Annals of Combinatorics 11 (2007) 69-77
0218-0006/07/010069-9
DOI 10.1007/s00026-007-0306-1

Annals of Combinatorics

Multiplicity Free Expansions of Schur *P*-Functions

Kristin M. Shaw and Stephanie van Willigenburg
Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
{krishaw, steph}@math.ubc.ca

Received February 28, 2005

AMS Subject Classification: 05E05, 05A17, 05A19, 05E10

Abstract. After deriving inequalities on coefficients arising in the expansion of a Schur *P*-function in terms of Schur functions we give criteria for when such expansions are multiplicity free. From here we study the multiplicity of an irreducible spin character of the twisted symmetric group in the product of a basic spin character with an irreducible character of the symmetric group, and determine when it is multiplicity free.

Keywords: multiplicity free, Schur functions, Schur *P*-functions, spin characters, staircase partitions

1. Introduction

In [4], Stembridge determined when the product of two Schur functions is multiplicity free, which yielded when the outer product of characters of the symmetric groups did not have multiplicities. Meanwhile, in [1], Bessenrodt determined when the product of two Schur *P*-functions is multiplicity free. This led to an analogous classification with respect to projective outer products of spin characters of double covers of the symmetric groups. In this article we interpolate between these two results to determine when a Schur *P*-function expanded in terms of Schur functions is multiplicity free. As an application we give criteria for when the multiplicity of an irreducible spin character of the twisted symmetric groups in the product of a basic spin character with an irreducible character of the symmetric groups is 0 or 1.

The remainder of this paper is structured as follows. We review the necessary definitions in the rest of this section. Then in Section 2 we derive some equalities and inequalities concerning certain coefficients. In Section 3 we give criteria for a Schur *P*-function to have a multiplicity free Schur function expansion before applying this to character theory in Section 4.

1.1. Partitions

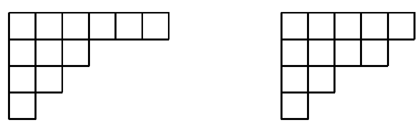
A *partition* $\lambda = \lambda_1\lambda_2\cdots\lambda_k$ of n is a list of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ whose sum is n, denoted $\lambda \vdash n$. We say k is the *length* of λ denoted by $l(\lambda)$, n is the *size* of λ and call the λ_i *parts*. Also denote the set of all partitions of n by $P(n)$. Contained in $P(n)$ is the subset of partitions $D(n)$ consisting of all the partitions whose parts are distinct, i.e., $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$. We call such partitions *strict*. A strict partition that will be of particular interest to us will be the *staircase* (of length k): $k(k-1)(k-2)\cdots 321$. Two other partitions that will be of interest to us are $\lambda + 1^r$ and $\lambda \cup r$ for any given partition λ, and positive integer r. The partition $\lambda + 1^r$ is formed by adding 1 to the parts $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $\lambda \cup r$ is formed by sorting the multiset of the union of parts of λ and r. With these concepts in mind we are able to define a final partition that will be of interest to us, known as a near staircase. A partition is a *near staircase* if it is of the form $\lambda + 1^r$, $1 \le r \le k$ or $\lambda \cup r$, $r \ge k+1$ and λ is the staircase of length k.

Example 1.1. 6321 and 5421 are both near staircases of 12.

1.2. Diagrams and Tableaux

For any partition $\lambda \vdash n$ the associated (*Ferrers*) *diagram*, also denoted by λ, is an array of left justified boxes with λ_i boxes in the i-th row, for $1 \le i \le l(\lambda)$. Observe that in terms of diagrams a near staircase is more easily visualised as a diagram of a strict partition such that the deletion of exactly one row or column yields the diagram of a staircase.

Example 1.2. The near staircases 6321 and 5421.



Given a diagram λ then the *conjugate* diagram of λ, λ', is formed by transposing the rows and columns of λ. The resulting partition λ' is also known as the conjugate of λ. The *shifted* diagram of λ, $S(\lambda)$, is formed by shifting the i-th row $(i-1)$ boxes to the right. If we are given two diagrams λ and μ such that if μ has a box in the (i, j)-th position then λ has a box in the (i, j)-th position then the *skew* diagram λ/μ is formed by the array of boxes

$$\{c \,|\, c \in \lambda, c \notin \mu\}.$$

Now that we have introduced the necessary diagrams we are now in a position to fill the boxes and form tableaux.

