 \oplus D_2$ by identifying the copy of ω in the northeast of D_1 with the copy of ω in the southwest of D_2.

Example 7.12. Consider the skew diagram

$$D = \begin{array}{cccc} & \times & \times & \times \\ \times & \times & \times & \end{array}.$$

Then D has $\omega = \times$ protruding from the top and bottom. Furthermore,

$$D \amalg_\omega D = \begin{array}{ccccccc} & & & & \times & \times & \times \\ & \times & \times & o & \times & \times & \\ \times & \times & \times & & & & \end{array}$$

and the copies of ω that have been amalgamated are indicated with the letter o.

Definition 7.13. When ω protrudes from the top of D_1 and bottom of D_2, one can form the *outer* (*respectively inner*) *projection of* D_1 *onto* D_2 *with respect to* ω. This is a new diagram in the plane, not necessarily skew, obtained from the disjoint union $D_1 \oplus D_2$ by translating D_2 and D_1

until the two copies of ω in D_1, D_2 are adjacent and occupy the same set of diagonals, and the copy of ω in D_1 is immediately northwest (respectively southeast) of ω in D_2.

Observe that at most one of these two projections can be a skew diagram (and possibly neither one is). When one of them is a skew diagram, denote it by $D_1 \cdot_\omega D_2$, and say that $D_1 \cdot_\omega D_2$ *is defined* in this case.

Example 7.14. Let D, ω be as in the previous example. Then the outer and inner projections of D onto D with respect to ω are

$$\begin{array}{cccc} & \times & \times & o \\ \times & \times & \times & \end{array} \qquad \begin{array}{cccc} & \times & \times & \times \\ o & \times & \times & \end{array} \qquad\qquad \begin{array}{cccccc} & & & \times & \times & \times \\ & & o & \times & \times & \\ & \times & \times & o & & \\ \times & \times & \times & & & \end{array} = D \cdot_\omega D$$

and only the latter is a skew diagram.

Definition 7.15. Given a skew diagram D, and ω a ribbon protruding from both the top and bottom of D, one can define

$$D^{\amalg_\omega n} = \underbrace{D \amalg_\omega D \amalg_\omega \cdots \amalg_\omega D}_{n \text{ factors}} := \big((D \amalg_\omega D) \amalg_\omega D\big) \amalg_\omega \cdots \amalg_\omega D.$$

If one assumes that $D \cdot_\omega D$ is also defined then by considering the northeasternmost copy of D in $D^{\amalg_\omega m}$ and the southwesternmost copy of D in $D^{\amalg_\omega n}$ for any positive integers m, n, we have $(D^{\amalg_\omega m}) \cdot_\omega (D^{\amalg_\omega n})$ is also defined. Under this assumption, for any ribbon $\alpha = (\alpha_1, \dots, \alpha_\ell)$, define the *amalgamated composition* of α and D with respect to ω to be the diagram

$$\alpha \circ_\omega D := \big(D^{\amalg_\omega \alpha_1}\big) \cdot_\omega \cdots \cdot_\omega \big(D^{\amalg_\omega \alpha_\ell}\big). \tag{7.5}$$

Example 7.16. Let D, ω be as in the previous example. Then we saw earlier that $D \cdot_\omega D$ is defined. Consider the ribbon

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = (2, 1, 3) = \begin{array}{cccc} & \times & \times & \times \\ & \times & & \\ \times & \times & & \end{array}.$$

Then one has

$$\begin{aligned} \alpha \circ_\omega D &= \big(D^{\amalg_\omega 2}\big) \cdot_\omega \big(D^{\amalg_\omega 1}\big) \cdot_\omega \big(D^{\amalg_\omega 3}\big) \\ &= \begin{array}{ccccccccccccccccc} & & & & & & & & & & & & & & 3 & 3 & 3 \\ & & & & & & & & & & & 3 & 3 & 3 & 3 & 3 & \\ & & & & & & & & 3 & 3 & 3 & 3 & 3 & & & & \\ & & & & & & & 3 & 3 & 3 & & & & & & & \\ & & & & & & 2 & 2 & 2 & & & & & & & & \\ & & & & & 2 & 2 & 2 & & & & & & & & & \\ & & & & 1 & 1 & 1 & & & & & & & & & & \\ & 1 & 1 & 1 & 1 & 1 & & & & & & & & & & & \\ 1 & 1 & 1 & & & & & & & & & & & & & & \end{array}. \end{aligned}$$

We now show that the operation $\alpha \circ_\omega D$ and the $\alpha \circ D$ operation defined in Section 7.1 associate with each other in a natural way.

Proposition 7.17. *When α, β, ω are ribbons and D is a skew diagram such that the appropriate operations are well-defined, one has*

$$(\alpha \circ \beta) \circ_\omega D = \alpha \circ_\omega (\beta \circ_\omega D).$$

Proof. This follows from the definitions since

$$\begin{aligned}(\alpha \circ \beta) \circ_\omega D &= \left(D^{\amalg_\omega \beta_1} \cdot_\omega \cdots \cdot_\omega D^{\amalg_\omega \beta_m}\right)^{\amalg_\omega \alpha_1} \cdot_\omega \cdots \cdot_\omega \left(D^{\amalg_\omega \beta_1} \cdot_\omega \cdots \cdot_\omega D^{\amalg_\omega \beta_m}\right)^{\amalg_\omega \alpha_\ell} \\ &= \alpha \circ_\omega (\beta \circ_\omega D). \qquad \square\end{aligned}$$

We now wish to interpret the diagrammatic operation $\alpha \circ_\omega D$ in terms of an algebraic operation, for certain skew diagrams D and ribbons ω.

Definition 7.18. Suppose that D is a skew diagram and ω a ribbon protruding from the top and bottom of D, so that $D^{\amalg_\omega r}$ is defined for all positive integers r. Define a map of sets

$$\begin{aligned}\Lambda &\xrightarrow{(-)\circ_\omega s_D} \Lambda \\ f &\mapsto f \circ_\omega s_D\end{aligned}$$

as the composite of two maps $\Lambda \to \Lambda[t] \to \Lambda$, which we now describe.

