

## Equality of Schur and Skew Schur Functions

Stephanie van Willigenburg\*

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada  
steph@math.ubc.ca

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**Abstract.** We determine the precise conditions under which any skew Schur function is equal to a Schur function over both infinitely and finitely many variables.

*Keywords:* Schur function, skew Schur function, Littlewood-Richardson coefficients

### 1. Introduction

Littlewood-Richardson coefficients arise in a variety of contexts. The first of these is that they are the structure constants in the algebra of symmetric functions with respect to the basis of Schur functions. Another instance is as the multiplicities of irreducible representations in the induced tensor product of representations of the symmetric group. A third occurrence is as intersection numbers in the Schubert calculus on a Grassmannian. Thus knowing their values has an impact on a number of fields. In this paper we calculate when certain coefficients are 0 or 1 by determining when a skew Schur function is equal to a Schur function. Although multiplicity free products have been studied in [4] our determination will reveal more precisely when certain coefficients are 0 and when they are 1. The related question of when two ribbon Schur functions are equal has been answered recently in [1], which revealed many new equalities of Littlewood-Richardson coefficients.

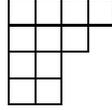
The remainder of this note is structured as follows. In the rest of Section 1 we review the definitions required. This is followed by the two main theorems in which we give straightforward conditions that prescribe when a skew Schur function is equal to a Schur function over both infinitely and finitely many variables.

#### 1.1. Schur and Skew Schur Functions

We say that a list of positive integers  $\lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$  whose sum is  $n$  is a *partition* of  $n$ , denoted  $\lambda \vdash n$ . We call the  $\lambda_i$  the *parts* of  $\lambda$ . A partition with at most one part size is called a *rectangle* and a partition with exactly two different part sizes

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is called a *fat hook*. If  $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k \vdash n$  then we define the (*Ferrers*) *diagram*  $D_\lambda$  to be the array of left justified boxes with  $\lambda_i$  boxes in the  $i$ -th row for  $1 \leq i \leq k$ . If we transpose  $D_\lambda$  we obtain another diagram  $D_{\lambda'}$  known as the *conjugate* of  $D_\lambda$  and we refer to  $\lambda$  and  $\lambda'$  as *conjugate partitions*. Furthermore, for any column  $c$  in  $D_\lambda$  we denote by  $l(c)$  the number of boxes in  $c$  and refer to  $l(c)$  as the *length* of  $c$ . Where the context is clear we abuse notation and refer to  $D_\lambda$  as  $\lambda$ .

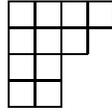


Example 1.1.  $D_{4322} =$

We define a (*Young*) *tableau*  $T$  of *shape*  $\lambda$  to be a filling of the boxes of  $D_\lambda$  with positive integers. If the filling is such that the integers in each row weakly increase, whilst the integers in each column strictly increase we say that  $T$  is a *semi-standard tableau*.

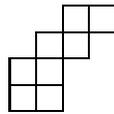
Example 1.2.  $\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & \\ 4 & 4 & & \\ 5 & 7 & & \end{array}$  is a semi-standard tableau of shape 4322.

If  $\mu = \mu_1 \cdots \mu_k \vdash m$ ,  $\lambda = \lambda_1 \cdots \lambda_l \vdash n$  where  $k \leq l$  and  $m \leq n$  such that  $\mu_i \leq \lambda_i$  for  $1 \leq i \leq k$  then we define the *skew diagram*  $D_{\lambda/\mu}$  to be the array of boxes that appear in  $D_\lambda$  but not in  $D_\mu$ . For our purposes  $D_{\lambda/\mu}$  will always be *connected*, that is to say, for any pair of adjacent rows in  $D_{\lambda/\mu}$  there exists at least one column in which they both have a box.



Example 1.3.  $D_{4322} =$

$D_{21} =$



$D_{4322/21} =$

Again when the context is clear we refer to  $D_{\lambda/\mu}$  as  $\lambda/\mu$ . Similarly we define *skew tableaux* and *semi-standard skew tableaux* by appropriately inserting the adjective skew in the above definitions for tableaux and semi-standard tableaux.