Consider the alphabet

$$1' < 1 < 2' < 2 < 3' < 3 < \cdots.$$

For convenience we call the integers $\{1, 2, 3, \ldots\}$ *unmarked* and the integers $\{1', 2', 3', \ldots\}$ *marked*. Any filling of the boxes of a diagram λ with letters from the above alphabet is called a *tableau* of shape λ. If we fill the boxes of a skew or shifted diagram we similarly obtain a skew or shifted tableau. Given any type of tableau, T, we define the *reading word* $w(T)$ to be the entries of T read from right to left and top to bottom, and define the *augmented reverse reading word* $\hat{w}(T)$ to be $w(T)$ read backwards with each entry increased by one according to the total order on our alphabet, e.g., if $T = \begin{matrix} 1' & 1 \\ 1 & 2' \\ 2 & \end{matrix}$ then $w(T) = 11'2'12$ and $\hat{w}(T) = 3'2'212'$. When there is no ambiguity concerning the tableau under discussion we refer to the reading word and augmented reverse reading word as w and $\hat{w}$ respectively. We also define the content of T, $c(T)$, to be the sequence of integers $c_1 c_2 \cdots$, where

$$c_i = |i| + \left|i'\right|$$

and $|i|$ is the number of is in $w(T)$ and $|i'|$ is the number of i's in $w(T)$. For our previous example $c(T) = 32$. Given a word we say it is *lattice* if as we read it if the number of is we have read is equal to the number of $(i+1)$s we have read then the next symbol we read is neither an $(i+1)$ nor an $(i+1)'$, e.g., $11'2'23$ is lattice, however, $11'22'3$ is not.

Let T be a (skew or shifted) tableau, then we say T is *amenable* if it satisfies the following conditions [2, p. 259]:

(1) The entries in each row of T weakly increase.
(2) The entries in each column of T weakly increase.
(3) Each row contains at most one i' for $i \geq 1$.
(4) Each column contains at most one i for each $i \geq 1$.
(5) The word $w\hat{w}$ is lattice.
(6) In w the rightmost occurrence of i is to the right of the rightmost occurrence of i' for all i.

Example 1.3. The first tableau is amenable whilst the second is not as it violates the lattice condition.

$$\begin{matrix} 1' & 1 \\ 1 & 2' \\ 2 & \end{matrix} \qquad \begin{matrix} 1' & 1 \\ 1 & 2 \\ 2 & \end{matrix}$$

Before we define Schur P-functions we make two observations about amenable tableaux.

Lemma 1.4. *Let T be an amenable tableau. If i or i' appears in row j then $j \geq i$.*

Proof. We proceed by induction on the number of rows of T. If T has one row then the result is clear. Assume the result holds up to row $(k-1)$. Consider row k. If it has an entry in it greater than k, l or l', then it must lie in the rightmost box since the rows of T weakly increase. However, this ensures that $w\hat{w}$ is not lattice as when we first read l or l' in w we will have read no $(l-1)$ or l. ∎

Lemma 1.5. *Let T be an amenable tableau with $c(T) = k(k-1)(k-2)\cdots(k-j)$, then in $w(T)$*

$$\left|i'\right| \leq \left|(i+1)'\right|, \quad 1 \leq i \leq j.$$

Proof. To prove this we consider the lattice condition on $w\hat{w}$. Assume $|i'| > |(i+1)'|$ then as we read w there will be a rightmost occurrence when $|i| = |(i+1)|$. However, because of $c(T)$ before we read another i in $w\hat{w}$ we must read $(i+1)$ or $(i+1)'$. ∎

1.3. Schur P-Functions

Given commuting variables $x_1, x_2, x_3, \dots$, let the r-th elementary symmetric function, e_r, be defined by

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Moreover, for any partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k$ let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k},$$

then the algebra of symmetric functions, Λ, is the algebra over $\mathbb{C}$ spanned by all e_λ where $\lambda \vdash n$, $n > 0$ and $e_0 = 1$. Another well-known basis for Λ is the basis of *Schur functions*, s_λ, defined by

$$s_\lambda = \det\left(e_{\lambda_i{}' - i + j}\right)_{1 \le i, j \le k},$$

where $e_r = 0$ if $r < 0$. It is these that can be used to define the subalgebra of Schur P-functions, Γ. More precisely, let λ be a strict partition of $n > 0$, then Γ is spanned by $P_0 = 1$ and all

$$P_\lambda = \sum_{\mu \vdash n} g_{\lambda\mu} s_\mu,$$

where $g_{\lambda\mu}$ is the number of amenable tableaux T of shape μ and content λ.