Thinking of Λ as the polynomial algebra $\mathbb{Z}[h_1, h_2, \ldots]$, we can temporarily grade Λ and $\Lambda[t]$ by setting $\deg(t) = \deg(h_r) = 1$ for all r. Note that this is *not* the usual grading on Λ, in which $\deg(h_r) = r$, and for which skew Schur functions s_D are homogeneous. In fact, s_D will generally be *inhomogeneous* with respect to this temporary grading. The first map $\Lambda \to \Lambda[t]$ simply homogenizes a polynomial in the h_rs with respect to this grading, using the variable t as the homogenization variable.

The second map is defined by

$$\begin{aligned}\Lambda[t] &\to \Lambda \\ h_r &\mapsto s_{D^{\amalg_\omega r}} \\ t &\mapsto s_\omega.\end{aligned}$$

Note that this composite map is not a ring homomorphism, nor even a map of $\mathbb{Z}$-modules, because these properties fail for the homogenization map $\Lambda \to \Lambda[t]$.

Before we state the next theorem we need some hypotheses.

Hypotheses 7.19. Suppose that D is a connected skew diagram and ω is a ribbon protruding from the top and bottom of D. We assume that D and ω satisfy the following conditions:

(i) $D \cdot_\omega D$ is defined,

(ii) the two copies of ω protruding from the top and bottom of D are separated by at least one diagonal, that is, there is a nonempty diagonal in D intersecting neither copy of ω.

Theorem 7.20. *Let D be a connected skew diagram, and ω a ribbon satisfying Hypotheses* 7.19. *Then for any ribbon α one has*

$$s_{\alpha \circ_\omega D} = s_\alpha \circ_\omega s_D.$$

Remark 7.21. In Theorem 7.20, some hypothesis about separating the two copies of ω within D is needed, as shown by the following example. Let α be the ribbon $(1, 1, 1)$, let D be the ribbon $(1, 1)$, and ω the single cell (1). In other words, let α, D, ω, respectively, be diagrams that consist of a single column, of sizes 3, 2, 1, respectively.

Then ω protrudes from the top and bottom of D, and one can check that

$$D \cdot_\omega D = \begin{array}{cc} \times & \times \\ \times & \times \end{array} \quad \text{and} \quad \alpha \circ_\omega D = \begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \end{array}$$

are defined. However, the two copies of ω within D occupy adjacent diagonals, so that they fail the separation hypothesis in the theorem. Correspondingly, one finds that

$$\begin{aligned} s_\alpha \circ_\omega s_D &= \det \begin{bmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{bmatrix} \circ_\omega s_D \\ &= \det \begin{bmatrix} s_D & s_{D \amalg_\omega D} & s_{D \amalg_\omega D \amalg_\omega D} \\ s_\omega & s_D & s_{D \amalg_\omega D} \\ 0 & s_\omega & s_D \end{bmatrix} \\ &= s_{\begin{array}{ccc} \times & \times & \times \\ \times & \times & \times \end{array}} - s_{\begin{array}{cc} \times & \times \\ \times & \times \\ \times & \times \end{array}} \\ &\neq s_{\alpha \circ_\omega D}. \end{aligned}$$

Proof of Theorem 7.20. We induct on the number of rows k in the ribbon α. In the base case $k = 1$, by Eq. (4.3) one has $s_\alpha = h_r$ for some r, and the assertion is trivial.

For the inductive step, let

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k), \\ \bar{\alpha} &= (\alpha_2, \alpha_3, \ldots, \alpha_k), \\ \hat{\alpha} &= (\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_k). \end{aligned}$$

Then expanding the Jacobi–Trudi determinant for s_α along its last row gives

$$s_\alpha = h_{\alpha_1} s_{\bar{\alpha}} - 1 \cdot s_{\hat{\alpha}}$$

and hence that

$$\begin{aligned} s_\alpha \circ_\omega s_D &= (h_{\alpha_1} s_{\bar\alpha} - 1 \cdot s_{\hat\alpha}) \circ_\omega s_D \\ &= s_{D^{\amalg_\omega \alpha_1}} (s_{\bar\alpha} \circ_\omega s_D) - s_\omega (s_{\hat\alpha} \circ_\omega s_D) \\ &= s_{D^{\amalg_\omega \alpha_1}} s_{\bar\alpha \circ_\omega D} - s_\omega s_{\hat\alpha \circ_\omega D} \end{aligned} \tag{7.6}$$

where the last equality uses the inductive hypothesis.

We wish to compare this last expression with an expansion for a certain Hamel–Goulden determinant computing $s_{\alpha\circ_\omega D}$. Note that the two copies of ω lying in the top and bottom of D are subribbons of the longest ribbon in the southeast decomposition Π of D, namely the cutting strip $\theta := \theta(\Pi)$. More generally, the two copies of ω in any diagram $D^{\amalg_\omega r}$ are subribbons of the longest ribbon in its southeast decomposition, namely $\theta^{\amalg_\omega r}$. One can then collate these southeast decompositions for $D^{\amalg_\omega \alpha_i}$ to produce an outside decomposition $(\theta_1, \ldots, \theta_n)$ for

$$\alpha \circ_\omega D = D^{\amalg_\omega \alpha_1} \cdot_\omega \cdots \cdot_\omega D^{\amalg_\omega \alpha_k}$$

in which the ribbons come in k different blocks, with those in the jth block comprising the subdiagram $D^{\amalg_\omega \alpha_j}$. Furthermore, because of the separation hypothesis about the two copies of ω in D, ribbons in different blocks will almost never share any nonempty diagonals, as this will only happen for the longest ribbon in two adjacent blocks. For notational purposes below, let m be the number of ribbons in the first block, and index the longest ribbons in the first and second blocks as θ_m and θ_{m+1}.