**Definition 1.4.** Given a (*skew*) *tableau*  $T$  and a set of variables  $x_1, x_2, \dots$ , we define the *monomial*  $x^T$  to be

$$x^T := x_1^{t_1} x_2^{t_2} \cdots$$

where  $t_i$  is the number of times  $i$  appears in  $T$ . Let  $\lambda, \mu$  be partitions such that  $\lambda/\mu$  is a (*skew*) *diagram*, then we define the corresponding (*skew*) Schur function  $s_{\lambda/\mu}$  to be

$$s_{\lambda/\mu} = \sum x^T$$

where the sum is over all semi-standard (skew) tableaux  $T$  of shape  $\lambda/\mu$ .

The set of all Schur functions  $s_\lambda$  (i.e.  $s_{\lambda/\mu}$  where  $\mu = \emptyset$ ) forms a basis for the algebra of symmetric functions,  $\Lambda$ , which is a subalgebra of  $\mathbb{Q}[x_1, x_2, \dots]$ .

Since it can be easily shown that skew Schur functions are symmetric it follows that skew Schur functions can be written as a linear combination of Schur functions. To be more precise we need to recall two more notions: that of the reading word and the content of a (skew) tableau. Firstly, given a (skew) tableau,  $T$ , we say its *reading word*,  $w(T)$ , is the entries of the tableau read from top to bottom and *right to left*. Given a reading word we say it is *lattice* if as we read it from left to right the number of  $i$ 's we have read is at least as large as the number of  $i + 1$ 's we have read e.g. 1213 is lattice, however, 1132 is not as when we have read 113 the number of 3's we have read is greater than the number of 2's. Secondly, the *content* of a (skew) tableau,  $c(T)$ , is a list  $t_1 t_2 t_3 \dots$  where, as before,  $t_i$  is the number of times  $i$  appears in  $T$ .

We are now ready to express any skew Schur function as a linear combination of Schur functions.

**Proposition 1.5.** [3, A1.3.3] *Let  $\lambda, \mu, \nu$  be partitions such that  $\lambda/\mu$  is a (skew) diagram, then*

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}$$

where  $c_{\mu\nu}^{\lambda}$  is the number of semi-standard (skew) tableaux  $T$  such that

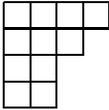
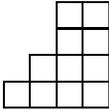
- (1) the shape of  $T$  is  $\lambda/\mu$ ,
- (2)  $c(T) = \nu$ ,
- (3)  $w(T)$  is lattice.

*Example 1.6.*  $s_{32/1} = s_{31} + s_{22}$ .

*Remark 1.7.* The  $c_{\mu\nu}^{\lambda}$  are known as Littlewood-Richardson coefficients, and the above method of computing them is known as the Littlewood-Richardson rule. There are many other methods for computing the  $c_{\mu\nu}^{\lambda}$  such as Zelevinsky's pictures or Remmel and Whitney's reverse numbering, and the interested reader may wish to consult, say, [2] for further details. However, it is the Littlewood-Richardson rule that will allow us to determine our results most succinctly.

## 2. Equality of Schur and Skew Schur Functions

Before we state our main result let us define an involution on diagrams. Given a diagram  $\lambda$  let  $\lambda^\circ$  be the (skew) diagram that is the diagram  $\lambda$  rotated by  $180^\circ$ .

*Example 2.1.*  $4322 =$   ,  $4322^\circ =$   .

**Theorem 2.2.** *For partitions  $\lambda, \mu, \nu$*

$$s_{\lambda/\mu} = s_{\nu} \text{ if and only if } \lambda/\mu = \nu \text{ or } \nu^\circ.$$

*Proof.* The reverse implication follows by [3, Exercise 7.56(a)], which yields that

$$s_{\mathbf{v}} = s_{\mathbf{v}^\circ}.$$

For the forward implication we need only show that if  $\lambda/\mu$  is not  $\mathbf{v}$  or  $\mathbf{v}^\circ$  for some diagram  $\mathbf{v}$  then  $\lambda/\mu$  has more than one filling whose reading word is lattice.

