OPERATORS ON COMPOSITIONS AND GENERALIZED SKEW PIERI RULES

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Abstract. Using operators on compositions we develop further both the theory of quasisymmetric Schur functions and of noncommutative Schur functions. By establishing relations between these operators, we show that the posets of compositions arising from the right and left Pieri rules for noncommutative Schur functions can each be endowed with both the structure of dual graded graphs and dual filtered graphs when paired with the poset of compositions arising from the Pieri rules for quasisymmetric Schur functions and its deformation. As a further application, we simplify the right Pieri rules for noncommutative Schur functions of Tewari. We then derive skew Pieri rules in the spirit of Assaf-McNamara for skew quasisymmetric Schur functions using the Hopf algebraic techniques of Lam-Lauve-Sottile, and recover the original rules of Assaf-McNamara as a special case. Finally we apply these techniques a second time to obtain skew Pieri rules for skew noncommutative Schur functions.

Contents

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1. Introduction

The Hopf algebra of symmetric functions, Sym, with their celebrated basis of Schur functions has been a vibrant area of research for centuries since Schur functions were introduced by Cauchy [6] and then cemented their role in the representation theory of $GL(n)$ early in the twentieth century [31]. Since then, Schur functions have played a pivotal role in Hilbert's 15th problem on Schubert calculus, for example [5], and more recently in quantum physics, for example [35]. One aspect of Schur functions that has, in particular, impacted all of these areas is the combinatorial rule for computing the multiplication of two Schur functions, namely the Littlewood-Richardson rule [21], and its specializations such as the classical Pieri rules [28] and much more recently Pieri rules for skew shapes [1]. A key ingredient of the first proofs of the Littlewood-Richardson rule [32, 38] was the RSK algortihm [18, 29, 30], from which the desire to understand and generalize the enumerative properties of the RSK algorithm led to the theory of differential posets [33] or dual graded graphs [10, 11]. Since then, dual graded graphs have become a fruitful avenue of research in their own right, being generalized to quantized versions [19], and most recently related to K-theory via dual filtered graphs [27]. Schur functions, themselves, have been generalized perhaps most famously as Macdonald polynomials [23], which are Schur functions with additional parameters, q, t such that the Schur functions are recovered when $q = t = 0$. Two more recent generalizations are a nonsymmetric generalization and a noncommutative generalization, known respectively as quasisymmetric Schur functions and noncommutative Schur functions.

More precisely, the Hopf algebra of quasisymmetric functions, QSym [14], is a nonsymmetric generalization of the Hopf algebra of symmetric functions, and is itself related to areas such as the representation theory of the 0-Hecke algebra [2, 7, 37] and probability [17, 34]. Recently the basis of quasisymmetric Schur functions was discovered [16], which arose from the combinatorics of Macdonald polynomials [15] and itself initiated a search for other Schur-like bases of QSym such as row-strict quasisymmetric functions [9, 26], dual immaculate quasisymmetric functions [2] and Young quasisymmetric Schur functions [22]. Their name was aptly chosen since these functions not only naturally refine Schur functions, but also generalize many Schur function properties, such as the Littlewood-Richardson rule [3, Theorem 3.5], Pieri rules [16, Theorem 6.3] and the RSK algorithm [25, Procedure 3.3].

Dual to QSym is the Hopf algebra of noncommutative symmetric functions, NSym [13], whose basis dual to that of quasisymmetric Schur functions is the basis of noncommutative Schur functions [3]. By duality this basis again has a Littlewood-Richardson rule and RSK algorithm, and, due to noncommutativity, two sets of Pieri rules, one arising from multiplication on the right [36, Theorem 9.3] and one from multiplication on the left [3, Corollary 3.8]. Therefore in both QSym and NSym a natural question arises: Are there dual graded and dual filtered graphs and skew Pieri rules for quasisymmetric and noncommutative Schur functions? In this paper we give such graphs and such rules that are analogous to that of Schur functions.

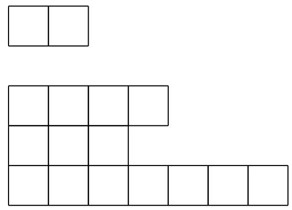
More precisely, the paper is structured as follows. In Section 2 we review necessary notions on compositions in order to define operators on them in Section 3. These operators are used to define four partially ordered sets in Subsection 3.1, $\mathcal{Q}_c$ and its deformation $\widetilde{\mathcal{Q}}_c$ that we later see arising in the Pieri rules for quasisymmetric Schur functions, and $\mathcal{R}_c$ and $\mathcal{L}_c$ that we later see respectively arising in the right and left Pieri rules for noncommutative Schur functions. We then establish useful relations satisfied by these operators in the remainder of Setion 3. In Section 4 we show that $\mathcal{R}_c$ and $\mathcal{Q}_c$, plus $\mathcal{L}_c$ and $\mathcal{Q}_c$, are each a pair of dual graded graphs in Theorems 4.3 and 4.13, and similarly $\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$, plus $\mathcal{L}_c$ and $\widetilde{\mathcal{Q}}_c$, are each a pair of dual filtered graphs in Theorems 4.5 and 4.15. In Section 5 we fully introduce QSym and NSym, the bases of quasisymmetric Schur functions and noncommutative Schur functions, and their respective Pieri rules. We then apply our operators to simplify the right Pieri rules for noncommutative Schur functions in Section 6. Finally in Section 7 we give skew Pieri rules for quasisymmetric Schur functions in Theorem 7.3 and recover the Pieri rules for skew shapes of Assaf and McNamara in Corollary 7.7. We close with skew Pieri rules for noncommutative Schur functions in Theorem 7.9.

2. Compositions and diagrams

A finite list of integers $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is called a *weak composition* if $\alpha_1, \ldots, \alpha_\ell$ are nonnegative, is called a *composition* if $\alpha_1, \ldots, \alpha_\ell$ are positive, and is called a *partition* if $\alpha_1 \geq \cdots \geq \alpha_\ell > 0$. Note that every weak composition has an underlying composition, obtained by removing all 0s, and in turn every composition has an underlying partition, obtained by reordering the list of integers into weakly decreasing order. Given $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ we call the α_i the *parts* of α, plus ℓ the *length* of α denoted by $\ell(\alpha)$, and the sum of the parts of α the *size* of α denoted by $|\alpha|$. If there exists $\alpha_{k+1} = \cdots = \alpha_{k+j} = i$ then we often abbreviate this to i^j. Also, given weak compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \ldots, \beta_m)$, we define the *concatenation* of α and β, denoted by $\alpha \bullet \beta$, to be the weak composition $(\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_m)$. For example, if $\alpha = (2, 1, 0, 3)$ and $\beta = (0, 4, 1)$, then $\alpha \bullet \beta = (2, 1, 0, 3, 0, 4, 1)$.

The *composition diagram* of a weak composition α, also denoted by α is the array of left-justified boxes with α_i boxes in row i from the *top*. We will often think of α as both a weak composition and as a composition diagram simultaneously, and hence future computations such as adding/subtracting 1 from the rightmost/leftmost part equalling i (as a weak composition) is synonymous with adding/removing a box from the bottommost/topmost row of length i (as a composition diagram).

Example 2.1. *The composition diagram of the weak composition of length 5, $\alpha = (2, 0, 4, 3, 6)$, is shown below.*



The composition of length 4 underlying α *is* $(2, 4, 3, 6)$*, and the partition of length 4 underlying it is* $(6, 4, 3, 2)$*. They all have size 15.*

3. Operators on compositions

In this section we will define four operators, which have already contributed to the theory of quasisymmetric and noncommutative Schur functions, and continue to cement their central role as we shall see later. Although originally defined on compositions, we will define them in the natural way on weak compositions to facilitate easier proofs. The first of these operators is the box removing operator $\mathfrak{d}$, which first appeared in the Pieri rules for quasisymmetric Schur functions [16]. The second of these is the appending operator a. These combine to define our third operator, the jeu de taquin or jdt operator $\mathfrak{u}$. This operator is pivotal in describing jeu de taquin slides on tableaux known as semistandard reverse composition tableaux and in describing the right Pieri rules for noncommutative Schur functions [36]. Our fourth and final operator is the box adding operator $\mathfrak{t}$, which plays the same role in the left Pieri rules for noncommutative Schur functions [3] as $\mathfrak{u}$ does in the aforementioned right Pieri rules. Each of these operators is defined on weak compositions for every integer $i \geq 0$ and we note that

$$\mathfrak{d}_0 = a_0 = \mathfrak{u}_0 = \mathfrak{t}_0 = Id$$

namely the identity map, which fixes the weak composition it is acting on. With this in mind we now define the remaining operators for $i \geq 1$, after establishing some set notation. Let $\mathbb{N}$ be the set of positive integers. Anytime we refer to a set $I \subset \mathbb{N}$, we implicitly assume that I is finite. Also, if we are given such a set I, then $I - 1$ is the set obtained by subtracting 1 from all the elements in I, and removing any 0s that might arise in so doing.

Example 3.1. *If* $I = \{1, 2, 4\}$*, then* $I - 1 = \{1, 3\}$*.*

By $[i]$ where $i \geq 1$, we mean the set $\{1, 2, \ldots, i\}$. We furthermore define $[0]$ to be the empty set. We will denote the maximum element (minimum element) of a set A by $\max(A)$ (respectively, $\min(A)$). If A is the empty set, by convention we have $\max(A) = \min(A) = 0$.

The first *box removing operator* on weak compositions, $\mathfrak{d}_i$ for $i \geq 1$, is defined as follows: Let α be a weak composition. Then

$$\mathfrak{d}_i(\alpha) = \alpha'$$

where α' is the weak composition obtained by subtracting 1 from the rightmost part equalling i in α. If there is no such part then we define $\mathfrak{d}_i(\alpha) = 0$.

Example 3.2. *Let* $\alpha = (2, 1, 2)$*. Then* $\mathfrak{d}_1(\alpha) = (2, 0, 2)$ *and* $\mathfrak{d}_2(\alpha) = (2, 1, 1)$*.*

Given a finite set $I = \{i_1 < \cdots < i_k\}$ of positive integers, we define

$$\mathfrak{d}_I = \mathfrak{d}_{i_1}\mathfrak{d}_{i_2}\cdots\mathfrak{d}_{i_k}.$$

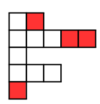
For convenience, we define $\mathfrak{d}_\emptyset = \mathfrak{d}_0$, as is the empty product of operators.