Let A be the Hamel–Goulden matrix for this outside decomposition of $\alpha \circ_\omega D$. We will do a generalized Laplace expansion [15, §1.8] of its determinant along the first m rows. Given subsets R, C of $[n] := \{1, 2, \ldots, n\}$, let $A_{R,C}$ be the submatrix of A having rows and columns indexed by R and C respectively. Then the generalized Laplace expansion says that

$$\det A = \sum_{\substack{C \subset [n] \\ |C| = m}} \epsilon_C \det(A_{[m],C}) \det(A_{[m+1,n],[n]\setminus C})$$

where $\epsilon_C = \pm 1$ is the sign of the permutation which sorts the concatenation of C and $[n] \setminus C$, both written in increasing order, to the sequence $1, 2, \ldots, n$.

The foregoing observations about separation of diagonals imply that $A_{[m+1,n],[n]\setminus C}$ will have a zero column (and hence vanishing determinant) unless the m-element subset C is chosen to contain all the columns $1, 2, \ldots, m-1$, so that for some $j \in [m, n]$, one has $C = [m-1] \cup \{j\}$ and hence $\epsilon_C = (-1)^{j-m}$. Thus

$$\begin{aligned} s_{\alpha\circ_\omega D} &= \sum_{j=m}^{n} (-1)^{j-m} \det(A_{[m],[m-1]\cup\{j\}}) \det(A_{[m+1,n],[m,n]\setminus\{j\}}) \\ &= s_{D^{\amalg_\omega \alpha_1}} s_{\bar\alpha \circ_\omega D} + \sum_{j=m+1}^{n} (-1)^{j-m} \det(A_{[m],[m-1]\cup\{j\}}) \cdot s_\omega \cdot \det(A_{[m+2,n],[m+1,n]\setminus\{j\}}) \end{aligned}$$

where the last equality uses the fact that the first column of $A_{[m+1,n],[m,n]}$ contains only one nonzero entry, namely $A_{m+1,m} = s_\omega$. Comparing this with Eq. (7.6), it only remains to show that

$$s_{\hat{\alpha}\circ_\omega D} = -\sum_{j=m+1}^{n} (-1)^{j-m} \det(A_{[m],[m-1]\cup\{j\}}) \det(A_{[m+2,n],[m+1,n]\setminus\{j\}}). \tag{7.7}$$

To see this, note that we can obtain an outside decomposition of $\hat{\alpha} \circ_\omega D$ by starting with the outside decomposition $(\theta_1, \dots, \theta_n)$ for $\alpha \circ_\omega D$ used above, and replacing the two ribbons θ_m, θ_{m+1} with a single ribbon $\theta_m \amalg_\omega \theta_{m+1} = \theta^{\amalg_\omega \alpha_1 + \alpha_2}$. Now expand the corresponding $(n-1) \times (n-1)$ Hamel–Goulden determinant for $s_{\hat{\alpha}\circ_\omega D}$ along its first m rows, and one obtains (7.7). □

We are now ready to state our second key way to create new skew-equivalences from known ones.

Theorem 7.22. *Let α, α' be ribbons with $\alpha \sim \alpha'$, and assume that D, ω satisfy Hypotheses* 7.19. *Then one has the following skew-equivalences*:

$$\alpha' \circ_\omega D \sim \alpha \circ_\omega D \sim \alpha \circ_{\omega^*} D^*.$$

Proof. Both skew-equivalences are immediate from Theorem 7.20. For the second, note that $(D^*)^{\amalg_{\omega^*} r} = (D^{\amalg_\omega r})^*$ for all r, so that the maps

$$\Lambda \xrightarrow{(-)\circ_\omega s_D} \Lambda,$$
$$\Lambda \xrightarrow{(-)\circ_{\omega^*} s_{D^*}} \Lambda$$

are the same. □

Remark 7.23. Theorem 7.22 is analogous to [2, Theorem 4.4, parts 1 and 2].

Theorem 7.24. *Let $\{\beta_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k$ be ribbons, and for each i either $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$. If the skew diagrams D, ω satisfy Hypotheses* 7.19, *then*

$$\begin{aligned} &\gamma_1 \circ_\omega \gamma_2 \circ_\omega \cdots \circ_\omega \gamma_k \circ_\omega D \\ &\quad \sim \beta_1 \circ_\omega \beta_2 \circ_\omega \cdots \circ_\omega \beta_k \circ_\omega D \\ &\quad \sim \beta_1 \circ_{\omega^*} \beta_2 \circ_{\omega^*} \cdots \circ_{\omega^*} \beta_k \circ_{\omega^*} D^* \end{aligned}$$

where all the operations $\circ_\omega$ or $\circ_{\omega^*}$ are performed from right to left.

Proof. By Theorems 7.2 and 7.22 we know

$$(\gamma_1 \circ \cdots \circ \gamma_k) \circ_\omega D \sim (\beta_1 \circ \cdots \circ \beta_k) \circ_\omega D \sim (\beta_1 \circ \cdots \circ \beta_k) \circ_{\omega^*} D^*.$$

From [2, Proposition 3.3] we know $\circ$ is associative, and by applying Proposition 7.17 repeatedly $k-1$ times the result follows. □

Remark 7.25. Theorem 7.24 is analogous to the reverse direction of [2, Theorem 4.1].

7.3. *Conjugation and ribbon staircases*

Recall from Definition 4.4 the *southeast decomposition* and *northwest decomposition* of a connected skew diagram. When either of these decompositions takes on a very special form, we will show that it gives rise to a nontrivial skew-equivalence, and in some cases to a skew-equivalence of the form $D \sim D^t$.