Consider the skew tableau  $T$  of shape  $\lambda/\mu$  where each column  $c$  is filled with the integers  $1, \dots, l(c)$  in increasing order. This filling is clearly lattice. Now since  $\lambda/\mu$  is not a diagram  $\mathbf{v}$  nor a (skew) diagram  $\mathbf{v}^\circ$  where  $\mathbf{v}$  is a diagram, consider the first row  $i$  where  $\lambda/\mu$  fails to be either  $\mathbf{v}$  or  $\mathbf{v}^\circ$  for some diagram  $\mathbf{v}$  (i.e. if  $\lambda/\mu$  is truncated at row  $i-1$  then we obtain a (rotated) diagram, but this is no longer true if  $\lambda/\mu$  is truncated at row  $i$ ). Moving from right to left note the first entry  $j$  in  $T$  which does not have  $i-1$  entries above it. Form the reading word of  $T$  up to this entry and note the smallest integer  $i \geq k > j$  for which the number of occurrences of  $k$  is strictly less than the number of occurrences of  $k-1$ . Change  $j$  to  $k$  and change all entries below it in that column by adding  $k-j$  to the existing entry to form a new skew tableau  $T'$  of shape  $\lambda/\mu$ . Since  $w(T')$  is clearly lattice, we are done. ■

### 3. Equality and $GL(n)$ or $SL(n)$ Characters

The set of all Schur functions restricted to the variables  $x_1, \dots, x_n$ , obtained by setting  $x_m = 0$  for  $m > n$ , forms a basis for the algebra of symmetric polynomials  $\Lambda_n$ . Skew Schur functions in  $\Lambda_n$  can be expressed in terms of the Schur functions by

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{\mathbf{v}} c_{\mu\mathbf{v}}^{\lambda} s_{\mathbf{v}}(x_1, \dots, x_n)$$

and thus we can ask when  $s_{\lambda/\mu}(x_1, \dots, x_n) = s_{\mathbf{v}}(x_1, \dots, x_n)$ . In terms of representation theory this yields when certain multiplicities in the tensor products of irreducible representations of  $SL(n, \mathbb{C})$  (or polynomial representations of  $GL(n, \mathbb{C})$ ) will be 0 and when they will be 1.

Clearly if  $s_{\lambda/\mu} = s_{\mathbf{v}}$  in  $\Lambda$  then the result holds in  $\Lambda_n$ , however the converse may not be true as an  $s_{\lambda/\mu}$  comprising of a sum of  $s_{\mathbf{v}}$  only one of which has less than  $n+1$  parts could exist. However, the search for such an  $s_{\lambda/\mu}$  is greatly reduced as the converse may not be true only when the length of the longest column in  $\lambda/\mu$  is equal to  $n$  by

**Lemma 3.1.** *Let  $\lambda, \mu$  be partitions such that the length of the longest column in  $\lambda/\mu$  is  $m$ .*

- (1) *If  $m > n$  then  $s_{\lambda/\mu} = 0$  in  $\Lambda_n$ .*
- (2) *If  $m < n$  then  $s_{\lambda/\mu} = s_{\mathbf{v}}$  in  $\Lambda_n$  if and only if  $s_{\lambda/\mu} = s_{\mathbf{v}}$  in  $\Lambda$ .*

*Proof.* The first result is immediate from the definitions. The reverse direction of the second result has already been discussed, thus it only remains to show that if  $m < n$  and  $s_{\lambda/\mu} \neq s_{\mathbf{v}}$  in  $\Lambda$  then  $s_{\lambda/\mu} \neq s_{\mathbf{v}}$  in  $\Lambda_n$ .