Amenable tableaux can also be used to describe the multiplication rule for the P_λ as follows. Let λ, μ, ν be strict partitions then

$$P_\mu P_\nu = \sum f^\lambda_{\mu\nu} P_\lambda,$$

where $f^\lambda_{\mu\nu}$ is the number of amenable shifted skew tableaux of shape λ/μ and content ν. Further details on Schur and Schur P-functions can be found in [2].

2. Relations on Stembridge Coefficients

The combinatorial descriptions of the $g_{\lambda\mu}$ and $f^\lambda_{\mu\nu}$ from the previous section were discussed by Stembridge [3] who also implicitly observed the following useful relationship between them.

Lemma 2.1. *If $\lambda \in D(n)$, $\mu \in P(n)$, and ν is a staircase of length $l(\mu)$ then*

$$f^{\mu+\nu}_{\lambda\nu} = g_{\lambda\mu}.$$

Proof. By definition $f^{\mu+\nu}_{\lambda\nu}$ is the number of amenable shifted skew tableaux of shape $(\mu+\nu)/\lambda$ and content ν. However, since $P_\lambda P_\nu = P_\nu P_\lambda$ we know $f^{\mu+\nu}_{\lambda\nu}$ is also the number of amenable shifted skew tableaux of shape $(\mu+\nu)/\nu$ and content λ. In addition, the shifted skew diagram $(\mu+\nu)/\nu$ is simply the diagram μ. Thus $f^{\mu+\nu}_{\lambda\nu}$ is the number of amenable tableaux of shape μ and content λ, and this is precisely $g_{\lambda\mu}$. ∎

There are also equalities between the $g_{\lambda\mu}$.

Lemma 2.2. *If $\lambda \in D(n)$ and $\mu \in P(n)$ then*

$$g_{\lambda\mu} = g_{\lambda\mu'}.$$

Proof. Let $\omega\colon \Lambda \to \Lambda$ be the involution on symmetric functions such that $\omega(s_\lambda) = s_{\lambda'}$. In [2, p. 259, Exercise 3] it was proved that $\omega(P_\lambda) = P_\lambda$. Consequently,

$$\omega(P_\lambda) = \sum g_{\lambda\mu}\omega(s_\mu) = \sum g_{\lambda\mu}s_{\mu'}$$

and the result follows. ∎

Finally, we present two inequalities that will be useful in the following sections and relate amenable tableaux of different shape and content.

Lemma 2.3. *Given $\lambda \in D(n)$ and $\mu \in P(n)$, if $r \le l(\lambda)+1$ and $s \ge \lambda_1 + 1$ then*

$$g_{\lambda\mu} \le g_{(\lambda+1^r)(\mu+1^r)}$$

and

$$g_{\lambda\mu} \le g_{(\lambda\cup s)(\mu\cup s)}.$$

Proof. Consider an amenable tableau T of shape μ and content λ. To prove the first inequality, for $1 \le i \le r$ append a box containing i to row i on the right side of T. By Lemma 1.4 it is straightforward to verify that this is an amenable tableau of shape $\mu + 1^r$ and content $\lambda + 1^r$. For the second inequality, replace each entry i with $(i+1)$ and each entry i' with $(i+1)'$ for $1 \le i \le l(\lambda)$ to form T'. Then append a row of s boxes each containing 1 to the top of T'. Again, it is straightforward to check this is an amenable tableau of shape $\mu \cup s$ and content $\lambda \cup s$. ∎

Example 2.4. Consider the amenable tableau $\begin{matrix} 1' & 1 & 1 \\ 1' & 2 & 2 \\ 1 & & \end{matrix}$. Then the following two tableaux illustrate the operations utilised in proving the first and second inequalities, respectively.

$$\begin{matrix} 1' & 1 & 1 & \mathbf{1} \\ 1' & 2 & 2 & \mathbf{2} \\ 1 & \mathbf{3} & & \end{matrix} \qquad \begin{matrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 2' & 2 & 2 & & & \\ 2' & 3 & 3 & & & \\ 2 & & & & & \end{matrix}$$

3. Multiplicity Free Schur Expansions

Despite the number of conditions amenable tableaux must satisfy, it transpires that most Schur P-functions do not have multiplicity free expansions in terms of Schur functions.