Definition 7.26. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_\ell)$ be ribbons. For an integer $m \geqslant 1$, say that the *m-intersection $\alpha \cap_m \beta$ exists* if there is a ribbon $\omega = (\omega_1, \dots, \omega_m)$ with m rows protruding from the top of α and the bottom of β for which $\omega_1 = \beta_1$ and $\omega_m = \alpha_k$; when $m = 1$, we set $\omega_1 := \min\{\alpha_k, \beta_1\}$. In this case, define the *m-intersection* $\alpha \cap_m \beta$ and the *m-union* $\alpha \cup_m \beta$ to be

$$\alpha \cap_m \beta := \omega,$$

$$\alpha \cup_m \beta := \alpha \amalg_\omega \beta.$$

If $\alpha \cup_m \beta = \alpha$ or β (respectively or $\alpha \cap_m \beta = \alpha$ or β) then we say the m-union (respectively m-intersection) is *trivial*. If α is a ribbon such that $\alpha \cap_m \alpha$ exists and is nontrivial then

$$\varepsilon_m^k(\alpha) := \underbrace{\alpha \cup_m \alpha \cup_m \cdots \cup_m \alpha}_{k \text{ factors}}$$

is the *ribbon staircase* of height k and depth m generated by α.

Example 7.27. Let α be the ribbon $(2, 3)$. Then

$$\varepsilon_1^3(\alpha) = \varepsilon_1^3\left(\begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \end{array}\right) = \begin{array}{cccccccc} & & & & & \times & \times & \times \\ & & & \times & \times & \times & & \\ & \times & \times & \times & & & & \\ \times & \times & & & & & & \end{array}.$$

Definition 7.28. Say that a skew diagram D has a *southeast ribbon staircase decomposition* if there exists an $m < \ell(\alpha)$ and a ribbon α such that all ribbons in the southeast decomposition of D are of the form $\alpha \cap_m \alpha$ or $\varepsilon_m^p(\alpha)$ for various integers $p \geqslant 1$.

In this situation, let k be the maximum value of p occurring among the $\varepsilon_m^p(\alpha)$ above, so that the largest ribbon θ equals $\varepsilon_m^k(\alpha)$. We will think of θ as containing k copies of α, numbered $1, 2, \dots, k$ from southwest to northeast. We now wish to define the *nesting* $\mathcal{N}$ associated to this decomposition. The nesting $\mathcal{N}$ is a word of length $k - 1$ using as letters the four symbols, *dot* ".", *left parenthesis* "(", *right parenthesis* ")" and *vertical slash* "|". Considering the ribbons in the southeast decomposition of D,

- a ribbon of the form $\varepsilon_m^p(\alpha)$ creates a pair of left and right parentheses in positions i and j if the ribbon occupies the same diagonals as the copies of α in θ numbered $i + 1, i + 2, \dots, j - 1, j$, while
- a ribbon of the form $\alpha \cap_m \alpha$ creates a vertical slash in position i if it occupies the same diagonals as the intersection of the $i, i + 1$ copies of α in θ, and
- all other letters in $\mathcal{N}$ are dots.

With this notation, say that $D = (\varepsilon_m^k(\alpha), \mathcal{N})_{\mathrm{se}}$. Analogously define the notation $D = (\varepsilon_m^k(\alpha), \mathcal{N})_{\mathrm{nw}}$ using the northwest decomposition.

Lastly, given a nesting $\mathcal{N}$, denote the *reverse nesting*, which is the reverse of the word $\mathcal{N}$, by $\mathcal{N}^*$.

Observe that a nesting is well-defined as if we wanted to place two different parentheses, or a parenthesis or a slash, in a given position then this would imply either that we did not have a ribbon staircase decomposition or that we did not have a skew diagram.

Example 7.29. Recall the southeast decomposition of a skew diagram D from Example 4.5:

$$D = \begin{array}{cccccccccccccccc}
 & & & & & & & & 4 & 4 & 3 & 3 & 3 & 1 & 1 & 1 \\
 & & & & & & & & 3 & 3 & 3 & 1 & 1 & 1 & & \\
 & & & & & & & 3 & 3 & 1 & 1 & 1 & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & & \\
 & & 2 & 2 & 2 & 1 & 1 & 1 & & & & & & & & \\
 & 2 & 2 & 1 & 1 & 1 & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & &
\end{array}.$$

This is a southeast ribbon staircase decomposition, in which $\alpha = (2, 3)$, $m = 1$, $k = 7$ and

$$\mathcal{N} = \begin{array}{cccccc} (&) & . & (& | &) \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

that is, $D = (\varepsilon_1^7(\alpha), \mathcal{N})_{\mathrm{se}}$. Here

$$\mathcal{N}^* = \begin{array}{cccccc} (& | &) & . & (&) \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

and $D' = (\varepsilon_1^7(\alpha), \mathcal{N}^*)_{\mathrm{se}}$ is the following skew diagram:

$$D' = \begin{array}{cccccccccccccccc}
 & & & & & & & & & & 2 & 2 & 2 & 1 & 1 & 1 \\
 & & & & & & & & & 2 & 2 & 1 & 1 & 1 & & \\
 & & & & & & & & & 1 & 1 & 1 & & & & \\
 & & 4 & 4 & 3 & 3 & 3 & 1 & 1 & 1 & & & & & & \\
 & & 3 & 3 & 3 & 1 & 1 & 1 & & & & & & & & \\
 & 3 & 3 & 1 & 1 & 1 & & & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & &
\end{array}.$$

We come now to the main result of this section.

Theorem 7.30. *Let α be a ribbon, and let*

$$D = \left(\varepsilon_m^k(\alpha), \mathcal{N}\right)_x,$$
$$D' = \left(\varepsilon_m^k(\alpha), \mathcal{N}^*\right)_x$$

where $m < \ell(\alpha)$ and $x = \mathrm{se}$ or nw. Then $D \sim D'$.

Proof. Assume that $x = \mathrm{se}$; the case where $x = \mathrm{nw}$ is analogous.