Consider the skew tableau  $T$  of shape  $\lambda/\mu$  where each column  $c$  is filled with the integers  $1, \dots, l(c)$  in increasing order. Since  $s_{\lambda/\mu} \neq s_{\mathbf{v}}$  in  $\Lambda$  this implies  $\lambda/\mu$  is not a diagram  $\mathbf{v}$  nor a (skew) diagram  $\mathbf{v}^\circ$ , so as in the proof of Theorem 2.2 consider the

first row  $i$  where  $\lambda/\mu$  fails to be  $\nu$  or  $\nu^\circ$  for some diagram  $\nu$ . Moving from right to left note the first entry  $j$  in column  $c$  of  $T$  that does not have  $i-1$  entries above it, form the reading word up to this entry,  $w'$ , and note the smallest integer  $i \geq k > j$  for which the number of occurrences of  $k$  is strictly less than the number of occurrences of  $k-1$ . If  $l(c) + k - j \leq n$  then change the  $j$  to  $k$  and change all entries below it in  $c$  by adding  $k-j$  to the existing entry to form a new skew tableau  $T'$  of shape  $\lambda/\mu$ . If not then find the largest entry in  $w'$ ,  $k' < n$ , and fill the  $n-k'$  lowest boxes in  $c$  with  $k'+1, \dots, n$  to form  $T'$ . Since  $w(T')$  is clearly lattice, the result follows. ■

*Example 3.2.* If  $n = 5$  then the following semi-standard skew tableaux illustrate  $T'$  in the situation  $l(c) + k - j \leq n$  and  $l(c) + k - j > n$  respectively.

$$\begin{array}{ccc} & & 1 \\ & 1 & 1 & 2 \\ & 2 & 2 & 3 \\ 1 & 3 & & 1 & 4 \\ 2 & 4 & & & 2 \\ & & & & 1 & 5 \end{array}$$

Thus, from here on we shall assume that the length of the longest column in  $\lambda/\mu$  is  $n$ . Before we reveal the analogous result to Theorem 2.2 let us define two operations on (skew) diagrams.

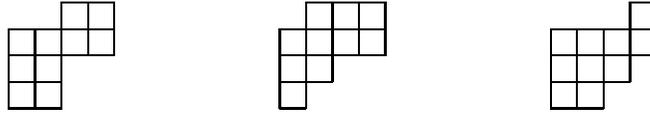
**Definition 3.3.** Let  $\lambda = \lambda_1 \cdots \lambda_k$  and  $\mu = \mu_1 \cdots \mu_l$  be partitions such that  $\lambda/\mu$  is a (skew) diagram. Let  $c$  be a column of longest length in  $\lambda/\mu$  and  $\bar{\lambda}$  and  $\bar{\mu}$  be partitions such that

$$\begin{aligned} (1) \quad \bar{\lambda}' &= (\lambda'_1 + r)(\lambda'_2 + r)(\lambda'_3 + r) \cdots (\lambda'_c + r)\lambda'_{c+1} \cdots \lambda'_{\lambda_1}, \\ \bar{\mu}' &= (\mu'_1 + r)(\mu'_2 + r)(\mu'_3 + r) \cdots (\mu'_c + r)\mu'_{c+1} \cdots \mu'_{\mu_1}, \text{ or} \\ (2) \quad \bar{\lambda}' &= (\lambda'_1 + r)(\lambda'_2 + r)(\lambda'_3 + r) \cdots (\lambda'_{c-1} + r)\lambda'_c \cdots \lambda'_{\lambda_1}, \\ \bar{\mu}' &= (\mu'_1 + r)(\mu'_2 + r)(\mu'_3 + r) \cdots (\mu'_{c-1} + r)\mu'_c \cdots \mu'_{\mu_1}, \end{aligned}$$

and  $\lambda/\mu$  is a (skew) diagram then we say  $\bar{\lambda}/\bar{\mu}$  is a shearing of  $\lambda/\mu$ .

*Remark 3.4.* Intuitively we can interpret this definition as creating a diagram  $\bar{\lambda}/\bar{\mu}$  from  $\lambda/\mu$  by choosing a column of longest length and sliding it and every column to the left of it down  $r$  boxes, or sliding it and every column to the right of it up  $r$  boxes.

*Example 3.5.* The first two skew diagrams are shearings of 442 whilst the third is not.