Example 3.3. *Let* $\alpha = (3, 5, 2, 4, 1, 2)$*. Then*

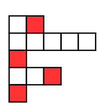
$$\begin{aligned}\mathfrak{d}_{[3]}(\alpha) &= \mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_3((3,5,2,4,1,2))\\ &= \mathfrak{d}_1\mathfrak{d}_2((2,5,2,4,1,2))\\ &= \mathfrak{d}_1((2,5,2,4,1,1))\\ &= (2,5,2,4,1,0).\end{aligned}$$

Now we will discuss two notions that will help us state our theorems in a concise way later, as well as connect our results to those in the classical theory of symmetric functions. Let $i_1 < \cdots < i_k$ be a sequence of positive integers, and let α be a weak composition. Consider the operator $\mathfrak{d}_{i_1}\cdots\mathfrak{d}_{i_k}$ acting on the weak composition α, and assume that the result is a valid weak composition. Then the boxes that are removed from α are said to form a *k-horizontal strip*, and we can think of the operator $\mathfrak{d}_{i_1}\cdots\mathfrak{d}_{i_k}$ as removing a k-horizontal strip. Similarly, given a sequence of positive integers $i_1 \geq \cdots \geq i_k$, consider the operator $\mathfrak{d}_{i_1}\cdots\mathfrak{d}_{i_k}$ acting on α and suppose that the result is a valid weak composition. Then the boxes that are removed from α are said to form a *k-vertical strip.* As before, we can think of the operator $\mathfrak{d}_{i_1}\cdots\mathfrak{d}_{i_k}$ as removing a k-vertical strip.

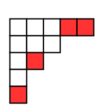
Example 3.4. *Consider* $\alpha = (2, 5, 1, 3, 1)$*. The operator* $\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_4\mathfrak{d}_5$ *removes the 4-horizontal strip shaded in red from* α*.*



The operator $\mathfrak{d}_3\mathfrak{d}_2\mathfrak{d}_1\mathfrak{d}_1$ *removes the 4-vertical strip shaded in red from* α*.*



Remark 3.5. If we consider partitions as Young diagrams (in English notation), then the above notions of horizontal and vertical strips coincide with their classical counterparts. For example, consider the operator $\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_4\mathfrak{d}_5$ acting on the partition $(5, 3, 2, 1, 1)$, in contrast to acting on the composition $(2, 5, 1, 3, 1)$ as in Example 3.4, then the 4-horizontal strip shaded in red is removed.



We now define the second *appending operator* on weak compositions, a_i for $i \geq 1$, as follows: Let $\alpha = (\alpha_1, \cdots, \alpha_{\ell(\alpha)})$ be a weak composition. Then

$$a_i(\alpha) = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, i)$$

namely, the weak composition obtained by appending a part i to the end of α. Note that a_0 can therefore also be thought of as adding a superfluous part 0 to the end of α.

Example 3.6. *Let $\alpha = (2, 1, 3)$. Then $a_2((2, 1, 3)) = (2, 1, 3, 2)$. However, $a_j\mathfrak{d}_2((3, 5, 1)) = 0$ for all $j \geq 0$ since $\mathfrak{d}_2((3, 5, 1)) = 0$.*

With the definitions of a_i and $\mathfrak{d}_i$ we define the third *jeu de taquin* or *jdt operator* on weak compositions, $\mathfrak{u}_i$ for $i \geq 1$, as follows:

$$\mathfrak{u}_i = a_i\mathfrak{d}_{[i-1]}.$$

Example 3.7. *We will compute $\mathfrak{u}_4(\alpha)$ where $\alpha = (3, 5, 2, 4, 1, 2)$. This corresponds to computing $a_4\mathfrak{d}_{[3]}(\alpha)$. We have already computed $\mathfrak{d}_{[3]} = (2, 5, 2, 4, 1, 0)$ in Example 3.3. Hence $\mathfrak{u}_4(\alpha) = (2, 5, 2, 4, 1, 0, 4)$.*

For any set of finite positive integers $I = \{i_1 < \cdots < i_k\}$, we define

$$\mathfrak{u}_I = \mathfrak{u}_{i_k} \cdots \mathfrak{u}_{i_1}.$$

For convenience, we define $\mathfrak{u}_\emptyset = \mathfrak{u}_0$, as is the empty product of operators. Note further that the order of indices in $\mathfrak{d}_I$ is the reverse of that in $\mathfrak{u}_I$. Let $i_1 < \cdots < i_k$ be a sequence of positive integers, and let α be a weak composition. Consider the operator $\mathfrak{u}_{i_k} \cdots \mathfrak{u}_{i_1}$ acting on the weak composition α, and assume that the result is a valid weak composition. Then the boxes that are added α are said to form a *k-right horizontal strip*, and we can think of the operator $\mathfrak{u}_{i_k} \cdots \mathfrak{u}_{i_1}$ as adding a k-right horizontal strip. Similarly, given a sequence of positive integers $i_1 \geq \cdots \geq i_k$, consider the operator $\mathfrak{u}_{i_k} \cdots \mathfrak{u}_{i_1}$ acting on α and suppose that the result is a valid weak composition. Then the boxes that are added to α are said to form a *k-right vertical strip*. As before, we can think of the operator $\mathfrak{u}_{i_k} \cdots \mathfrak{u}_{i_1}$ as adding a k-right vertical strip.

Lastly, we define the fourth *box adding operator* on weak compositions, $\mathfrak{t}_i$ for $i \geq 1$, as follows: let $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$ be a weak composition. Then

$$\mathfrak{t}_1(\alpha) = (1, \alpha_1, \ldots, \alpha_{\ell(\alpha)})$$

and for $i \geq 2$

$$\mathfrak{t}_i(\alpha) = (\alpha_1, \ldots, \alpha_j + 1, \ldots \alpha_{\ell(\alpha)})$$

where α_j is the leftmost part equalling $i - 1$ in α. If there is no such part, then we define $\mathfrak{t}_i(\alpha) = 0$.

Example 3.8. *Consider the composition $\alpha = (3, 2, 3, 1, 2)$. Then $\mathfrak{t}_1(\alpha) = (1, 3, 2, 3, 1, 2)$, $\mathfrak{t}_2(\alpha) = (3, 2, 3, 2, 2)$, $\mathfrak{t}_3(\alpha) = (3, 3, 3, 1, 2)$, $\mathfrak{t}_4(\alpha) = (4, 2, 3, 1, 2)$ and $\mathfrak{t}_i(\alpha) = 0$ for all $i \geq 5$.*

As with the jdt operators let $i_1 < \cdots < i_k$ be a sequence of positive integers, and let α be a weak composition. Consider the operator $\mathfrak{t}_{i_k} \cdots \mathfrak{t}_{i_1}$ acting on the weak composition α, and assume that the result is a valid weak composition. Then the boxes that are added α are said to form a *k-left horizontal strip*, and we can think of the operator $\mathfrak{t}_{i_k} \cdots \mathfrak{t}_{i_1}$ as adding a k-left horizontal strip. Likewise, given a sequence of positive integers $i_1 \geq \cdots \geq i_k$, consider the operator $\mathfrak{t}_{i_k} \cdots \mathfrak{t}_{i_1}$ acting on α and suppose that the result is a valid weak composition. Then the boxes that are added to α are said to form a *k-left vertical strip*, and we can think of the operator $\mathfrak{t}_{i_k} \cdots \mathfrak{t}_{i_1}$ as adding a k-left vertical strip.

3.1. **Composition posets.** With our operators we will now define four partially ordered sets on compositions noting that *if any parts of size 0 arise during computation, then they are ignored.* The adjectives right and left in the first two are not only used to distinguish between the posets, but also to refer to their roles in the right and left Pieri rules for noncommutative Schur functions in [36, Theorem 9.3] and [3, Corollary 3.8] respectively, and whose notation we follow now.

Definition 3.9. *The* right composition poset*, denoted by $\mathcal{R}_c$, is the poset consisting of all compositions with cover relation $\lessdot_r$ such that for compositions α, β*

$$\beta \lessdot_r \alpha \text{ if and only if } \alpha = \mathfrak{u}_i(\beta)$$

for some $i \geq 1$. Meanwhile the left composition poset*, denoted by $\mathcal{L}_c$, is the poset consisting of all compositions with cover relation $\lessdot_c$ such that for compositions α, β*

$$\beta \lessdot_c \alpha \text{ if and only if } \alpha = \mathfrak{t}_i(\beta)$$

for some $i \geq 1$.

The order relation $<_r$ in $\mathcal{R}_c$ (respectively, $<_c$ in $\mathcal{L}_c$) is obtained by taking the transitive closure of the cover relation $\lessdot_r$ (respectively, $\lessdot_c$).

Example 3.10. *Let $\beta = (3,1,4,2,1)$, $\alpha^R = (2,1,4,1,4)$ and $\alpha^L = (4,1,4,2,1)$. Then $\beta \lessdot_r \alpha^R = \mathfrak{u}_4(\beta)$ and $\beta \lessdot_c \alpha^L = \mathfrak{t}_4(\beta)$.*

Our third poset, meanwhile, stems from the Pieri rules for quasisymmetric Schur functions [16, Theorem 6.3], hence its name.

Definition 3.11. *The* quasisymmetric composition poset*, denoted by $\mathcal{Q}_c$, is the poset consisting of all compositions with cover relation $\lessdot_q$ such that for compositions α, β*

$$\beta \lessdot_q \alpha \text{ if and only if } \mathfrak{d}_i(\alpha) = \beta$$

for some $i \geq 1$.

Again, the order relation $<_q$ in $\mathcal{Q}_c$ is obtained by taking the transitive closure of the cover relation $\lessdot_q$.

Example 3.12. *Let $\beta = (4,1,3,2,1)$, and $\alpha = (4,1,4,2,1)$. Then $\mathfrak{d}_4(\alpha) = \beta \lessdot_q \alpha$.*

Lastly, our fourth poset is a deformation of $\mathcal{Q}_c$, and in contrast to definitions of the other posets above, we define it by stating the order relations explicitly.

Definition 3.13. *The* deformed quasisymmetric composition poset*, denoted by $\widetilde{\mathcal{Q}}_c$, is the poset consisting of all compositions with order relation $<_{\widetilde{q}}$ such that for compositions α, β*

$$\beta <_{\widetilde{q}} \alpha \text{ if and only if } \mathfrak{d}_I(\alpha) = \beta$$

for some finite $\emptyset \neq I \subset \mathbb{N}$.