Index the ribbons in the southeast ribbon staircase decompositions of D, D' so that the largest ribbon, which is the cutting strip θ, comes first in each case. Index the remaining ribbons so that they correspond under the natural bijection between the letters in the words $\mathcal{N}$ and $\mathcal{N}^*$. One can then check that the associated Hamel–Goulden matrices are transposes of each other, and hence have the same determinant. □

Corollary 7.31. *Let D be a connected skew diagram with a ribbon staircase decomposition, that is, $D = (\varepsilon_m^k(\alpha), \mathcal{N})_x$ for some ribbon α, with $m < l(\alpha)$ and $x = \mathrm{se}$ or nw. Then D^t also has a ribbon staircase decomposition, specifically*

$$D^t = \left(\varepsilon_{m'}^k\left(\alpha^t\right), \mathcal{N}^*\right)_x$$

where $m' = |\alpha \cap_m \alpha| - (m-1)$. Furthermore, if $\alpha = \alpha^t$, then $m' = m$ and $D^t \sim D$.

Proof. The first assertion is a straightforward verification, in which one must treat the cases $k = 1, 2$ separately.

For the second assertion, when $\alpha = \alpha^t$ it is similarly straightforward to check that $m = m'$, and then one has

$$D^t = \left(\varepsilon_{m'}^k\left(\alpha^t\right), \mathcal{N}^*\right)_x = \left(\varepsilon_m^k(\alpha), \mathcal{N}^*\right)_x \sim D$$

by Theorem 7.30. □

We close this section with an interesting special case of Corollary 7.31, which was first pointed out to us by John Stembridge and for which we offer two proofs.

Corollary 7.32. *For any Ferrers diagram μ contained in the staircase partition $\delta_n := (n-1, n-2, \dots, 1) \vdash \binom{n}{2}$, one has*

$$\delta_n/\mu \sim (\delta_n/\mu)^t.$$

Proof 1. Check that the southeast decomposition of δ_n/μ is always a southeast ribbon staircase decomposition of the form $\delta_n/\mu = (\varepsilon_1^{n-2}(\alpha), \mathcal{N})_{\mathrm{se}}$, in which α is the self-conjugate ribbon $(1, 2)$. Then apply Corollary 7.31. □

Proof 2. (Cf. [16, Proposition 7.17.7].) Since all border strips in $D = \delta/\mu$ have odd size, when one expands

$$s_D = \sum_\lambda z_\lambda^{-1} \chi^D(\lambda) p_\lambda$$

in terms of power sum symmetric functions as in [16, 7.17.5], the Murnaghan–Nakayama formula [16, Theorem 7.17.3] for the coefficient $\chi^D(\lambda)$ shows that it vanishes when λ has any even parts. Hence s_D is a polynomial in the odd power sums $p_1, p_3, p_5, \dots$. Since the involution ω satisfies $\omega(p_r) = (-1)^{r-1} p_r$, one has $s_{D^t} = \omega(s_D) = s_D$. □

8. Necessary conditions

We now present some combinatorial invariants for the skew-equivalence relation $D_1 \sim D_2$ on connected skew diagrams.

8.1. Frobenius rank

The *Durfee* or *Frobenius rank* of a skew diagram D is defined to be the minimum number of ribbons in any decomposition of D into ribbons. The rank of D is an invariant of s_D which can be extracted in at least two ways using recent results. Firstly, Stanley has pointed out to us that the discussion at the beginning of [17, §5] implies the rank of D is the minimum length $\ell(\nu)$ among partitions ν which appear when expanding s_D uniquely as a sum of power sum symmetric functions p_ν. Secondly, it was recently conjectured by Stanley [17], and proven by Chen and Yang [3], that the rank coincides with the highest power of t dividing the polynomial $s_D(1, 1, \ldots, 1, 0, 0, \ldots)$, where t of the variables have been set to 1, and the rest to zero. Either of these implies the following.

Corollary 8.1. *Frobenius rank is an invariant of skew-equivalence, that is two skew-equivalent diagrams must have the same Frobenius rank.*

In particular, skew-equivalence restricts to the subset of ribbons as they are the skew diagrams of Frobenius rank 1.

8.2. Overlaps

Data about the amount of overlap between sets of rows or columns in the skew diagram D can be recovered from its skew Schur function s_D.

Definition 8.2. Let D be a skew diagram occupying r rows. For each k in $\{1, 2, \ldots, r\}$, define the *k-row overlap composition*

$$r^{(k)} = \left(r_1^{(k)}, \ldots, r_{r-k+1}^{(k)}\right)$$

to be the sequence where $r_i^{(k)}$ is the number of columns occupied in common by the rows $i, i+1, \ldots, i+k-1$. Let $\rho^{(k)}$ be the *k-row overlap partition* that is the weakly decreasing rearrangement of $r^{(k)}$. Similarly define column overlap compositions $c^{(k)}$ and column overlap partitions $\gamma^{(k)}$.

Example 8.3. If $D = \lambda/\mu$ with $\ell := \ell(\lambda)$, then the 1-row and 2-row overlap compositions are the sequences

$$r^{(1)} = (\lambda_1 - \mu_1, \ldots, \lambda_\ell - \mu_\ell),$$
$$r^{(2)} = (\lambda_2 - \mu_1, \lambda_3 - \mu_2, \ldots, \lambda_\ell - \mu_{\ell-1})$$

that played an important role in Proposition 6.2.

It transpires that the row overlap partitions $(\rho^{(k)})_{k\geqslant 1}$ and the column overlap partitions $(\gamma^{(k)})_{k\geqslant 1}$ determine each other uniquely. To see this, we define a third form of data on a skew diagram D, which mediates between the two, and which is more symmetric under conjugation.