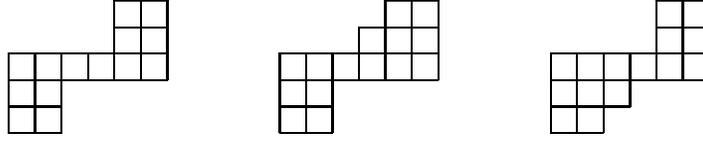
**Definition 3.6.** Let  $\underline{\lambda}$  and  $\underline{\mu}$  be partitions such that

$$\begin{aligned} (1) \quad \underline{\lambda}' &= (a+b)^i c^{j+k} \text{ and } \underline{\mu}' = b^{i+k} \text{ if } a \neq c \text{ or,} \\ (2) \quad \underline{\lambda}' &= (a+b)^i (v_1 + b) \cdots (v_k + b) c^j \text{ and } \underline{\mu}' = b^{i+k}, \text{ where } \nu = \nu_1 \cdots \nu_k \text{ is a partition,} \\ &\text{if } a = c, \end{aligned}$$

and  $\underline{\lambda}/\underline{\mu}$  is a (skew) diagram then we say  $\underline{\lambda}/\underline{\mu}$  is a fattening of  $(a^i c^j)'$  and  $(\underline{\lambda}/\underline{\mu})^\circ$  is a fattening of  $((a^i c^j)')^\circ$ .

*Remark 3.7.* Intuitively we can interpret this definition as creating a diagram say  $\underline{\lambda}/\underline{\mu}$  from a fat hook or rectangle in the following way. If we have a fat hook  $(a^i c^j)'$  then we shear the rightmost column of length  $a$  and all the columns to the left of it down by  $b$  boxes. We then insert a rectangle  $k^{c-b}$  such that the result is a (skew) diagram. If we have a rectangle  $(a^{i+j})'$  then we shear a column and all the columns to the left of it down by  $b$  boxes. We then insert a diagram such that the result is a (skew) diagram. A similar interpretation follows for  $(\underline{\lambda}/\underline{\mu})^\circ$ .

*Example 3.8.* All three skew diagrams are fattenings of 444.



For convenience we extend the notion of fattening to all (skew) diagrams by defining the fattening of  $\lambda/\mu$  to be  $\lambda/\mu$  if  $\lambda/\mu$  is any (skew) diagram other than those referred to in Definition 3.6. In addition, for clarity of exposition, we denote by  $\widetilde{\lambda}/\mu$  any (skew) diagram that has been derived from  $\lambda/\mu$  via some combination of shearings or fattenings.

**Theorem 3.9.** For partitions  $\lambda, \mu, \nu, \eta = \eta_1 \eta_2 \dots$  where  $\eta_i$  is the number of columns of length at least  $i$  in  $\lambda/\mu$

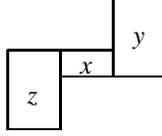
$$s_{\lambda/\mu} = s_\eta \text{ in } \Lambda_n \text{ if and only if } \lambda/\mu = \widetilde{\nu} \text{ or } \widetilde{\nu}^\circ.$$

*Proof.* For the reverse implication it is straightforward to check that if  $\lambda/\mu = \widetilde{\nu}$  or  $\widetilde{\nu}^\circ$  then the only semi-standard skew tableau  $T$  of shape  $\lambda/\mu$  whose reading word is lattice has each column  $c$  filled with the integers  $1, \dots, l(c)$  in increasing order.

The forward implication will follow once we show that if  $\lambda/\mu \neq \widetilde{\nu}$  or  $\widetilde{\nu}^\circ$  for some diagram  $\nu$  then  $\lambda/\mu$  has more than one filling utilising the integers  $1, \dots, n$  whose reading word is lattice.