Example 3.14. *Let $\beta = (4, 1, 3, 1, 1)$, and $\alpha = (4, 1, 4, 2, 1)$. Then $\mathfrak{d}_{\{2,4\}}(\alpha) = \beta <_{\widetilde{q}} \alpha$.*

3.2. **Relations satisfied by operators of type $\mathfrak{u}$ and $\mathfrak{d}$.** We will now prove a variety of lemmas regarding the jdt operators and box removing operators, which will be useful in proving our main theorems later. Hence this subsection can be safely skipped for now and referred to later. In all the proofs we assume that α is a weak composition.

Lemma 3.15. *For $i \geq 0$ we have $a_i = \mathfrak{d}_{i+1} a_{i+1}$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Then $a_{i+1}(\alpha) = (\alpha_1, \ldots, \alpha_\ell, i+1)$. This implies by definition that $\mathfrak{d}_{i+1} a_{i+1}(\alpha) = (\alpha_1, \ldots, \alpha_\ell, i) = a_i(\alpha)$. □

As a corollary we obtain the following.

Corollary 3.16. *For positive integers i and j satisfying $i \geq j$, we have*

$$\mathfrak{d}_j \mathfrak{d}_{j+1} \cdots \mathfrak{d}_{i-1} \mathfrak{d}_i a_i = a_{j-1}.$$

Lemma 3.17. *Let $i \neq j$ be positive integers. Then*

$$\mathfrak{d}_i a_j = a_j \mathfrak{d}_i.$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. Let $\beta = a_j(\alpha) = (\alpha_1, \ldots, \alpha_\ell, j)$. If α does not have a part equalling i, then neither does β, as $i \neq j$. Thus in this case we have $\mathfrak{d}_i a_j(\alpha) = \mathfrak{d}_i(\beta) = 0 = a_j \mathfrak{d}_i(\alpha)$. Now, assume that α_r is the rightmost part equalling i in α. Then $a_j \mathfrak{d}_i(\alpha) = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r - 1, \ldots, \alpha_\ell, j)$. Since $i \neq j$, we are guaranteed that $\mathfrak{d}_i(\beta) = (\alpha_1, \ldots, \alpha_{r-1}, \alpha_r - 1, \ldots, \alpha_\ell, j)$. Thus we have $\mathfrak{d}_i a_j(\alpha) = a_j \mathfrak{d}_i(\alpha)$ in this case as well, and we are done. □

Lemma 3.18. *Let i and j be distinct positive integers such that $|i - j| \geq 2$. Then*

$$\mathfrak{d}_i \mathfrak{d}_j = \mathfrak{d}_j \mathfrak{d}_i.$$

Proof. Without loss of generality, assume that $i > j$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. First assume that α does not have a part equalling j. Then $\mathfrak{d}_i \mathfrak{d}_j(\alpha) = 0$. Since $i - j \geq 2$, we have that either $\mathfrak{d}_i(\alpha) = 0$ or $\mathfrak{d}_i(\alpha)$ does not have a part equalling j. In both the cases we have that $\mathfrak{d}_j \mathfrak{d}_i(\alpha) = 0 = \mathfrak{d}_i \mathfrak{d}_j(\alpha)$. Now assume that α does not have a part equalling i. Then $\mathfrak{d}_j \mathfrak{d}_i(\alpha) = 0$. Similarly to before, we have that either $\mathfrak{d}_j(\alpha) = 0$ or $\mathfrak{d}_j(\alpha)$ does not have a part equalling i. Therefore $\mathfrak{d}_i \mathfrak{d}_j(\alpha) = 0 = \mathfrak{d}_j \mathfrak{d}_i(\alpha)$ again.

Now assume that α has parts equalling i and j. Let α_r and α_s be the rightmost parts equalling i and j in α, respectively. The condition $i-j\geq 2$ guarantees that both $\mathfrak{d}_i\mathfrak{d}_j(\alpha)$ and $\mathfrak{d}_j\mathfrak{d}_i(\alpha)$ equal the composition

$$\begin{cases} (\alpha_1,\ldots,\alpha_r-1,\ldots,\alpha_s-1,\ldots,\alpha_\ell) & r<s \\ (\alpha_1,\ldots,\alpha_s-1,\ldots,\alpha_r-1,\ldots,\alpha_\ell) & r>s. \end{cases}$$

Thus we have shown that $\mathfrak{d}_i\mathfrak{d}_j(\alpha)=\mathfrak{d}_j\mathfrak{d}_i(\alpha)$ for all α. □

Lemma 3.19. *Let $i\geq 1$. Then $\mathfrak{d}_i^2\mathfrak{d}_{i+1}=\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i$.*

Proof. Let $\alpha=(\alpha_1,\ldots,\alpha_\ell)$. First assume that α does not have a part equalling $i+1$. Then $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)=0$. Notice also that if α does not have a part equalling $i+1$, then neither does $\mathfrak{d}_i(\alpha)$ (assuming $\mathfrak{d}_i(\alpha)$ is not 0 already). Thus in this case $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)=0=\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$.

Now assume that α does not have a part equalling i. Then $\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)=0$. We also know that since α does not have any part equalling i, $\mathfrak{d}_{i+1}(\alpha)$ has exactly one part equalling i (assuming $\mathfrak{d}_{i+1}(\alpha)$ is not 0 already). Then $\mathfrak{d}_i^2(\mathfrak{d}_{i+1}(\alpha))=0$, and we have the claimed equality in this case as well.

We can now turn our focus to the case where α has parts equalling i and $i+1$. We will first deal with the situation where α has exactly one part equalling i. Let $\alpha_r=i$ and let α_s be the rightmost part equalling $i+1$ in α.

If $r<s$, then

$$\begin{aligned} \mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha) &= (\alpha_1,\ldots,\alpha_r,\ldots,\alpha_s-2,\ldots,\alpha_\ell) \\ \mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha) &= (\alpha_1,\ldots,\alpha_r-1,\ldots,\alpha_s-1,\ldots,\alpha_\ell). \end{aligned}$$

Notice that α_r is the rightmost part equalling i in $\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ while α_s-1 is the rightmost part equalling i in $\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$. Thus both $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ equal

$$(\alpha_1,\ldots,\alpha_r-1,\ldots,\alpha_s-2,\ldots,\alpha_\ell).$$

If $r>s$, then both $\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ equal

$$\alpha=(\alpha_r,\ldots,\alpha_s-1,\ldots,\alpha_r-1,\ldots,\alpha_\ell).$$

Thus $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)=\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ are equal in this case as well.

Finally, assume that α has more than one part equalling i. Let α_r and α_s be the two rightmost instances of a part equalling i in α with $r<s$, and let α_t be the rightmost part equalling $i+1$ in α. If $r<s<t$ or $r<t<s$, then the part α_r remains unaltered and does not play a role in the computation of $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ and we perform the same manipulation as in the case prior to this. So we will focus on the case where $t<r<s$. Then it is easily seen that both $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ equal

$$(\alpha_1,\ldots,\alpha_t-1,\ldots,\alpha_r-1,\ldots,\alpha_s-1,\ldots,\alpha_\ell).$$

Now we have covered all possible cases and thus established that $\mathfrak{d}_i^2\mathfrak{d}_{i+1}(\alpha)=\mathfrak{d}_i\mathfrak{d}_{i+1}\mathfrak{d}_i(\alpha)$ for all α. □

Lemma 3.20. *Let $i\geq 1$. Then $\mathfrak{d}_i\mathfrak{d}_{i+1}^2=\mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_\ell)$. First assume that the number of parts equalling $i+1$ in α is ≤ 1. Then clearly $\mathfrak{d}_i\mathfrak{d}_{i+1}^2(\alpha) = 0$. Furthermore, $\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ has no parts equalling $i+1$ (assuming it is not 0 already). Thus $\mathfrak{d}_{i+1}(\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)) = 0$ as well. Hence $\mathfrak{d}_i\mathfrak{d}_{i+1}^2(\alpha) = \mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ in this case.

Now assume that α has at least two parts equalling $i+1$, and let α_r and α_s denote the two rightmost instances of a part equalling $i+1$, with $r < s$. First assume there is no part equalling i. Then both $\mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}^2(\alpha)$ equal

$$(\alpha_1, \dots, \alpha_r - 1, \dots, \alpha_s - 2, \dots, \alpha_\ell).$$

Now assume that the rightmost part equalling i in α is α_t. If $t < r < s$ or $r < t < s$, then α_t remains unaltered by both $\mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}^2$, and the result is the same as the one in the case prior to this. Hence we can assume that $r < s < t$. Then it can be easily seen that both $\mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ and $\mathfrak{d}_i\mathfrak{d}_{i+1}^2(\alpha)$ equal

$$(\alpha_1, \dots, \alpha_r - 1, \dots, \alpha_s - 1, \dots, \alpha_t - 1, \dots, \alpha_\ell).$$

Thus we have shown that $\mathfrak{d}_i\mathfrak{d}_{i+1}^2(\alpha) = \mathfrak{d}_{i+1}\mathfrak{d}_i\mathfrak{d}_{i+1}(\alpha)$ for all α. ☐

Lemma 3.21. *Let $i \neq j$ be positive integers. Then*

$$\mathfrak{u}_i\mathfrak{d}_j = \mathfrak{d}_j\mathfrak{u}_i.$$

Proof. Let us first consider the case $1 \leq i \leq j-1$. Then by Lemmas 3.17 and 3.18, we have that $\mathfrak{d}_j$ commutes with $a_i, \mathfrak{d}_1, \dots, \mathfrak{d}_{i-1}$. Hence $\mathfrak{u}_i\mathfrak{d}_j = \mathfrak{d}_j\mathfrak{u}_i$ in this case.