Proposition 8.4. *Given a skew diagram D, consider the doubly-indexed array $(a_{k,\ell})_{k,\ell\geqslant 1}$ where $a_{k,\ell}$ is defined to be the number of $k\times\ell$ rectangular subdiagrams contained inside D. Then we have*

$$a_{k,\ell}=\sum_{\ell'\geqslant\ell}\bigl(\rho^{(k)}\bigr)^t_{\ell'}=\sum_{k'\geqslant k}\bigl(\gamma^{(\ell)}\bigr)^t_{k'}.$$

Consequently, any one of the three forms of data

$$\bigl(\rho^{(k)}\bigr)_{k\geqslant 1},\qquad \bigl(\gamma^{(k)}\bigr)_{k\geqslant 1},\qquad (a_{k,\ell})_{k,\ell\geqslant 1}$$

on D determines the other two uniquely.

Proof. It suffices to prove the first equation, since exchanging rows and columns gives the second. Every $k\times\ell$ rectangular subdiagram of D occupies a particular k-tuple of rows, and the corresponding entry of $\rho^{(k)}$ coming from that k-tuple of rows must be of size $\ell'\geqslant\ell$. This part ℓ' corresponds to a total of $\ell'-\ell+1$ such $k\times\ell$ subdiagrams, and hence

$$a_{k,\ell}=\sum_{\text{parts }\ell'\geqslant\ell\text{ in }\rho^{(k)}}(\ell'-\ell+1)=\sum_{\ell'\geqslant\ell}\bigl(\rho^{(k)}\bigr)^t_{\ell'}.$$

Since this relationship is invertible, the two forms of data determine each other. $\square$

Note that the data of the first two row-overlap compositions $r^{(1)}, r^{(2)}$ are enough to recover the skew diagram D up to translation within $\mathbb{Z}^2$, and similarly for the compositions $c^{(1)}, c^{(2)}$. Thus one cannot expect to recover $r^{(1)}, r^{(2)}$ or $c^{(1)}, c^{(2)}$ from the skew Schur function s_D.

However, it turns out one *can* recover all of the row overlap *partitions* $(\rho^{(k)})_{k\geqslant 1}$ (or the column overlap partitions $(\gamma^{(k)})_{k\geqslant 1}$) from s_D; see Corollary 8.11 below, whose proof is our goal for the remainder of this section.

Lemma 8.5. *Given any skew diagram D, there is a unique skew diagram $\hat{D}$ satisfying $r^{(k)}(\hat{D})=r^{(k+1)}(D)$ for all k.*

Proof. Observe that $\hat{D}$ is obtained from D by removing the top cell from every column of D. $\square$

Example 8.6. Let D be the skew diagram from Example 5.3, shown below with its row overlap compositions depicted vertically to its right:

$$D=\begin{array}{cccc ccccc}
 & & & & r^{(1)} & r^{(2)} & r^{(3)} & r^{(4)} & r^{(5)}\\
 & & & \times & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & \times & 1 & & 0 & & \\
 & & & & & 0 & & 0 & \\
 & \times & \times & & 2 & & 0 & & 0\\
 & & & & & 2 & & 0 & \\
\times & \times & \times & & 3 & & 1 & & \\
 & & & & & 2 & & & \\
\times & \times & & & 2 & & & &
\end{array}.$$

Then $\hat{D}$ is as shown below:

$$\hat{D} = \begin{array}{cccccccc} & & & & r^{(1)} & r^{(2)} & r^{(3)} & r^{(4)} \\ & & & \times & 1 & & & \\ & & & & & 0 & & \\ & & & & 0 & & 0 & \\ & & & & & 0 & & 0 \\ & \times & \times & & 2 & & 0 & \\ & & & & & 1 & & \\ \times & \times & & & 2 & & & \end{array} .$$

Note in particular that the second row of $\hat{D}$ is *empty*.

We now prove a crucial lemma.

Lemma 8.7. *Let D be a skew diagram with c nonempty columns. The map on column-strict tableaux that removes the first row and lowers all other entries by* 1 *restricts to a bijection*

$$\{T \in \text{Pictures}(D)\colon\ \lambda_1(T) = c\} \to \text{Pictures}(\hat{D}).$$

Proof. Let $T_{\text{row}}(D)$ be the row filling of D, and let t_j denote the topmost entry in column j of $T_{\text{row}}(D)$ for $j = 1, 2, \ldots, c$. The following facts are then easy to verify from the definitions:

(i) the row filling $T_{\text{row}}(\hat{D})$ is obtained from the row filling $T_{\text{row}}(D)$ by removing the entries $t_1, t_2, \ldots, t_c$ and then lowering the remaining entries by 1,
(ii) a picture T for D will have $\lambda_1(T) = c$ if and only if the first row of T is exactly $(t_c, t_{c-1}, \ldots, t_2, t_1)$,
(iii) the map on pictures for $\hat{D}$ that raises all entries by 1 and then adds a new top row $(t_c, t_{c-1}, \ldots, t_2, t_1)$ gives a well-defined map into the pictures for D, and is the inverse of the map defined in the lemma. $\square$

Example 8.8. Let D and $\hat{D}$ be as in Example 8.6, with row fillings as shown here, and the topmost elements $(t_1, t_2, t_3, t_4) = (4, 3, 3, 1)$ in each column of $T_{\text{row}}(D)$ shown in bold:

$$T_{\text{row}}(D) = \begin{array}{cccc} & & & \mathbf{1} \\ & & & 2 \\ & \mathbf{3} & \mathbf{3} & \\ \mathbf{4} & 4 & 4 & \\ 5 & 5 & & \end{array}, \qquad T_{\text{row}}(\hat{D}) = \begin{array}{cccc} & & & 1 \\ & & & \\ & 3 & 3 & \\ 4 & 4 & & \end{array}.$$