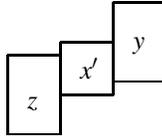
Consider the skew tableau  $T$  of such a shape  $\lambda/\mu$  whose reading word is lattice where each column  $c$  is filled with the integers  $1, \dots, l(c)$  in increasing order. If  $\lambda/\mu$  has a column  $c_1$  such that  $l(c_1) < n$  and  $c_1$  contains at least one box with no box to the right of it, and a column  $c_2$  to the right of  $c_1$  such that  $l(c_2) < n$  then if  $l(c_1) \leq l(c_2)$  change the entry in the last box of  $c_1$  to  $l(c_2) + 1$  otherwise, unless the box at the head of  $c_1$  and the column immediately to the right of it are in the same row, change the entry  $l(c_2)$  in  $c_1$  to  $l(c_2) + 1$  and increase all entries below it in  $c_1$  by 1 to form a new skew tableau  $T'$  of shape  $\lambda/\mu$  whose reading word is lattice.

Thus if  $\lambda/\mu$  does not satisfy these criteria then  $\lambda/\mu$  must be of the form



where  $x$  is a rectangle, and by Lemma 3.1 and what we have already proved  $y$  (and  $z$ ) must consist of a non-rectangular (skew) diagram  $\delta$  or  $\delta^\circ$ , for some diagram  $\delta$ , whose column lengths are all less than  $n$  with a non-negative number of columns to the right or left of it that consist of  $n$  boxes. If  $c_s$  is the column of shortest length  $l(c_s)$  in  $y$  then change the entry  $l(c_s)$  in the first column of  $x$  to  $l(c_s) + 1$  and increase all the entries below it in that column by 1. If the first column of  $x$  does not contain the entry  $l(c_s)$  then change the entry in the last box to  $l(c_s) + 1$ . In both instances we form a new skew tableau  $T'$  of shape  $\lambda/\mu$  whose reading word is lattice.

Hence the number of columns in  $x$  must be zero and  $\lambda/\mu$  must be of the form



where the length of every column of  $x'$  is  $n$  and  $y$  (and  $z$ ) consists of a non-rectangular (skew) diagram  $\delta$  or  $\delta^\circ$ , for some diagram  $\delta$ , whose column lengths are all less than  $n$  with a non-negative number of columns to the right (respectively left) that consist of  $n$  boxes. For clarity of exposition we identify  $y$  and  $z$  with the sub (skew) diagram  $\delta$  or  $\delta^\circ$  that they contain. Let  $\delta$  and  $\varepsilon$  be diagrams and  $c_l, c_s$  be the columns of longest or shortest length in  $\delta$  respectively.

If  $y = \delta$  and  $z = \varepsilon$  then it follows that  $x'$  must contain zero columns otherwise we can change the entries in the first column of  $\varepsilon$ ; every column in  $\varepsilon$  must be at least as long as the longest column of  $\delta$ ; and the box at the head of the leftmost column of  $\delta$  and the rightmost column of  $\varepsilon$  are not in the same row. Change the entry  $l(c_l)$  in the first column of  $\varepsilon$  to  $l(c_l) + 1$  and increase all the entries below it in that column by 1.

If  $y = \delta^\circ$  and  $z = \varepsilon^\circ$  then  $x'$  must contain zero columns; every column in  $\varepsilon^\circ$  must be no longer than the shortest column in  $\delta^\circ$ ; and the box at the base of the leftmost column of  $\delta^\circ$  and the rightmost column of  $\varepsilon^\circ$  are not in the same row. Change the last entry in the first column of  $\varepsilon^\circ$  to  $l(c_l) + 1$ .

If  $y = \delta$  and  $z = \varepsilon^\circ$  then either  $x'$  must contain zero columns or the box at the base of the rightmost column of  $\varepsilon^\circ$  is in the same row as the adjacent column of length  $n$ . In either case we can apply an argument similar to that above to either change the entries of the rightmost column of  $\varepsilon^\circ$  from  $l(c_s)$  downwards by increasing them by 1 or change the entry in the last box to  $l(c_s) + 1$ .

Finally if  $y = \delta^\circ$  and  $z = \varepsilon$  then the same conditions must be satisfied as for the case  $y = \delta$  and  $z = \varepsilon$ . However, we can change the entries of the rightmost column of  $\varepsilon$  from  $l(c_l)$  downwards by increasing them by 1.

In each situation we have been able to create a new skew tableau  $T'$  of shape  $\lambda/\mu$  whose reading word is lattice, and having eliminated all possibilities the result follows. ■

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