Now consider the case where $i > j \geq 1$. Then $\mathfrak{d}_j\mathfrak{u}_i = \mathfrak{d}_j a_i \mathfrak{d}_1\mathfrak{d}_2 \cdots \mathfrak{d}_{i-1}$. Again, using Lemmas 3.17 and 3.18, we can write this as

$$a_i\mathfrak{d}_1 \cdots \mathfrak{d}_{j-2}\mathfrak{d}_j\mathfrak{d}_{j-1}\mathfrak{d}_j \cdots \mathfrak{d}_{i-1}.$$

Using Lemma 3.20, we can write the above as

$$a_i\mathfrak{d}_1 \cdots \mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_j\mathfrak{d}_{j+1} \cdots \mathfrak{d}_{i-1}.$$

Notice at this stage, if we assume $j = i-1$, then we have shown that $\mathfrak{u}_i\mathfrak{d}_j = \mathfrak{d}_j\mathfrak{u}_i$. So let us assume $i - j \geq 2$. Using Lemma 3.19, we can transform the above expression to

$$a_i\mathfrak{d}_1 \cdots \mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_{j+1}\mathfrak{d}_j\mathfrak{d}_{j+2} \cdots \mathfrak{d}_{i-1}.$$

Now Lemma 3.18 easily establishes that the above expression equals

$$a_i\mathfrak{d}_1 \cdots \mathfrak{d}_{j-2}\mathfrak{d}_{j-1}\mathfrak{d}_j\mathfrak{d}_{j+1}\mathfrak{d}_{j+2} \cdots \mathfrak{d}_{i-1}\mathfrak{d}_j,$$

and we are done. ☐

Lemma 3.22. *Let $i \geq 1$. Then $\mathfrak{u}_i\mathfrak{d}_i = \mathfrak{d}_{i+1}\mathfrak{u}_{i+1}$.*

Proof. Notice that $\mathfrak{u}_i\mathfrak{d}_i = a_i\mathfrak{d}_{[i]}$. Furthermore, Lemma 3.15 states that $a_i = \mathfrak{d}_{i+1}a_{i+1}$, and hence $\mathfrak{u}_i\mathfrak{d}_i = \mathfrak{d}_{i+1}a_{i+1}\mathfrak{d}_{[i]}$. Since $\mathfrak{u}_{i+1} = a_{i+1}\mathfrak{d}_{[i]}$, by definition, the claim follows. ☐

Lemma 3.23. *Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be a set of positive integers. Then*

$$\mathfrak{u}_I = a_{i_k}\mathfrak{d}_{[i_k]\setminus I}.$$

Proof. We will proceed by induction on size of the set I. The claim is clearly true if $|I| = 1$ as that is the definition of $\mathfrak{u}_{i_1}$. Assume the claim holds for all I satisfying $|I| < k$. Let $I' = I \setminus \{i_k\}$. Then we have

$$\begin{aligned}
\mathfrak{u}_I &= \mathfrak{u}_{i_k} \overbrace{\mathfrak{u}_{i_{k-1}} \cdots \mathfrak{u}_{i_1}} \\
&\overset{\text{(induction hypothesis)}}{=} \mathfrak{u}_{i_k} a_{i_{k-1}} \mathfrak{d}_{[i_{k-1}]\setminus I'} \\
&= a_{i_k} \overbrace{\mathfrak{d}_{[i_k-1]} a_{i_{k-1}}} \mathfrak{d}_{[i_{k-1}]\setminus I'} \\
&\overset{\text{(Lemma 3.17, Corollary 3.16)}}{=} a_{i_k} \overbrace{\mathfrak{d}_{[i_k-1]\setminus[i_{k-1}]} \mathfrak{d}_{[i_{k-1}]\setminus I'}} \\
&\overset{\text{(Lemma 3.18)}}{=} a_{i_k} \mathfrak{d}_{[i_{k-1}]\setminus I'} \mathfrak{d}_{[i_k-1]\setminus[i_{k-1}]} \\
&= a_{i_k} \mathfrak{d}_{[i_k]\setminus I}
\end{aligned}$$

as desired. $\square$

Example 3.24. *To illustrate the above proof technique let $I = \{1, 4, 6, 7, 10\}$. Then*

$$\begin{aligned}
\mathfrak{u}_I &= \mathfrak{u}_{10}\mathfrak{u}_7\mathfrak{u}_6 \overbrace{\mathfrak{u}_4\mathfrak{u}_1} \\
&= \mathfrak{u}_{10}\mathfrak{u}_7\mathfrak{u}_6\ a_4 \cancel{\mathfrak{d}_1} \mathfrak{d}_2\mathfrak{d}_3 \cancel{a_1} \\
&= \mathfrak{u}_{10}\mathfrak{u}_7\ \overbrace{\mathfrak{u}_6 a_4}\ \mathfrak{d}_2\mathfrak{d}_3 \\
&= \mathfrak{u}_{10}\mathfrak{u}_7\ a_6 \cancel{\mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}_3\mathfrak{d}_4} \mathfrak{d}_5 \cancel{a_4}\ \mathfrak{d}_2\mathfrak{d}_3 \\
&= \mathfrak{u}_{10}\ \overbrace{\mathfrak{u}_7 a_6}\ \mathfrak{d}_5\ \mathfrak{d}_2\mathfrak{d}_3 \\
&= \mathfrak{u}_{10} a_7\ \mathfrak{d}_5\ \mathfrak{d}_2\mathfrak{d}_3 \\
&= a_{10}\ \mathfrak{d}_8\mathfrak{d}_9\ \mathfrak{d}_5\ \mathfrak{d}_2\mathfrak{d}_3 \\
&= a_{10}\mathfrak{d}_2\mathfrak{d}_3\mathfrak{d}_5\mathfrak{d}_8\mathfrak{d}_9 \\
&= a_{10}\mathfrak{d}_{[10]\setminus I}.
\end{aligned}$$

The above lemma allows us to give a description for $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$, which will be stated in Section 6, and simplifies the results in [36].

3.3. **Relations satisfied by operators of type $\mathfrak{t}$ and $\mathfrak{d}$.** As with the previous subsection, we again prove a variety of useful lemmas, but this time regarding the box adding and box removing operators. Again, if desired, this subsection can be safely skipped for now and referred to later. In all the proofs we assume that α is a weak composition.

Lemma 3.25. *Let $i \neq j$ be positive integers. Then*

$$\mathfrak{t}_i\mathfrak{d}_j = \mathfrak{d}_j\mathfrak{t}_i.$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. First consider the case $i = 1$. If α does not have a part equalling j, then $\mathfrak{t}_1\mathfrak{d}_j(\alpha) = 0$. Note now that, since $j \neq 1$, we have $\mathfrak{d}_j\mathfrak{t}_1(\alpha) = \mathfrak{d}_j((1, \alpha_1, \ldots, \alpha_\ell)) = 0$ as well.

Hence we can assume that $i \geq 2$. If α does not have a part equalling $i-1$, then using the fact that $i \neq j$, we get that $\mathfrak{d}_j(\alpha)$ does not have a part equalling $i-1$ either (assuming it does not equal 0 already). This implies that $\mathfrak{t}_i\mathfrak{d}_j(\alpha) = 0$. Our assumption that α has no part equalling $i-1$ also implies that $\mathfrak{d}_j\mathfrak{t}_i(\alpha) = 0$.

Finally assume that α does have a part equalling $i-1$, and let α_r denote the leftmost such part. Then

$$\mathfrak{t}_i(\alpha) = (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_\ell).$$

If α does not have a part equalling j, then neither does $\mathfrak{t}_i(\alpha)$. This follows from the fact that $i \neq j$. This immediately implies that $\mathfrak{t}_i\mathfrak{d}_j(\alpha) = \mathfrak{d}_j\mathfrak{t}_i(\alpha) = 0$ in this case. If α does have a part equalling j, then let α_s denote the rightmost such part. Note that α_s continues to be the rightmost part equalling j in $\mathfrak{t}_i(\alpha)$ as well. Again, this follows since $i \neq j$. Thus we get that

$$\mathfrak{t}_i\mathfrak{d}_j(\alpha) = \mathfrak{d}_j\mathfrak{t}_i(\alpha) = (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)$$

if $r < s$ and

$$\mathfrak{t}_i\mathfrak{d}_j(\alpha) = \mathfrak{d}_j\mathfrak{t}_i(\alpha) = (\alpha_1, \ldots, \alpha_s - 1, \ldots, \alpha_r + 1, \ldots, \alpha_\ell)$$

if $s < r$. □

Lemma 3.26. *Let i be a positive integer. Then $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha)$ if one of the following conditions holds.*

1. *We have $i \geq 2$ and α has parts equalling i and $i-1$.*
2. *We have $i = 1$ and α has at least one part equalling 1.*

Proof. Assume first that $i \geq 2$ and that $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ has parts equalling i and $i-1$. Furthermore suppose that α_r is the leftmost part equalling $i-1$, and α_s is the rightmost part equalling i in α.

If $r < s$, then we have the following.

$$\begin{aligned}\mathfrak{d}_i\mathfrak{t}_i(\alpha) &= \mathfrak{d}_i\mathfrak{t}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s, \ldots, \alpha_\ell)) \\ &= \mathfrak{d}_i((\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s, \ldots, \alpha_\ell)) \\ &= (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)\end{aligned}$$

Also, the following sequence of equalities holds.

$$\begin{aligned}\mathfrak{t}_i\mathfrak{d}_i(\alpha) &= \mathfrak{t}_i\mathfrak{d}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s, \ldots, \alpha_\ell)) \\ &= \mathfrak{t}_i((\alpha_1, \ldots, \alpha_r, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)) \\ &= (\alpha_1, \ldots, \alpha_r + 1, \ldots, \alpha_s - 1, \ldots, \alpha_\ell)\end{aligned}$$

Thus if $r < s$ we clearly have that $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha)$.

If $r > s$, then we have the following.

$$\begin{aligned}\mathfrak{d}_i\mathfrak{t}_i(\alpha) &= \mathfrak{d}_i\mathfrak{t}_i((\alpha_1, \ldots, \alpha_s, \ldots, \alpha_r, \ldots, \alpha_\ell)) \\ &= \mathfrak{d}_i((\alpha_1, \ldots, \alpha_s, \ldots, \alpha_r + 1, \ldots, \alpha_\ell)) \\ &= (\alpha_1, \ldots, \alpha_s, \ldots, \alpha_r, \ldots, \alpha_\ell)\end{aligned}$$

Also, the following sequence of equalities holds.

$$\begin{aligned}\mathfrak{t}_i\mathfrak{d}_i(\alpha) &= \mathfrak{t}_i\mathfrak{d}_i((\alpha_1,\dots,\alpha_s,\dots,\alpha_r,\dots,\alpha_\ell))\\ &= \mathfrak{t}_i((\alpha_1,\dots,\alpha_s-1,\dots,\alpha_r,\dots,\alpha_\ell))\\ &= (\alpha_1,\dots,\alpha_s,\dots,\alpha_r,\dots,\alpha_\ell)\end{aligned}$$

Thus in this case we see that $\mathfrak{d}_i\mathfrak{t}_i(\alpha) = \mathfrak{t}_i\mathfrak{d}_i(\alpha) = \alpha$.