Note that removing the entries (t_1, t_2, t_3, t_4) in $T_{\text{row}}(D)$ and lowering the remaining entries by 1 gives $T_{\text{row}}(\hat{D})$. The set Pictures(D) was shown in Example 5.3, where only the last five of these pictures T have $\lambda_1(T) = c = 4$:

$$\left\{ \begin{array}{cccc} 1 & 3 & 3 & 4 \\ 2 & 4 & & \\ 4 & 5 & & \\ 5 & & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 3 & 4 \\ 2 & 4 & 5 & \\ 4 & & & \\ 5 & & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 3 & 4 \\ 2 & 4 & 5 & \\ 4 & 5 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 3 & 4 \\ 2 & 4 & 4 & \\ 5 & 5 & & \end{array}, \quad \begin{array}{cccc} 1 & 3 & 3 & 4 \\ 2 & 4 & 4 & 5 \\ 5 & & & \end{array} \right\}.$$

On the other hand, for $\hat{D}$ one has pictures

$$\left\{\begin{matrix}1&3\\3&4\\4&\end{matrix}\quad,\quad\begin{matrix}1&3&4\\3&&\\4&&\end{matrix}\quad,\quad\begin{matrix}1&3&4\\3&4&\end{matrix}\quad,\quad\begin{matrix}1&3&3\\4&4&\end{matrix}\quad,\quad\begin{matrix}1&3&3&4\\4&&&\end{matrix}\right\}.$$

Note the bijection from the top set to the bottom set, obtained by removing the first row and lowering the remaining entries by 1.

Definition 8.9. Given a positive integer ℓ, define a $\mathbb{Z}$-linear map $\phi_\ell : \Lambda \to \Lambda$ by

$$\phi_\ell(s_\lambda) := \begin{cases} s_{\lambda+1^\ell} & \text{if } \ell(\lambda) \leqslant \ell, \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda + 1^\ell := (\lambda_1 + 1, \ldots, \lambda_\ell + 1)$. Also, given $f \in \Lambda = \mathbb{Z}[h_1, h_2, \ldots]$, let $[h_r](f)$ denote the polynomial in the h_i which gives the coefficient of h_r in the expansion of f.

Theorem 8.10. *Let D be a skew diagram with ℓ nonempty rows and c nonempty columns. Then*

$$s_{\hat{D}} = [h_{\ell+c}]\phi_\ell(s_D).$$

Proof. Recall the Littlewood–Richardson expansion (5.1)

$$s_D = \sum_{T \in \text{Pictures}(D)} s_{\lambda(T)}.$$

Since D has ℓ nonempty rows, any picture T for D will have at most ℓ rows, and hence $\phi_\ell(s_{\lambda(T)}) = s_{\lambda(T)+1^\ell}$. Thus

$$\phi_\ell(s_D) = \sum_{T \in \text{Pictures}(D)} s_{\lambda(T)+1^\ell},$$

$$[h_{\ell+c}]\phi_\ell(s_D) = \sum_{T \in \text{Pictures}(D)} [h_{\ell+c}]s_{\lambda(T)+1^\ell}$$

and it remains to extract the coefficient of $h_{\ell+c}$ in each term $s_{\lambda(T)+1^\ell}$.

It was noted in the proof of Corollary 6.3 that the Jacobi–Trudi expansion for $s_{\nu/\mu}$ takes a certain form; when μ is empty this form specializes to

$$s_\nu = s_{\hat{\nu}} \cdot h_{L(\nu)} + r$$

where

$$L(\nu) := \ell(\nu) + \nu_1 - 1,$$
$$\hat{\nu} := (\nu_2 - 1, \nu_3 - 1, \ldots, \nu_{\ell(\nu)} - 1)$$

and the remainder r is a polynomial in h_k for $k < L(\nu)$. Note that since D has c nonempty columns, any picture T for D will have at most c nonempty columns, and hence $\nu := \lambda(T) + 1^\ell$ will have $L(\nu) \leqslant \ell + c$, with equality if and only if $\lambda_1(T) = c$. Hence we have

$$[h_{\ell+c}]s_{\lambda(T)+1^\ell} = \begin{cases} s_{(\lambda_2(T),\ldots,\lambda_\ell(T))} & \text{if } \lambda_1(T) = c, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$[h_{\ell+c}]\phi_\ell(s_D) = \sum_{\substack{T \in \mathrm{Pictures}(D): \\ \lambda_1(T)=c}} s_{(\lambda_2(T),\ldots,\lambda_\ell(T))},$$

which equals $s_{\hat{D}}$ by Lemma 8.7. □

We are now ready to state our main necessary condition for skew-equivalence.

Corollary 8.11. *The skew Schur function s_D determines the row overlap partition $(\rho^{(k)})_{k\geqslant 1}$ data. Consequently, if $D \sim D'$, then D, D' must have the same row overlap partitions.*

Proof. Induct on the number ℓ of nonempty rows in D. Proposition 6.2(ii) showed that $\rho^{(1)}$ can be recovered as the dominance-smallest partition occurring among the subscripts of monomials in the h_r-expansion of s_D.

From Theorem 8.10 we know that s_D determines $s_{\hat{D}}$. By induction, $s_{\hat{D}}$ determines its own row overlap partitions, which by Lemma 8.5 coincide with the rest of the row overlap partitions $\rho^{(2)}, \rho^{(3)}, \ldots$ for D. □

Example 8.12. Unfortunately, having the same row and column overlap partitions $\rho^{(k)}, \gamma^{(k)}$ is not sufficient for the skew-equivalence of two skew diagrams. For example,