Example 3.1. Neither P_{541} nor P_{654} is multiplicity free. We see this in the first case by observing there are at least two amenable tableaux of shape 4321 and content 541:

$$\begin{matrix} 1' & 1 & 1 & 1 \\ 1 & 2' & 2 & \\ 2 & 2 & & \\ 3 & & & \end{matrix} \qquad \begin{matrix} 1' & 1 & 1 & 1 \\ 1 & 2' & 2 & \\ 2' & 3 & & \\ 2 & & & \end{matrix}\,.$$

In the second case we note that there are at least two amenable tableaux of shape 54321 and content 654:

$$\begin{array}{lllll} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2' & 3' & 3 & & \\ 2 & 3' & & & \\ 3 & & & & \end{array} \qquad \begin{array}{lllll} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 2 & 3' & & \\ 3' & 3 & & & \\ 3 & & & & \end{array} .$$

However, in some cases it is easy to deduce a certain Schur P-function expansion is multiplicity free as we can give a precise description of it in terms of Schur functions.

Proposition 3.2.

$$P_n = \sum_{1\leq k\leq n} s_{k1^{n-k}}$$

and

$$P_{(n-1)1} = \sum_{1\leq k\leq n-1} s_{k1^{n-k}} + \sum_{2\leq k\leq n-2} s_{k21^{n-k-2}}.$$

Proof. For the first result observe that $g_{n(k1^{n-k})} = 1$ since the tableau filled with one $1'$ and $(k-1)$ 1s in the first row and one 1 and $(n-k)$ $1'$s in the first column is the only amenable tableau of shape $k1^{n-k}$ and content n. Then observe that $g_{n\mu} = 0$ for any other μ since we have no way to fill a 2×2 rectangle with only 1 or $1'$ and create an amenable tableau.

For the second result note that since the content of any tableau we create is $(n-1)1$ we will be filling our diagram with $(n-1)$ 1 or $1'$s and one 2. As in the previous case if our resulting tableau is to be amenable the 1s must appear in the first row and column marked or unmarked as necessary. The unmarked 2 can now only appear in one of two places, either at the end of the second row, or the end of the first column. ∎

A third multiplicity free expansion is obtained from the determination of when a Schur P-function is equal to a Schur function.

Theorem 3.3.

$$P_\lambda = s_\lambda \text{ if and only if } \lambda \text{ is a staircase.}$$

Proof. The reverse implication is proved in [2, p. 259, Exercise 3(b)]. For the forward implication assume that $\lambda = \lambda_1 \cdots \lambda_k \in D(n)$ is not a staircase. It follows there must exist at least one $1 \leq i \leq k$ for which $\lambda_i \geq \lambda_{i+1} + 2$. We are going to show that in this situation there are at least two amenable tableaux with content λ.

Consider the tableau, T, of shape and content λ where the j-th row is filled with unmarked js. Clearly T is amenable. Now consider the first row i for which $\lambda_i \geq \lambda_{i+1} + 2$. Delete the rightmost box from this row and append it to the first column of T to form a tableau T' of shape $\lambda_1\lambda_2\cdots\lambda_i - 1\cdots\lambda_k 1$ and content λ. Now alter the entries in the first column of T' as follows. In row i change i to i' and in rows $i+1 \leq j \leq k$ change j to $j-1$. Finally in row $k+1$ change i to k. Now T' is an amenable tableau, and we are done. ∎

As we will see, staircases play an important role in the determination of multiplicity free expansions of Schur P-functions.

Theorem 3.4. *For $\lambda \in D(n)$ the Schur function expansion of P_λ is multiplicity free if and only if λ is one of the following partitions*:

(1) *staircase,*
(2) *near staircase,*
(3) $k(k-1)(k-2)\cdots 43$,
(4) $k(k-1)$.

Proof. If $\lambda \in D(n)$ is not one of the partitions listed in Theorem 3.4 then it must satisfy one of the following conditions:

(1) For all $1 \le i < l(\lambda)$, $\lambda_i = \lambda_{i+1} + 1$ and $\lambda_{l(\lambda)} \ge 4$ and $l(\lambda) \ge 3$.
(2) There exists exactly one $1 < i < l(\lambda)$ such that $\lambda_i \ge \lambda_{i+1} + 3$ and $\lambda_j = \lambda_{j+1} + 1$ for all $1 \le j < l(\lambda)$, $j \ne i$ and $\lambda_{l(\lambda)} = 1$.
(3) There exists $i < j$ such that $\lambda_i \ge \lambda_{i+1} + 2$ and $\lambda_j \ge \lambda_{j+1} + 2$ and $\lambda_{l(\lambda)} = 1$.
(4) There exists $1 \le i < l(\lambda)$ such that $\lambda_i \ge \lambda_{i+1} + 2$ and $\lambda_{l(\lambda)} \ge 2$.