Assume now that $i = 1$, and that α has at least one part equalling 1. Furthermore, suppose that α_r is the rightmost part equalling 1. Then it can be easily verified that $\mathfrak{d}_1\mathfrak{t}_1(\alpha) = \mathfrak{t}_1\mathfrak{d}_1(\alpha) = (1,\alpha_1,\dots,\alpha_{r-1},0,\alpha_{r+1},\dots,\alpha_\ell)$. This finishes the proof of the claim. □

4. Dual graphs from composition posets

As our first application of relations between our operators we show that our composition posets $\mathcal{R}_c$ and $\mathcal{L}_c$ when paired with our composition posets $\mathcal{Q}_c$ and $\widetilde{\mathcal{Q}}_c$ are each dual graded graphs and dual filtered graphs, respectively.

Recall that a graph G consisting of a set of vertices P endowed with a rank function $\rho : P \to \mathbb{Z}$ with $x, y \in P$ and y is of rank weakly greater than x is called a *graded graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) = \rho(x) + 1$. The graph G is called a *weak filtered graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) \geq \rho(x)$, and a *strong filtered graph* when the edge set E satisfies if $(x, y) \in E$ then $\rho(y) > \rho(x)$. Now given a field K of characteristic 0, the vector space KP is the space consisting of all formal linear combinations of vertices of G. Then we define the up and down operators $U, D \in End(KP)$ associated with G to be

$$U(x) = \sum_y m(x,y)y$$

$$D(y) = \sum_x m(x,y)x$$

where x and y are vertices of G, y is of weakly greater rank than x, and $m(x, y)$ is the number of edges connecting x and y. With this in mind, let G_1 be a graded graph with up operator U and G_2 be a graded graph with down operator D such that G_1 and G_2 have a common vertex set P and rank function ρ. Then G_1 and G_2 are *dual graded graphs* if and only if on KP

$$DU - UD = Id \tag{4.1}$$

where Id is the identity operator on KP. Similarly let $\widetilde{G_1}$ be a weak filtered graph with up operator $\widetilde{U}$ and $\widetilde{G_2}$ be a strong filtered graph with down operator $\widetilde{D}$ such that $\widetilde{G_1}$ and $\widetilde{G_2}$ have a common vertex set P and rank function ρ. Then $\widetilde{G_1}$ and $\widetilde{G_2}$ are *dual filtered graphs* if and only if on KP

$$\widetilde{D}\widetilde{U} - \widetilde{U}\widetilde{D} = \widetilde{D} + Id. \tag{4.2}$$

4.1. **Dual graphs and the right composition poset.** Observe that our composition posets $\mathcal{R}_c$ and $\mathcal{Q}_c$ defined in Subsection 3.1 with vertex set being the set of all compositions and whose rank function is given by the size of a composition are both examples of graded graphs. By the definition of the cover relation $\lessdot_r$ it follows that the up operator associated with $\mathcal{R}_c$ is given by

$$U = \sum_{i\geq 1} \mathfrak{u}_i. \tag{4.3}$$

Example 4.1. *Let α be the composition $(2,1,3)$. Then*

$$U((2,1,3)) = (2,1,3,1)+(2,0,3,2)+(1,0,3,3)+(2,1,0,4) = (2,1,3,1)+(2,3,2)+(1,3,3)+(2,1,4).$$

Similarly, by the definition of the cover relation $\lessdot_q$ it follows that the down operator associated with $\mathcal{Q}_c$ is given by

$$D = \sum_{i\geq 1} \mathfrak{d}_i. \tag{4.4}$$

Example 4.2. *Let α be the composition $(2,1,3)$. Then*

$$D((2,1,3)) = (2,0,3) + (1,1,3) + (2,1,2) = (2,3) + (1,1,3) + (2,1,2).$$

Moreover we have the following.

Theorem 4.3. *$\mathcal{R}_c$ and $\mathcal{Q}_c$ are dual graded graphs, that is, on compositions*

$$DU - UD = Id.$$

Proof. Notice that

$$DU = \sum_{\substack{i\neq j\\ i,j\geq 1}} \mathfrak{d}_j\mathfrak{u}_i + \sum_{k\geq 1} \mathfrak{d}_k\mathfrak{u}_k,$$

and

$$UD = \sum_{\substack{i\neq j\\ i,j\geq 1}} \mathfrak{u}_i\mathfrak{d}_j + \sum_{k\geq 1} \mathfrak{u}_k\mathfrak{d}_k.$$

Using Lemma 3.21 and Lemma 3.22, we reach the conclusion that

$$DU - UD = \mathfrak{d}_1\mathfrak{u}_1.$$

But, by Lemma 3.15, $\mathfrak{d}_1\mathfrak{u}_1 = a_0 = Id$. This finishes the proof. $\square$

Example 4.4. *Let $\alpha = (2,1,3)$. Then suppressing commas and parentheses for ease of comprehension, we have that*

$$\begin{aligned} DU(\alpha) &= D(2131 + 2032 + 1033 + 2104) \\ &= 2130 + 1131 + 2121 + 2031 + 2022 \\ &\quad +0033 + 1032 + 2004 + 1104 + 2103 \end{aligned}$$

and

$$\begin{aligned} UD(\alpha) &= U(203+113+212) \\ &= 2031+0033+2004+1131+1032 \\ &\quad +1104+2121+2022+2103. \end{aligned}$$

Thus $(DU-UD)(\alpha) = 213 = Id(\alpha)$.

We also have that $\widetilde{\mathcal{Q}}_c$ with vertex set and rank function being that of $\mathcal{R}_c$ (and $\mathcal{Q}_c$) is clearly an example of a strong filtered graph, whose order relation $<_{\widetilde{q}}$ yields that the down operator associated with $\widetilde{\mathcal{Q}}_c$ is given by

$$\widetilde{D} = \sum_{I\subset\mathbb{N}} \mathfrak{d}_I \tag{4.5}$$

where the sum is over all finite but nonempty susbsets of $\mathbb{N}$. Hence we can relate $\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$ as follows, since any graded graph, such as $\mathcal{R}_c$, is also a weak filtered graph.

Theorem 4.5. *$\mathcal{R}_c$ and $\widetilde{\mathcal{Q}}_c$ are dual filtered graphs, that is, on compositions*

$$\widetilde{D}U - U\widetilde{D} = \widetilde{D} + Id.$$

Proof. First note that the operator $\widetilde{D}U$ has the following expansion.

$$\begin{aligned} \widetilde{D}U &= \sum_{\substack{i\geq 1\\ I\subset\mathbb{N}}} \mathfrak{d}_I\mathfrak{u}_i \\ &= \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{u}_i + \sum_{\substack{I\subset\mathbb{N}\\ i\geq 1, i\notin I}} \mathfrak{d}_I\mathfrak{u}_i \end{aligned}$$

In a similar manner, we obtain the following expansion for $U\widetilde{D}$.

$$\begin{aligned} U\widetilde{D} &= \sum_{\substack{i\geq 1\\ I\subset\mathbb{N}}} \mathfrak{u}_i\mathfrak{d}_I \\ &= \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{u}_i\mathfrak{d}_I + \sum_{\substack{I\subset\mathbb{N}\\ i\geq 1, i\notin I}} \mathfrak{u}_i\mathfrak{d}_I \end{aligned}$$

Using Lemma 3.21, we obtain that

$$\widetilde{D}U - U\widetilde{D} = \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{u}_i - \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{u}_i\mathfrak{d}_I.$$

Consider now a fixed set $I \subset \mathbb{N}$ and $i \in I$. We will next show that the operator $\mathfrak{d}_I\mathfrak{u}_i$ corresponds to either to a unique operator $\mathfrak{u}_{i'}\mathfrak{d}_{I'}$ where $i' \in I'$, or an operator $a_0\mathfrak{d}_{I'}$ where I' might be the empty set.

Let $j \in I$ be the smallest positive integer such that $j-1 \notin I$ but every integer k satisfying $j \leq k \leq i$ belongs to I. Consider the following sets.

$$\begin{aligned} A &= \{k \mid k \in I, k < j\} \\ B &= \{k \mid j \leq k \leq i\} \\ C &= \{k \mid k \in I, k > i\} \end{aligned}$$

Clearly, we have that $I = A \amalg B \amalg C$ where $\amalg$ denotes disjoint union. Define the set I' to be $A \amalg (B-1) \amalg C$. Notice that I' can be the empty set (precisely in the case where A and C are empty, while $B = \{1\}$). Now we have the following sequence of equalities using Lemma 3.21 and Lemma 3.22.

$$\begin{aligned} \mathfrak{d}_I \mathfrak{u}_i &= \mathfrak{d}_A \mathfrak{d}_B \mathfrak{d}_C \mathfrak{u}_i \\ &= \mathfrak{d}_A \mathfrak{d}_B \mathfrak{u}_i \mathfrak{d}_C \\ &= \mathfrak{d}_A \mathfrak{u}_{j-1} \mathfrak{d}_{B-1} \mathfrak{d}_C \\ &= \mathfrak{u}_{j-1} \mathfrak{d}_A \mathfrak{d}_{B-1} \mathfrak{d}_C \\ &= \mathfrak{u}_{j-1} \mathfrak{d}_{I'} \end{aligned}$$

Given the invertibility of our computation, it is clear how to recover $\mathfrak{d}_I \mathfrak{u}_i$ starting from $\mathfrak{u}_{j-1}\mathfrak{d}'_I$. Furthermore, if $j \neq 1$, then we clearly have that $j-1 \in I'$. The above thus implies that

$$\sum_{\substack{I \subset \mathbb{N} \\ i \in I}} \mathfrak{d}_I \mathfrak{u}_i - \sum_{\substack{I \subset \mathbb{N} \\ i \in I}} \mathfrak{u}_i \mathfrak{d}_I = a_0 + a_0 \widetilde{D},$$

thereby finishing the proof. □

Example 4.6. *Let $\alpha = (1,2)$. Then suppressing commas and parentheses as before, we have that*

$$\widetilde{D}(\alpha) = (02 + 11 + 10).$$

Plus

$$\begin{aligned} \widetilde{D}U(\alpha) &= \widetilde{D}(121 + 022 + 103) \\ &= 120 + 111 + 110 + 021 + 020 \\ &\quad +003 + 102 + 002 + 101 + 100 \end{aligned}$$

and

$$\begin{aligned} U\widetilde{D}(\alpha) &= U(02 + 11 + 10) \\ &= 021 + 003 + 111 + 102 + 101 + 002. \end{aligned}$$

Thus $(\widetilde{D}U - U\widetilde{D})(\alpha) = 12 + 2 + 11 + 1 = (\widetilde{D} + Id)(\alpha)$.

