

A Proof of Solomon's Rule

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Abstract

We put forward a proof of Solomon's rule, in terms of matrices, for multiplication in the descent algebra of the symmetric group. Our proof exploits the graphs that we can obtain from all the subsets of the set of transpositions, $\{(i, i + 1)\}_{i=1}^{n-1}$.

Let W be a Coxeter group with generating set, S , of fundamental reflections. If J is any subset of S , let W_J be the subgroup generated by J . Let X_J be the unique set of minimal length left coset representatives of W_J . Note that $X_J^{-1} = \{x^{-1} | x \in X_J\}$ is the unique set of minimal length representatives for the right cosets of W_J . Let $l(y)$ denote the length of y in W .

Solomon [9] then gives us the following theorem:

Theorem 1 *For every subset K of S , let*

$$\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.$$

Then for subsets J and K in S

$$\mathcal{X}_J \mathcal{X}_K = \sum_{x \in X_J^{-1} \cap X_K} \mathcal{X}_{x^{-1} J x \cap K}$$

From this it follows that the set of all \mathcal{X}_K form a basis for an algebra, the *descent algebra* of W .

Independently, interpretations of this theorem involving certain matrices have been developed for the descent algebras of the Coxeter groups of types A and B as a means of obtaining further results about these algebras, [6],[5],[1], [2], [3]. However, no such matrix interpretation was known for the Coxeter groups

of type D . In this paper we shall develop some tools and a lemma from which we can easily deduce the matrix interpretation of Theorem 1 for the descent algebra of the Coxeter groups of type A . The purpose of this paper, however, is not just to give another proof of a well known result, instead it is to act as a precursor to a subsequent paper in which we formulate the missing matrix interpretation for the descent algebras of the Coxeter groups of type D , [4].

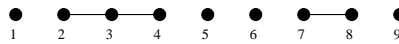
To develop our tools, let us take our Coxeter group W to be the Coxeter group of type A with $n - 1$ fundamental reflections, that is the symmetric group S_n . More specifically, let us take S_n to be the group of permutations acting on the set $N = \{1, \dots, n\}$, with generating set S , where S is the set of $n - 1$ transpositions s_1, s_2, \dots, s_{n-1} , such that $s_i = (i, i + 1)$.

If J is a subset of S , then we define the *graph* $\mathcal{J} = (N, \mathcal{E})$ of J to be the graph with vertex set $\{1, \dots, n\}$, and edge set $\mathcal{E} = \{(i, i + 1) \in J\}$. In general we shall use roman capitals J, K, \dots for subsets of S , and their calligraphic counterparts $\mathcal{J}, \mathcal{K}, \dots$ for their associated graphs. Suppose now that \mathcal{J} has r connected components. A set of vertices is associated with each component, and we can order these sets by their least elements in a natural way. Once ordered, we can label them $\mathcal{J}_1, \dots, \mathcal{J}_r$ such that $1 \in \mathcal{J}_1$, and define the *ordered presentation* of \mathcal{J} to be the ordered list

$$(\mathcal{J}_1, \dots, \mathcal{J}_r).$$

Note that this is the canonical ordered set partition associated to J , and that if $u \in \mathcal{J}_i$ and $v \in \mathcal{J}_j$, and $i < j$, then $u < v$.

Example 1 In S_9 . If $J = \{(2, 3), (3, 4), (7, 8)\}$, then \mathcal{J} is



with

$$\mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2, 3, 4\}, \mathcal{J}_3 = \{5\}, \mathcal{J}_4 = \{6\}, \mathcal{J}_5 = \{7, 8\}, \mathcal{J}_6 = \{9\}.$$

The ordered presentation of \mathcal{J} is

$$(\{1\}, \{2, 3, 4\}, \{5\}, \{6\}, \{7, 8\}, \{9\}).$$

Let us define $\mathcal{W}_{\mathcal{J}_i}$ to be the subgroup of S_n that consists of all permutations of S_n that fix all points outside of \mathcal{J}_i , and let

$$\mathcal{W}_{\mathcal{J}} = \mathcal{W}_{\mathcal{J}_1} \times \dots \times \mathcal{W}_{\mathcal{J}_r}.$$

Observe that $\mathcal{W}_{\mathcal{J}} = \mathcal{W}_J$. If we let κ be a composition of n , with components $\kappa_1, \kappa_2, \dots, \kappa_r$, and define

$$S_{\kappa} = S_{\kappa_1} \times \dots \times S_{\kappa_r},$$

then $S_\kappa \cong \mathcal{W}_\mathcal{J}$, where the sets \mathcal{J}_i of \mathcal{J} satisfy

$$|\mathcal{J}_i| = \kappa_i. \quad (1)$$

Let us now take J and K to be any subsets of S , and $x \in W$, and let $x\mathcal{J}$ denote the image of the graph \mathcal{J} under x ; that is $(x(i), x(j))$ is an edge in $x\mathcal{J}$ if and only if (i, j) is an edge in \mathcal{J} . Let $\mathcal{J} \cap \mathcal{K}$ be the graph with vertex set N whose edges are those present in both \mathcal{J} and \mathcal{K} . The ordered presentation of $x^{-1}\mathcal{J} \cap \mathcal{K}$, where $x \in X_J^{-1} \cap X_K$, is given by the following lemma.