If λ satisfies the first criterion then by observing P_{654} is not multiplicity free and Lemma 2.3, it follows that P_λ is not multiplicity free. Similarly, if λ satisfies the second criterion then by observing that P_{541} has multiplicity and Lemma 2.3, again P_λ has multiplicity. If λ satisfies the third criterion then consider the partition $\nu = k(k-2)(k-3)\cdots 431 \in D(n)$ and the partition $\mu = (k-1)(k-2)(k-3)\cdots 432 \in D(n)$ for $k \ge 5$. We now show that $g_{\nu\mu} > 1$. Take a diagram μ and fill the first row with one $1'$ and $(k-2)$ 1s. For $i > 1$ fill the i-th row with one $(i-1)$ and the rest is. This is clearly an amenable tableau T. If we now change the $(k-3)$ to a $(k-3)'$ in the second column of the penultimate row of T we obtain another amenable tableau of shape μ and content ν. Thus $g_{\nu\mu} > 1$ and P_ν is not multiplicity free. This combined with Lemma 2.3 yields that P_λ is not multiplicity free if λ satisfies the third criterion. Lastly, if λ satisfies the fourth criterion then consider the partition $\nu = k(k-2)(k-3)\cdots 432 \in D(n)$ and the staircase $\mu = (k-1)(k-2)(k-3)\cdots 4321 \in D(n)$ for $k \ge 4$. We now prove that $g_{\nu\mu} > 1$. Take the staircase μ and fill the first row with one $1'$ and $(k-2)$ 1s. For $1 < i < k-1$ fill the i-th row with one $(i-1)$ and the rest is. Fill the last row with one $(k-2)$. It is straightforward to see that this is an amenable tableau T. If we now change the $(k-2)$ to a $(k-2)'$ in the second column of the penultimate row of T we obtain another amenable tableau of shape μ and content ν and $g_{\nu\mu} > 1$ as desired. Consequently, P_ν is not multiplicity free, which combined with Lemma 2.3 shows that P_λ is not multiplicity free if λ satisfies the fourth criterion.

Finally it remains to show that if $\lambda \in D(n)$ is one of the partitions listed in Theorem 3.4 then $g_{\lambda\mu} \le 1$ for all μ. If λ is a staircase the result follows from Theorem 3.3. If λ is a near staircase of the form $mk(k-1)\cdots 321$ then by Lemma 1.5 it follows that any amenable tableau of content λ must contain no i' for $i > 1$ and thus there can exist at most one amenable tableau of shape μ and content λ for any given μ. Consequently $g_{\lambda\mu} \le 1$ for all μ. Similarly if λ is the other type of near staircase or of the form $k(k-1)\cdots 43$ then by Lemma 2.1 we can calculate $g_{\lambda\mu}$ by enumerating all amenable shifted skew tableaux, T, of shape $(\mu+\nu)/\lambda$ and *staircase* content. By Lemma 1.5 it follows that no entry in T can be marked. From this we can deduce that there can exist at most one amenable shifted skew tableau and so $g_{\lambda\mu} \le 1$. Finally a proof similar to that of Proposition 3.2 yields that $P_{k(k-1)}$ is multiplicity free. ∎

Remark 3.5. The reverse direction of the above theorem can also be proved via Lemma 2.1 and [1, Theorem 2.2].