Remark 4.7. We find it worthwhile to mention the connection between our results here and Fomin's work in [12]. In particular, note that the relations [12, Equation 1.9] satisfied by his box adding and box removing operators on partitions (denoted therein by u and d

respectively) are the same as those satisfied by the jdt operators and box removing operators on compositions. The relations are indeed easier to establish in the case of partitions, but as can be seen from calculations above that deriving the same relations in the case of compositions is more delicate.

Fomin then uses these operators to define generating functions $A(x)$ and $B(y)$ that add or remove horizontal strips in all possible ways respectively, and then uses [12, Equation 1.9] to prove the following commutation relation [12, Theorem 1.2].

$$A(x)B(y) = B(y)A(x)(1-xy)^{-1}$$

He later notes that the dual graded graph nature of Young's lattice is encoded in the aforementioned identity. More precisely it follows from comparing the coefficient of xy on either side [12, Equation 1.13]. In fact, one can obtain various identities by comparing coefficients. The reader is invited to verify that the relations describing dual filtered graphs can be obtained by setting $y = 1$ and then subsequently comparing the coefficient of x on either side. Thus in a sense, Fomin's relations uniformly establish both the dual graded graph and the dual filtered graph structures on Young's lattice and $\mathcal{R}_c$.

We now proceed to discuss $\mathcal{L}_c$ defined using box adding operators. We will establish that this poset can also be endowed with a structure of a dual graded graph and a dual filtered graph. But the relations satisfied in this case are different than the ones we have encountered, and we can not use Fomin's commutation relation in this setting. In fact, as the reader will see, the cancellations in the case of $\mathcal{L}_c$ are more subtle despite the simplicity of the action of $\mathfrak{t}$ compared to the action of $\mathfrak{u}$.

4.2. **Dual graphs and the left composition poset.** As in the previous subsection, our composition poset $\mathcal{L}_c$ with vertex set being the set of all compositions whose rank function is given by the size of a composition is a graded graph and hence also a weak filtered graph. By the definition of the cover relation $\lessdot_c$ it follows that the up operator associated with $\mathcal{L}_c$ is given by

$$U_t = \sum_{i\geq 1} \mathfrak{t}_i. \tag{4.6}$$

Example 4.8. *Let α be the composition $(2,1,3)$. Then*

$$U_t((2,1,3)) = (1,2,1,3) + (2,2,3) + (3,1,3) + (2,1,4).$$

Plus $\mathcal{Q}_c$ and $\widetilde{\mathcal{Q}}_c$ are respectively a graded graph and a strong filtered graph with respective down operators D and $\widetilde{D}$.

For the remainder of this section, we will fix a composition α. This given define the sets X and Y as follows.

$$X = \{\mathfrak{d}_I\mathfrak{t}_i \mid I \subset \mathbb{N}, i \in I, \mathfrak{d}_I\mathfrak{t}_i(\alpha) \neq 0\}$$
$$Y = \{\mathfrak{t}_i\mathfrak{d}_I \mid I \subset \mathbb{N}, i \in I, \mathfrak{t}_i\mathfrak{d}_I(\alpha) \neq 0\}$$

Consider $w = \mathfrak{t}_i \mathfrak{d}_I \in Y$. Decompose $I = A \amalg \{i\} \amalg B$ where

$$A = \{j \in I \mid j < i\}$$
$$B = \{j \in I \mid j > i\}.$$

By Lemma 3.25, we have that $w = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B$. Let k denote the largest part of α that is strictly less than i. If such a part does not exist, we define k to be 0. Let $i' = k + 1$. Now let $I' = A \amalg \{i'\} \amalg B$ and

$$\Phi(w) = \mathfrak{d}_{I'} \mathfrak{t}_{i'} = \mathfrak{d}_A \mathfrak{d}_{i'} \mathfrak{t}_{i'} \mathfrak{d}_B.$$

Lemma 4.9. *Let $w = \mathfrak{t}_i \mathfrak{d}_I = \mathfrak{t}_i \mathfrak{d}_A \mathfrak{d}_i \mathfrak{d}_B = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B \in Y$ and let $w' = \Phi(w)$. Then the following statements hold.*

1. *$w'(\alpha) = w(\alpha)$ if $i = 1$.*
2. *$w'(\alpha) = w(\alpha)$ if $i \geq 2$ and i is not the smallest part of $\mathfrak{d}_B(\alpha)$.*
3. *$w'(\alpha) = (0, w(\alpha))$ if $i \geq 2$ and i is the smallest part of $\mathfrak{d}_B(\alpha)$.*

Proof. To prove this, we will perform a case analysis. Assume first that $i = 1$. Then it follows that A is the empty set, and thus $w = \mathfrak{t}_1 \mathfrak{d}_1 \mathfrak{d}_B$ and, by our definition of the map Φ, we have $w' = \mathfrak{d}_1 \mathfrak{d}_B \mathfrak{t}_1$. Since $w \in Y$, we know that $\mathfrak{d}_B(\alpha)$ has a part equalling 1. But then Lemma 3.26 implies that $w(\alpha) = \mathfrak{d}_1 \mathfrak{t}_1 \mathfrak{d}_B(\alpha)$. Now, in view of Lemma 3.25, we have that $\mathfrak{t}_1 \mathfrak{d}_B(\alpha) = \mathfrak{d}_B \mathfrak{t}_1(\alpha)$, thereby establishing that $w(\alpha) = \mathfrak{d}_1 \mathfrak{d}_B \mathfrak{t}_1(\alpha) = w'(\alpha)$.

Now assume that $i \geq 2$ and is such that smallest part of $\mathfrak{d}_B(\alpha)$ does not equal i. Let k be the largest part of α strictly less than i. We know that it exists (that is, does not equal 0) by our hypothesis on i. First we will establish that $k \geq \max(A)$.

If $k = i-1$, then this is immediate. Hence assume that $k \leq i-2$. If $\max(A) > k$, consider the composition $\mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B(\alpha) = \gamma$. By the definition of k, we know that this composition does not have parts r satisfying $k < r < i$. But then $\mathfrak{d}_A(\gamma) = 0$, contradicting our assumption that $w \in Y$. Thus we must have $k \geq \max(A)$.

Next we claim that $\mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B(\alpha) = \mathfrak{d}_{k+1} \mathfrak{t}_{k+1} \mathfrak{d}_B(\alpha)$. If $k = i-1$, then this follows from Lemma 3.26 since $w \in Y$. Therefore we assume that $k \leq i-2$. But in this case, notice that $\mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B(\alpha)$ is exactly $\mathfrak{d}_B(\alpha)$. Also since $\mathfrak{d}_B(\alpha)$ has no part equalling r where $k < r < i$, but does have a part equalling k, we get that $\mathfrak{d}_{k+1} \mathfrak{t}_{k+1} \mathfrak{d}_B(\alpha)$ equals $\mathfrak{d}_B(\alpha)$. Thus we obtain that in all cases

$$\mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B(\alpha) = \mathfrak{d}_A \mathfrak{d}_{k+1} \mathfrak{t}_{k+1} \mathfrak{d}_B(\alpha).$$

Using Lemma 3.25, we get that the right hand side equals $\mathfrak{d}_A \mathfrak{d}_{k+1} \mathfrak{d}_B \mathfrak{t}_{k+1}(\alpha) = w'(\alpha)$. This finishes this case.

Finally assume that $i \geq 2$ and that i is the smallest part of $\mathfrak{d}_B(\alpha)$. Since i is the smallest part of $\mathfrak{d}_B(\alpha)$ and $w \in Y$ it follows that A is the empty set. Thus we have $w = \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B$. Since i is the smallest part of $\mathfrak{d}_B(\alpha)$, we easily see that $w(\alpha) = \mathfrak{d}_B(\alpha)$. On the other hand in this case, we have that $\Phi(w) = w' = \mathfrak{d}_1 \mathfrak{d}_B \mathfrak{t}_1$. By Lemma 3.25 we get that $w'(\alpha) = \mathfrak{d}_1 \mathfrak{t}_1 \mathfrak{d}_B(\alpha)$, and this clearly equals $(0, \mathfrak{d}_B(\alpha)) = (0, w(\alpha))$. This finishes the proof. $\square$

Remark 4.10. Observe from the above lemma that it is implicit that $\Phi : Y \to X$, and that at the level of compositions we have that $w(\alpha) = \Phi(w)(\alpha)$ for all $w \in Y$.

Lemma 4.11. *Φ is an injection from Y to X.*

Proof. Let $w_1 = \mathfrak{t}_i\mathfrak{d}_I$ and $w_2 = \mathfrak{t}_j\mathfrak{d}_J$ be distinct elements of Y. We have

$$w_1 = \mathfrak{t}_i\mathfrak{d}_A\mathfrak{d}_i\mathfrak{d}_B$$
$$w_2 = \mathfrak{t}_j\mathfrak{d}_C\mathfrak{d}_j\mathfrak{d}_D$$

and

$$\Phi(w_1) = \mathfrak{d}_A\mathfrak{d}_{k+1}\mathfrak{d}_B\mathfrak{t}_{k+1}$$
$$\Phi(w_2) = \mathfrak{d}_C\mathfrak{d}_{m+1}\mathfrak{d}_D\mathfrak{t}_{m+1}$$

where

$$k = \text{ largest part} < i \text{ in } \mathfrak{d}_B(\alpha) \text{ (or 0)}$$
$$m = \text{ largest part} < j \text{ in } \mathfrak{d}_D(\alpha) \text{ (or 0).}$$

Assuming that $\Phi(w_1) = \Phi(w_2)$ we must have $k = m$. Furthemore, note that this also implies the following key fact.

$$A \amalg B = C \amalg D \tag{4.7}$$

The two facts together yield that $A = C$ and $B = D$. So all we need to finish the proof is to establish that $i = j$.

At this point assume without loss of generality that $i > j$. Then we have the following.

$$w_1 = \mathfrak{t}_i\mathfrak{d}_A\mathfrak{d}_i\mathfrak{d}_B$$
$$w_2 = \mathfrak{t}_j\mathfrak{d}_A\mathfrak{d}_j\mathfrak{d}_B$$

Since both these words belong to Y, we know that they act on α and give a valid (nonzero) composition. However, $w_2 \in Y$ implies that the largest part of α that is strictly less than i is weakly greater than j. But this implies that $\Phi(w_1) \neq \Phi(w_2)$, which is contrary to our assumption. Hence i must equal j, and we are done. $\square$

The next step for us is to identify the image of Y under the map Φ. The image of Y is a very special subset of X, which has the following explicit description. Let the largest part of α be m. Define Z as follows.

$$Z = \{\mathfrak{d}_I\mathfrak{t}_i \in X \mid i \leq m\}$$

Thus in other words, Z is the subset comprising of words that never add a box to the largest part. Note that by the definition of Φ we have that $\Phi(Y) \subseteq Z$ since if $w \in Y$ and $\Phi(w)$ has rightmost operator $\mathfrak{t}_j$ then $j \leq m$. Our next aim is to show that Φ actually bijects Y onto Z.