Lemma 1 *Let J and K be subsets of S , and let $x \in X_J^{-1} \cap X_K$. Let the ordered presentation of \mathcal{J} be $(\mathcal{J}_1, \dots, \mathcal{J}_r)$, and \mathcal{K} be $(\mathcal{K}_1, \dots, \mathcal{K}_s)$. Then the ordered presentation of $x^{-1}\mathcal{J} \cap \mathcal{K}$ is*

$$\begin{aligned} & (x^{-1}\mathcal{J}_1 \cap \mathcal{K}_1, x^{-1}\mathcal{J}_2 \cap \mathcal{K}_1, \dots, x^{-1}\mathcal{J}_r \cap \mathcal{K}_1, \\ & x^{-1}\mathcal{J}_1 \cap \mathcal{K}_2, x^{-1}\mathcal{J}_2 \cap \mathcal{K}_2, \dots, x^{-1}\mathcal{J}_r \cap \mathcal{K}_2, \\ & \quad \dots, \\ & x^{-1}\mathcal{J}_1 \cap \mathcal{K}_s, x^{-1}\mathcal{J}_2 \cap \mathcal{K}_s, \dots, x^{-1}\mathcal{J}_r \cap \mathcal{K}_s) \end{aligned} \quad (2)$$

with empty sets removed.

PROOF To prove that (2) is the ordered presentation of $x^{-1}\mathcal{J} \cap \mathcal{K}$, it is sufficient to prove the following two statements

1. The elements of each set are less than those that appear in any set later in the list.
2. Each non-empty set $x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ is indeed the vertex set of a connected component of $x^{-1}\mathcal{J} \cap \mathcal{K}$.

To prove statement 1 we must show that

1.1 The elements in $x^{-1}\mathcal{J}_r \cap \mathcal{K}_m$ are less than those in $x^{-1}\mathcal{J}_1 \cap \mathcal{K}_{m+1}$

1.2 The elements in $x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ are less than those in $x^{-1}\mathcal{J}_{q+1} \cap \mathcal{K}_m$.

Case 1.1 follows immediately, since all vertices in \mathcal{K}_m are less than those in \mathcal{K}_{m+1} by definition.

To prove case 1.2 we need only show that if the vertex $i \in x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ and $j \in x^{-1}\mathcal{J}_{q+1} \cap \mathcal{K}_m$, then $i < j$. To do this we shall first prove that all vertices in \mathcal{K}_m appear from left to right in increasing order in the list $x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n)$.

From the definition of X_K as a set of minimal length coset representatives, it follows that if $x \in X_K$ then $l(xk) > l(x)$ for all $k \in K$. In S_n , $l(x)$ is the number

of inversions in x , that is the number of $h < l$ for which $x(l) < x(h)$, [7]. Hence it follows that for all $k = (h, h + 1) \in K$ we have $x(h) < x(h + 1)$, since xk , x differ only in the reversing of h and $h + 1$. From this we can deduce that h is to the left of $h + 1$ in the list

$$x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n).$$

Now suppose that $i \in x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$, $j \in x^{-1}\mathcal{J}_{q+1} \cap \mathcal{K}_m$. Then $x(i) = u \in \mathcal{J}_q$, and $x(j) = v \in \mathcal{J}_{q+1}$. It follows that $u < v$, and so $x^{-1}(u)$ appears before $x^{-1}(v)$ in $x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n)$. However, $x^{-1}(u) = i$ and $x^{-1}(v) = j$, and since we know that the vertices of \mathcal{K}_m appear in increasing order from left to right in $x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n)$, it follows that $i < j$.

Statement 2 will follow if we can prove the following assertions.

2.1 The sets $x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ are all disjoint.

2.2 No edge in $x^{-1}\mathcal{J} \cap \mathcal{K}$ connects vertices in different subsets $x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ and $x^{-1}\mathcal{J}_{q'} \cap \mathcal{K}_{m'}$.

2.3 For every $i, i + 1 \in x\mathcal{J}_q \cap \mathcal{K}_m$, an edge exists in $x^{-1}\mathcal{J} \cap \mathcal{K}$ between $i, i + 1$.

Again, assertion 2.1 follows since all \mathcal{J}_q and \mathcal{K}_m are disjoint and x is a bijection from N to itself.

To prove assertion 2.2, let (u, v) be an edge in $x^{-1}\mathcal{J} \cap \mathcal{K}$, such that $u \in x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ and $v \in x^{-1}\mathcal{J}_{q'} \cap \mathcal{K}_{m'}$. We know that \mathcal{J}_q and $\mathcal{J}_{q'}$ are vertex sets of connected components of \mathcal{J} , so $x^{-1}\mathcal{J}_q$ and $x^{-1}\mathcal{J}_{q'}$ must be vertex sets of connected components of $x^{-1}\mathcal{J}$. Hence, $q = q'$. Similarly, \mathcal{K}_m and $\mathcal{K}_{m'}$ are vertex sets of connected components of \mathcal{K} , and so $m = m'$.