4. Multiplicity Free Spin Character Expansions

The twisted symmetric group $\tilde{S}_n$ is presented by

$$\left\langle z, t_1, t_2, \ldots, t_{n-1} \,\middle|\, z^2 = 1, t_i^2 = (t_i t_{i+1})^3 = (t_i t_j)^2 = z \; |i-j| \geq 2 \right\rangle.$$

Moreover, the ordinary representations of $\tilde{S}_n$ are equivalent to the projective representations of the symmetric group S_n and in [3] Stembridge determined the product of a basic spin character of $\tilde{S}_n$ with an irreducible character of S_n, whose description we include here for completeness. If $\lambda = \lambda_1 \cdots \lambda_k \in D(n)$ then define

$$\varepsilon_\lambda = \begin{cases} 1, & \text{if } n-k \text{ is even,} \\ \sqrt{2}, & \text{if } n-k \text{ is odd.} \end{cases}$$

Let ϕ^λ be an irreducible spin character of $\tilde{S}_n$, χ^μ for $\mu \in P(n)$ be an irreducible character of S_n and $\langle \cdot, \cdot \rangle$ be defined on Λ by $\langle s_\mu, s_\nu \rangle = \delta_{\mu\nu}$ then we have

Theorem 4.1. [3, Theorem 9.3] *If $\lambda \in D(n)$, $\mu \in P(n)$ then*

$$\left\langle \phi^n \chi^\mu, \phi^\lambda \right\rangle = \frac{1}{\varepsilon_\lambda \varepsilon_n} 2^{(l(\lambda)-1)/2} g_{\lambda\mu} \tag{4.1}$$

unless $\lambda = n$, n even, and $\mu = k1^{n-k}$ in which case the multiplicity is 0 *or* 1.

Using this formula we can deduce

Theorem 4.2. *If $\lambda \in D(n), \mu \in P(n)$ then the coefficient of ϕ^λ in $\phi^n \chi^\mu$ is multiplicity free for all μ if and only if λ is one of the following*:

(1) n,
(2) $(n-1)1$,
(3) $k(k-1)$,
(4) $(2k+1)21$,
(5) 543,
(6) 431.

Proof. Considering Equation 4.1 we first show no λ exists such that $g_{\lambda\mu} \geq 2$ but $\langle \phi^n \chi^\mu, \phi^\lambda \rangle$ is multiplicity free. If such a λ did exist then

$$2^{(l(\lambda)-1)/2} < \varepsilon_\lambda \varepsilon_n,$$

where $\varepsilon_\lambda \varepsilon_n = 1, \sqrt{2}, 2$ depending on λ and its size. However, if $\varepsilon_\lambda \varepsilon_n = 1$ then $l(\lambda) = 0$ and if $\varepsilon_\lambda \varepsilon_n = \sqrt{2}$ then $l(\lambda) = 1$ so $\lambda = n$ but then $\varepsilon_\lambda \varepsilon_n \neq \sqrt{2}$ so we must have that

$$2^{(l(\lambda)-1)/2} < 2,$$

and hence $l(\lambda) < 3$. Since $\varepsilon_\lambda, \varepsilon_n = \sqrt{2}$ it follows n is even and we must in fact have $\lambda = n$. By Proposition 3.2 we know in this case $g_{\lambda\mu} \leq 1$ for all μ and we have our desired contradiction. Consequently if $\left\langle \phi^n \chi^\mu, \phi^\lambda \right\rangle$ is multiplicity free then $g_{\lambda\mu} \leq 1$. Additionally we must have $2^{(l(\lambda)-1)/2} = \varepsilon_\lambda \varepsilon_n$ and so it remains for us to check three cases.

(1) $\varepsilon_\lambda \varepsilon_n = 1$. We have $l(\lambda) = 1$ and so by Theorem 3.4 $\lambda = n$, n odd.

(2) $\varepsilon_\lambda \varepsilon_n = \sqrt{2}$. We have $l(\lambda) = 2$ and so by Theorem 3.4 $\lambda = (n-1)1$, or $\lambda = k(k-1)$.

(3) $\varepsilon_\lambda \varepsilon_n = 2$. We have $l(\lambda) = 3$ and since $\varepsilon_n = \sqrt{2}$ it follows that n is even. Hence by Theorem 3.4 $\lambda = (2k+1)21$, $\lambda = 543$, or $\lambda = 431$. ∎

Acknowledgments. The authors would like to thank the referee and Ron King for their valuable comments. Both authors were supported in part by the National Sciences and Engineering Research Council of Canada.

References

1. C. Bessenrodt, On multiplicity-free products of Schur *P*-functions, Ann. Combin. **6** (2002) 119–124.
2. I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd Ed., Oxford University Press, Oxford, 1995.
3. J. Stembridge, Shifted tableaux and the projective representations of symmetric groups, Adv. Math. **74** (1989) 87–134.
4. J. Stembridge, Multiplicity-free products of Schur functions, Ann. Combin. **5** (2001) 113–121.