Consider $w \in Z = \mathfrak{d}_I\mathfrak{t}_i$. Writing $I = A \amalg \{i\} \amalg B$ in the usual way, and using Lemma 3.25 allows us to write w as shown below.

$$w = \mathfrak{d}_A\mathfrak{d}_i\mathfrak{t}_i\mathfrak{d}_B$$

Let i'' be the smallest part of $\mathfrak{d}_B(\alpha)$ weakly greater than i. This always exists by our hypothesis that $w \in Z$ thereby implying that i is not the largest part of α. We define $\Psi(w)$ as follows.

$$\begin{aligned}\Psi(w) &= \mathfrak{d}_A \mathfrak{t}_{i''} \mathfrak{d}_{i''} \mathfrak{d}_B \\ &= \mathfrak{t}_{i''} \mathfrak{d}_{I''}\end{aligned}$$

where $I'' = A \amalg \{i''\} \amalg B$. It is straightforward to see that if k is the largest part of α strictly less thank i, $i' = k+1$, and i'' is the smallest part of $\mathfrak{d}_B(\alpha)$ weakly greater than i', then $i'' = i$ and hence

$$\Psi(\Phi(w)) = \mathfrak{d}_A \mathfrak{t}_i \mathfrak{d}_i \mathfrak{d}_B = w$$

so Ψ is the inverse of Φ.

Example 4.12. *Consider the composition $\alpha = (2,6,1,4)$ and let $w = \mathfrak{t}_4 \mathfrak{d}_{\{1,4,5,6\}}$. Then $w(\alpha) = (2,4,0,4)$ so $w \in Y$. We have the following decomposition for w.*

$$w = \mathfrak{d}_{\{1\}} \mathfrak{t}_4 \mathfrak{d}_4 \mathfrak{d}_{\{5,6\}}$$

Then the corresponding A, B and i are $\{1\}$, $\{5,6\}$ and 4 respectively. Our method for constructing $\Phi(w)$ demands of us that first we find the largest part k strictly less than i in $\mathfrak{d}_B(\alpha)$. Since $\mathfrak{d}_B(\alpha) = (2,4,1,4)$ it follows that $k = 2$. This implies that

$$\Phi(w) = \mathfrak{d}_{\{1,3,5,6\}} \mathfrak{t}_3$$

and hence $\Phi(w) = (2,4,0,4) = w(\alpha)$ and $\Phi(w) \in Z$. Lastly note that $i'' = 4$ and

$$\Psi(\Phi(w)) = \Psi(\mathfrak{d}_{\{1\}} \mathfrak{t}_4 \mathfrak{d}_4 \mathfrak{d}_{\{5,6\}}) = \mathfrak{d}_{\{1\}} \mathfrak{t}_4 \mathfrak{d}_4 \mathfrak{d}_{\{5,6\}} = w$$

as desired.

Consider the sets P and Q defined as follows.

$$\begin{aligned}P &= \{\mathfrak{d}_i \mathfrak{t}_i \mid i \geq 1, \mathfrak{d}_i \mathfrak{t}_i(\alpha) \neq 0\} \\ Q &= \{\mathfrak{t}_i \mathfrak{d}_i \mid i \geq 1, \mathfrak{t}_i \mathfrak{d}_i(\alpha) \neq 0\}\end{aligned}$$

Clearly, $P \subset X$ and $Q \subset Y$. Furthermore, we have that $\Phi(Q)$ maps into P. In fact, a stronger claim holds from the discussion prior to this: $\Phi(Q) = P \setminus \{\mathfrak{d}_{m+1} \mathfrak{t}_{m+1}\}$ where m is the largest part of α.

Then utilising all of the above we have the following two theorems.

Theorem 4.13. *$\mathcal{L}_c$ and $\mathcal{Q}_c$ are dual graded graphs, that is, on compositions*

$$DU_t - U_t D = Id.$$

Proof. Firstly note that DU_t corresponds to the following expansion.

$$DU_t = \sum_{i,j \geq 1} \mathfrak{d}_i \mathfrak{t}_j = \sum_{i,j \geq 1, i \neq j} \mathfrak{d}_i \mathfrak{t}_j + \sum_{k \geq 1} \mathfrak{d}_k \mathfrak{t}_k$$

Also the operator U_tD corresponds to the expansion below.

$$U_tD = \sum_{i,j\geq 1} \mathfrak{t}_j\mathfrak{d}_i = \sum_{i,j\geq 1, i\neq j} \mathfrak{t}_j\mathfrak{d}_i + \sum_{k\geq 1} \mathfrak{t}_k\mathfrak{d}_k$$

Then, on using Lemma 3.25, we obtain the following.

$$DU_t - U_tD = \sum_{k\geq 1} \mathfrak{d}_k\mathfrak{t}_k - \sum_{k\geq 1} \mathfrak{t}_k\mathfrak{d}_k \tag{4.8}$$

Taking α into account we can rewrite the above equation as stating the following.

$$(DU_t - U_tD)(\alpha) = \sum_{w\in P} w(\alpha) - \sum_{w\in Q} w(\alpha) = \mathfrak{d}_{m+1}\mathfrak{t}_{m+1}(\alpha) + \sum_{w\in Q} (\Phi(w) - w)(\alpha)$$

Now we have $\sum_{w\in Q}(\Phi(w) - w)(\alpha) = 0$ by Lemma 4.9, and $\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}(\alpha) = \alpha$. This implies the claim. □

Example 4.14. *Let $\alpha = (2, 1, 3)$. Then suppressing commas and parentheses as before, we have that*

$$\begin{aligned} DU_t(\alpha) &= D(1213 + 223 + 313 + 214) \\ &= 1203 + 1113 + 1212 + 213 + 222 \\ &\quad +303 + 312 + 204 + 114 + 213 \end{aligned}$$

and

$$\begin{aligned} U_tD(\alpha) &= U_t(203 + 113 + 212) \\ &= 1203 + 303 + 204 + 1113 + 213 \\ &\quad +114 + 1212 + 222 + 312. \end{aligned}$$

Thus $(DU_t - U_tD)(\alpha) = 213 = Id(\alpha)$.

Theorem 4.15. *$\mathcal{L}_c$ and $\widetilde{\mathcal{Q}}_c$ are dual filtered graphs, that is, on compositions*

$$\widetilde{D}U_t - U_t\widetilde{D} = \widetilde{D} + Id.$$

Proof. The beginning of the proof is very similar to that in Theorem 4.5 but with $\mathfrak{t}_i$ instead of $\mathfrak{u}_i$. Using Lemma 3.25 we obtain the following equality.

$$\widetilde{D}U_t - U_t\widetilde{D} = \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{d}_I\mathfrak{t}_i - \sum_{\substack{I\subset\mathbb{N}\\ i\in I}} \mathfrak{t}_i\mathfrak{d}_I$$

Now for the fixed composition α, we can rewrite the above equation as follows.

$$\begin{aligned} (\widetilde{D}U_t - U_t\widetilde{D})(\alpha) &= \sum_{w\in X} w(\alpha) - \sum_{w\in Y} w(\alpha) \\ &= \sum_{w\in X\setminus Z} w(\alpha) + \sum_{w\in Z} w(\alpha) - \sum_{w\in Y} w(\alpha) \end{aligned} \tag{4.9}$$

At the level of compositions, Lemma 4.9 implies that

$$\sum_{w\in Y}(\Phi(w)(\alpha) - w(\alpha)) = 0.$$

Using the above in Equation (4.9) at the level of compositions gives

$$(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = \sum_{w\in X\setminus Z} w(\alpha).$$

Observe now that every element of $X \setminus Z$ has the form $\mathfrak{d}_A\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}$ where A consists only of instances of $\mathfrak{d}_i$ where $i \leq m$ and m is the largest part of α. Furthermore we do have the possibility that A is empty. Additionally, it is easy to see that $\mathfrak{d}_{m+1}\mathfrak{t}_{m+1}$ is the identity map. The preceding discussion allows us to conclude the following equality at the level of compositions, thereby finishing the proof.

$$(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = (\widetilde{D} + Id)(\alpha)$$

□

Example 4.16. *Let $\alpha = (1, 2)$. Then suppressing commas and parentheses as before, we have that*

$$\widetilde{D}(\alpha) = (02 + 11 + 10).$$

Plus

$$\begin{aligned}\widetilde{D}U_t(\alpha) &= \widetilde{D}(112 + 22 + 13)\\ &= 012 + 111 + 110 + 21 + 20\\ &\quad +03 + 12 + 02 + 11 + 10\end{aligned}$$

and

$$\begin{aligned}U_t\widetilde{D}(\alpha) &= U_t(02 + 11 + 10)\\ &= 102 + 03 + 111 + 21 + 110 + 20.\end{aligned}$$

Thus $(\widetilde{D}U_t - U_t\widetilde{D})(\alpha) = 12 + 2 + 11 + 1 = (\widetilde{D} + Id)(\alpha)$.

5. Quasisymmetric and noncommutative symmetric functions

Our next applications of operators on compositions are more algebraic, and to this end we now recall the Hopf algebra of quasisymmetric functions and its dual, the Hopf algbera of noncommutative symmetric functions. For more details see [8, 13, 14, 24]. The Hopf algebra of quasisymmetric functions QSym is a subalgebra of $\mathbb{C}[[x_1, x_2, \ldots]]$ with basis given by the following functions, which in turn are reliant on the natural bijection between compositions and sets: Given a composition $\alpha = (\alpha_1, \ldots, \alpha_{\ell(\alpha)})$, there is a natural subset of $[|\alpha| - 1]$ corresponding to it, namely,

$$\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)-1}\}.$$

Conversely, given a subset $S = \{s_1, \dots, s_{|S|}\} \subseteq [i-1]$, there is a natural composition of size i corresponding to it, namely

$$\mathrm{comp}(S) = (s_1, s_2 - s_1, \dots, i - s_{|S|}).$$

Definition 5.1. *Let $\alpha = (\alpha_1, \dots, \alpha_\ell)$ be a composition. Then the* fundamental quasisymmetric function F_α *is defined to be $F_0 = 1$ and*

$$F_\alpha = \sum x_{i_1} \cdots x_{i_{|\alpha|}}$$

where the sum is over all $|\alpha|$-tuples $(i_1, \dots, i_{|\alpha|})$ of indices satisfying

$$i_1 \leq \cdots \leq i_{|\alpha|} \text{ and } i_j < i_{j+1} \text{ if } j \in \mathrm{set}(\alpha).$$

Example 5.2. $F_{(1,2)} = x_1x_2^2 + x_1x_3^2 + \cdots + x_1x_2x_3 + x_1x_2x_4 + \cdots$.