For assertion 2.3, let $i, i + 1 \in x^{-1}\mathcal{J}_q \cap \mathcal{K}_m$ and let $x(i) = u$, and $x(i + 1) = u + l$. Since we know from the proof of case 1.2 that all $i \in \mathcal{K}_m$ appear in increasing order from left to right in the list $x^{-1}(1), x^{-1}(2), \dots, x^{-1}(n)$, we can deduce that $l \geq 1$. We can also deduce that because $X_{\mathcal{J}}^{-1}$ is defined as a set of minimal length right coset representatives, we have that $x^{-1}(v) < x^{-1}(v + 1)$ for all $(v, v + 1) \in \mathcal{J}$.

Therefore, since $u, u + l \in \mathcal{J}_q$, we have that $u + k \in \mathcal{J}_q$ for all $k = 0, \dots, l$ such that

$$x^{-1}(u) < x^{-1}(u + 1) < \dots < x^{-1}(u + l - 1) < x^{-1}(u + l).$$

However, $x^{-1}(u) = i$, $x^{-1}(u + l) = i + 1$, so it follows that $l = 1$, and so, by definition $(u, u + l)$ is an edge in \mathcal{J} . Therefore, since $x^{-1}(u) = i$ and $x^{-1}(u + l) = i + 1$, it follows that $(i, i + 1)$ is an edge in $x^{-1}\mathcal{J} \cap \mathcal{K}$, and we are done. ■

As a consequence of Lemma 2 [9], and our Lemma 1

$$\begin{aligned}
x^{-1}W_Jx \cap W_K &= W_{x^{-1}Jx \cap K} \\
&= \mathcal{W}_{x^{-1}\mathcal{J} \cap \mathcal{K}} \\
&= \mathcal{W}_{x^{-1}\mathcal{J}_1 \cap \mathcal{K}_1} \times \dots \times \mathcal{W}_{x^{-1}\mathcal{J}_r \cap \mathcal{K}_s} \\
&= (x^{-1}\mathcal{W}_{\mathcal{J}_1}x \cap \mathcal{W}_{\mathcal{K}_1}) \times \dots \times (x^{-1}\mathcal{W}_{\mathcal{J}_r}x \cap \mathcal{W}_{\mathcal{K}_s}) \\
&\cong (x^{-1}S_{\kappa_1}x \cap S_{\nu_1}) \times \dots \times (x^{-1}S_{\kappa_r}x \cap S_{\nu_s})
\end{aligned}$$

where κ and ν are suitable compositions of n determined by J , K respectively, according to condition (1). Note that the final isomorphism symbol is an equality if $x^{-1}S_{\kappa_i}x \cap S_{\nu_j}$ is regarded as the group of permutations on $x^{-1}\mathcal{J}_i \cap \mathcal{K}_j$.

Let

$$z_{ij} = |x^{-1}\mathcal{J}_i \cap \mathcal{K}_j|$$

then, by Theorem 1.3.10 [8], we have a bijective mapping

$$\zeta : x \mapsto (z_{ij})$$

from $X_J^{-1} \cap X_K$ into the set of $s \times r$ matrices with non-negative integer entries, $\mathbf{z} = (z_{ij})$, which satisfy

$$\sum_i z_{ij} = \kappa_j, \quad \sum_j z_{ij} = \nu_i.$$

Observe that reading the non-zero entries of the matrix \mathbf{z} by row give a composition, η , of n . We say that η is the *reading word* of \mathbf{z} , and note that S_η is isomorphic to $W_{x^{-1}Jx \cap K}$. We also observe that each matrix corresponds to one $x \in X_J^{-1} \cap X_K$, given in Solomon's Theorem. Therefore, if we now rename the basis elements such that X_J becomes B_κ , where the components of κ in order are the sizes of the vertex sets of \mathcal{J} taken in the natural order, we can recast Solomon's Theorem in terms of compositions and matrices as follows.

Theorem 2 *For every composition ν of n , let X_ν be the unique set of minimal length left coset representatives of S_n/S_ν . Let*

$$B_\nu = \sum_{\sigma \in X_\nu} \sigma.$$

If κ, ν are compositions of n , then

$$B_\kappa B_\nu = \sum_{\mathbf{z}} B_\eta$$

where the sum is over all matrices $\mathbf{z} = (z_{ij})$ with non-negative integer entries that satisfy

1. $\sum_i z_{ij} = \kappa_j,$
2. $\sum_j z_{ij} = \nu_i.$

For each matrix, z , η is the reading word of z .

This is precisely the classical matrix interpretation of Solomon's Theorem for the symmetric groups, for instance Proposition 1.1, [6].

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