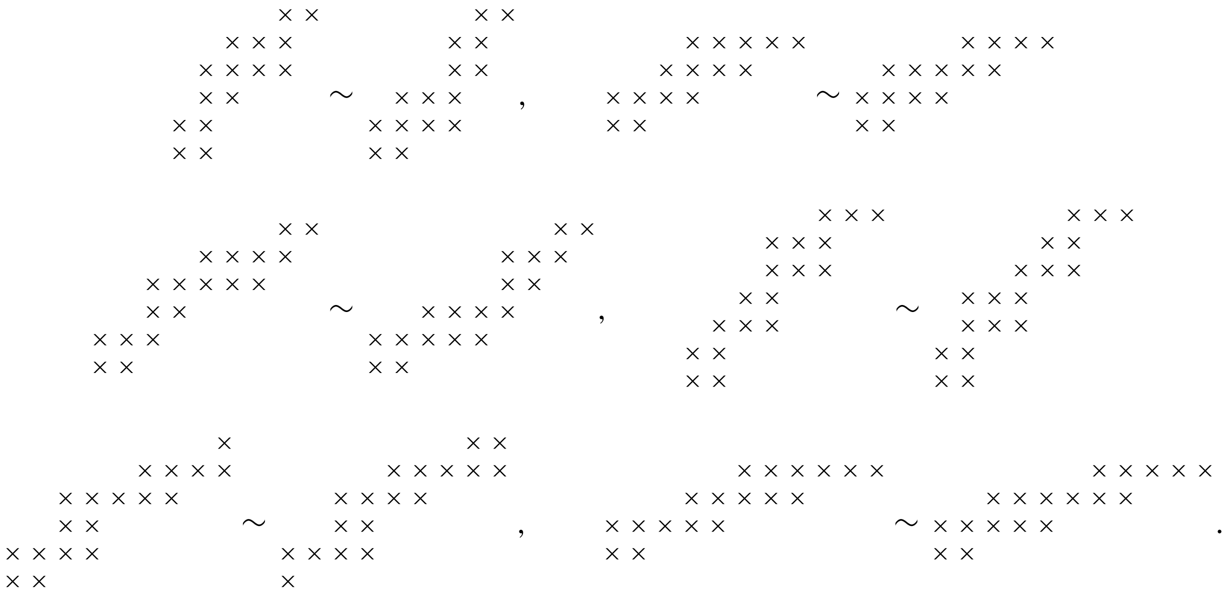
$$\begin{array}{cccc} & & \times & \times \\ \times & \times & \times & \\ \times & & & \end{array} \quad \not\sim \quad \begin{array}{cccc} & \times & \times & \times \\ \times & \times & & \\ \times & & & \end{array}$$

even though they have the same row and column overlap partitions $\rho^{(k)}, \gamma^{(k)}$ for every k.

Remark 8.13. Corollary 8.11 gives an alternate proof of the second assertion in Corollary 8.1, that is, that ribbons can only be skew-equivalent to other ribbons, since a skew diagram D is a ribbon if and only if it is connected and its 2-row overlap partition $\rho^{(2)}$ has the form $(1, 1, \ldots, 1)$.

9. Complete classification

The sufficient conditions discussed in this paper explain all but the following six skew-equivalences among skew diagrams with up to 18 cells, up to antipodal rotation and/or conjugation. However, one way to explain both these skew-equivalences and the phenomenon occurring in Remark 7.10 has recently been discovered in [11] and extends the definitions and results of Section 7.2 naturally.



Note that these skew-equivalences occur in pairs. This leads us to end with the following conjecture that holds for all skew diagrams with up to 18 cells.

Conjecture 9.1. *Every skew-equivalence class of skew diagrams has cardinality a power of* 2.

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References

[1] F. Bergeron, R. Biagioli, M.H. Rosas, Inequalities between Littlewood–Richardson coefficients, J. Combin. Theory Ser. A 113 (2006) 567–590.
[2] L.J. Billera, H. Thomas, S. van Willigenburg, Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions, Adv. Math. 204 (2006) 204–240.
[3] W.Y.C. Chen, A.L.B. Yang, Stanley's zrank problem on skew partitions, Trans. Amer. Math. Soc., in press.
[4] W.Y.C. Chen, G.-G. Yan, A.L.B. Yang, Transformations of border strips and Schur function determinants, J. Algebraic Combin. 21 (2005) 379–394.
[5] S. Fomin, W. Fulton, C.-K. Li, Y.-T. Poon, Eigenvalues, singular values, and Littlewood–Richardson coefficients, Amer. J. Math. 127 (2005) 101–127.
[6] I.M. Gel'fand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.
[7] A. Hamel, I. Goulden, Planar decompositions of tableaux and Schur function determinants, European J. Combin. 16 (1995) 461–477.
[8] T. Lam, A. Postnikov, P. Pylyavskyy, Schur positivity and Schur log-concavity, Amer. J. Math., in press.
[9] A. Lascoux, P. Pragacz, Équerres et fonctions de Schur, C. R. Acad. Sci. Paris Ser. I Math. 299 (1984) 955–958.
[10] I. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, New York, USA, 1995.
[11] P.R.W. McNamara, S. van Willigenburg, A combinatorial classification of skew Schur functions, preprint, math. CO/0608446.
[12] A. Okounkov, Log-concavity of multiplicities with application to characters of $U(\infty)$, Adv. Math. 127 (1997) 258–282.

[13] B.E. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions, second ed., Grad. Texts in Math., vol. 203, Springer-Verlag, New York, USA, 2001.

[14] I. Schur, Über eine Klasse von Matrizen die sich einer gegebenen Matrix zuorden lassen, Inaugural-Dissertation, Berlin, 1901.

[15] G.E. Shilov, Linear Algebra, revised English edition, Dover Publications, New York, USA, 1977, translated from the Russian and edited by Richard A. Silverman.

[16] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, UK, 1999.

[17] R.P. Stanley, The rank and minimal border strip decompositions of a skew partition, J. Combin. Theory Ser. A 100 (2002) 349–375.

[18] G.-G. Yan, A.L.B. Yang, J. Zhou, The zrank conjecture and restricted Cauchy matrices, Linear Algebra Appl. 411 (2005) 371–385.

[19] A.V. Zelevinsky, A generalization of the Littlewood–Richardson rule and the Robinson–Schensted–Knuth correspondence, J. Algebra 69 (1981) 82–94.