Moreover, QSym is a graded Hopf algebra

$$\mathrm{QSym} = \bigoplus_{i \geq 0} \mathrm{QSym}^i$$

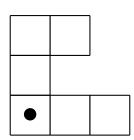
where

$$\mathrm{QSym}^i = \mathrm{span}\{F_\alpha \mid |\alpha| = i\}.$$

However, this is not the only basis of QSym that will be useful to us. For the second basis we will need to define skew composition diagrams and then standard skew composition tableaux.

For the first of these, let α, β be two compositions such that $\beta <_c \alpha$. Then we define the *skew composition diagram* $\alpha /\!\!/ \beta$ to be the array of boxes that are contained in α but not in β. That is, the boxes that arise in the chain $\beta \lessdot_c \cdots \lessdot_c \alpha$. We say the *size* of $\alpha /\!\!/ \beta$ is $|\alpha /\!\!/ \beta| = |\alpha| - |\beta|$. Note that if $\beta = \emptyset$ then we recover the composition diagram α.

Example 5.3. *The skew composition diagram $(2,1,3) /\!\!/ (1)$ is drawn below with β denoted by $\bullet$.*



We can now define standard skew composition tableaux. Recall that a saturated chain in a poset is a finite sequence of consecutive cover relations. Then given a saturated chain, C, in $\mathcal{L}_c$

$$\beta = \alpha^0 \lessdot_c \alpha^1 \lessdot_c \cdots \lessdot_c \alpha^{|\alpha /\!\!/ \beta|} = \alpha$$

we define the *standard skew composition tableau* τ_C of *shape* $\alpha /\!\!/ \beta$ to be the skew composition diagram $\alpha /\!\!/ \beta$ whose boxes are filled with integers such that the number $|\alpha /\!\!/ \beta| - i + 1$ appears in the box in τ_c that exists in α^i but not α^{i-1}. If $\beta = \emptyset$ then we say we have a *standard composition tableau*. Given a standard skew composition tableau, τ, of size i we say that the *descent set* of τ is

$$\mathrm{Des}(\tau) = \{j \mid j+1 \text{ appears weakly right of } j\} \subseteq [i-1]$$

and the corresponding *descent composition* of τ is $\mathrm{comp}(\tau) = \mathrm{comp}(\mathrm{Des}(\tau))$.

Example 5.4. *The saturated chain*

$$(1) \lessdot_c (2) \lessdot_c (1,2) \lessdot_c (1,1,2) \lessdot_c (1,1,3) \lessdot_c (2,1,3)$$

gives rise to the standard skew composition tableau τ *of shape* $(2,1,3)/\!\!/(1)$ *below.*

$$\begin{array}{|c|c|c|}\hline 3 & 1 & \\ \hline 4 & & \\ \hline \bullet & 5 & 2 \\ \hline \end{array}$$

Note that $\mathrm{Des}(\tau) = \{1,3,4\}$ *and hence* $\mathrm{comp}(\tau) = (1,2,1,1)$.

With this is mind we can now define skew quasisymmetric Schur functions.

Definition 5.5. *Let* $\alpha/\!\!/\beta$ *be a skew composition diagram. Then the* skew quasisymmetric Schur function $\mathcal{S}_{\alpha/\!\!/\beta}$ *is defined to be*

$$\mathcal{S}_{\alpha/\!\!/\beta} = \sum F_{\mathrm{comp}(\tau)}$$

where the sum is over all standard skew composition tableaux τ *of shape* $\alpha/\!\!/\beta$. *When* $\beta = \emptyset$ *we call* $\mathcal{S}_\alpha$ *a* quasisymmetric Schur function.

Example 5.6. *We can see that* $\mathcal{S}_{(n)} = F_{(n)}$ *and* $\mathcal{S}_{(1^n)} = F_{(1^n)}$ *and*

$$\mathcal{S}_{(2,1,3)/\!\!/(1)} = F_{(2,1,2)} + F_{(2,2,1)} + F_{(1,2,1,1)}$$

from the standard skew composition tableaux below.

$$\begin{array}{|c|c|c|}\hline 2 & 1 & \\ \hline 3 & & \\ \hline \bullet & 5 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 & \\ \hline 4 & & \\ \hline \bullet & 5 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 1 & \\ \hline 4 & & \\ \hline \bullet & 5 & 2 \\ \hline \end{array}$$

Moreover, the set of all quasisymmetric Schur functions forms another basis for QSym and

$$\mathrm{QSym}^i = \mathrm{span}\{\mathcal{S}_\alpha \mid |\alpha| = i\}.$$

As discussed in the introduction, quasisymmetric Schur functions have many interesting algebraic and combinatorial properties, one of the first of which to be discovered was the exhibition of Pieri rules that utilise our box removing operators [16, Theorem 6.3].

Theorem 5.7. (Pieri rules for quasisymmetric Schur functions) *Let* α *be a composition and* n *be a positive integer. Then*

$$\mathcal{S}_\alpha \cdot \mathcal{S}_{(n)} = \sum \mathcal{S}_{\alpha^+}$$

where α^+ *is a composition such that* α *can be obtained by removing an* n*-horizontal strip from it.*

Similarly,

$$\mathcal{S}_\alpha \cdot \mathcal{S}_{(1^n)} = \sum \mathcal{S}_{\alpha^+}$$

where α^+ *is a composition such that* α *can be obtained by removing an* n*-vertical strip from it.*

Dual to QSym is the Hopf algebra of noncommutative symmetric functions, itself a subalgebra of $\mathbb{C} << x_1, x_2, \ldots >>$ with many interesting bases [13]. The one of particular interest to us is the following.

Definition 5.8. *Let α be a composition. Then the* noncommutative Schur function $\mathbf{s}_\alpha$ *is the function under the duality paring $\langle , \rangle : \mathrm{QSym} \otimes \mathrm{NSym} \to \mathbb{C}$ that satisfies*

$$\langle \mathcal{S}_\alpha, \mathbf{s}_\beta \rangle = \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise.

Noncommutative Schur functions also have rich and varied algebraic and combinatorial properties, including Pieri rules, although due to the noncommutative nature of NSym there are now Pieri rules arising both from multiplication on the right [36, Theorem 9.3], and from multiplication on the left [3, Corollary 3.8].

Theorem 5.9. (Right Pieri rules for noncommutative Schur functions) *Let α be a composition and n be a positive integer. Then*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)} = \sum \mathbf{s}_{\alpha^+}$$

where α^+ is a composition such that it can be obtained by adding an n-right horizontal strip to α.

Similarly,

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum \mathbf{s}_{\alpha^+}$$

where α^+ is a composition such that it can be obtained by adding an n-right vertical strip to α.

Theorem 5.10. (Left Pieri rules for noncommutative Schur functions) *Let α be a composition and n be a positive integer. Then*

$$\mathbf{s}_{(n)} \cdot \mathbf{s}_\alpha = \sum \mathbf{s}_{\alpha^+}$$

where α^+ is a composition such that it can be obtained by adding an n-left horizontal strip to α.

Similarly,

$$\mathbf{s}_{(1^n)} \cdot \mathbf{s}_\alpha = \sum \mathbf{s}_{\alpha^+}$$

where α^+ is a composition such that it can be obtained by adding an n-left vertical strip to α.

Note that since quasisymmetric and noncommutative Schur functions are indexed by compositions, if any parts of size 0 arise during computation, then they are ignored. Before we reach our next application we recall one Hopf algebraic lemma, which will play a key role later. Let $\mathcal{H}$ and $\mathcal{H}^*$ be a pair of dual Hopf algebras under a duality pairing

$\langle,\rangle : \mathcal{H} \otimes \mathcal{H}^* \longrightarrow K$, where K is a field of characteristic 0. For $h \in \mathcal{H}$ and $a \in \mathcal{H}^*$, let the following be the respective coproducts in Sweedler notation.

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)}, \tag{5.1}$$

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}. \tag{5.2}$$

Now define left actions of $\mathcal{H}^*$ on $\mathcal{H}$ and $\mathcal{H}$ on $\mathcal{H}^*$, both denoted by $\rightharpoonup$, as follows.

$$a \rightharpoonup h = \sum_h \langle h_{(2)}, a\rangle h_{(1)}, \tag{5.3}$$

$$h \rightharpoonup a = \sum_a \langle h, a_{(2)}\rangle a_{(1)}, \tag{5.4}$$

where $a \in \mathcal{H}^*$, $h \in \mathcal{H}$. Then we have the following.

Lemma 5.11. [20] *For all $g, h \in \mathcal{H}$ and $a \in \mathcal{H}^*$, we have that*

$$(a \rightharpoonup g) \cdot h = \sum_h \big(S(h_{(2)}) \rightharpoonup a\big) \rightharpoonup \big(g \cdot h_{(1)}\big)$$

where $S : \mathcal{H} \longrightarrow \mathcal{H}$ is the antipode.

6. Right Pieri rules for noncommutative Schur functions

We now come to our next application of our lemmas in Section 3. More precisely, as we saw in the last section, in [36, Theorem 9.3], combinatorial rules are given for the products of noncommutative Schur functions $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ and $\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)}$, known as the *right Pieri rules for noncommutative Schur functions*, and for each summand in the product the rules require the application of n jdt operators to α. However, we can now recast the rules more simply, only requiring the removal of horizontal strips from α followed by appending parts to the result. As noted earlier, noncommutative Schur functions are indexed by compositions, so if any parts of size 0 arise during computation, then they are ignored.

Theorem 6.1. *Let α be a composition and n be a positive integer. For a positive integer i, define $A_{i,n}$ to be set of subsets of $[i-1]$ with cardinality $i-n$. Then we have the following expansion.*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)} = \sum_{\substack{i\geq 1,\ I\in A_{i,n} \\ a_i \mathfrak{d}_I(\alpha)=\beta}} \mathbf{s}_\beta$$

Proof. Note first that the statement of Theorem 5.9 (namely, [36, Theorem 9.3]) is equivalent to the following.

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)} = \sum_{\substack{J \subset \mathbb{N},\ |J|=n \\ \mathfrak{u}_J(\alpha)=\beta}} \mathbf{s}_\beta \tag{6.1}$$

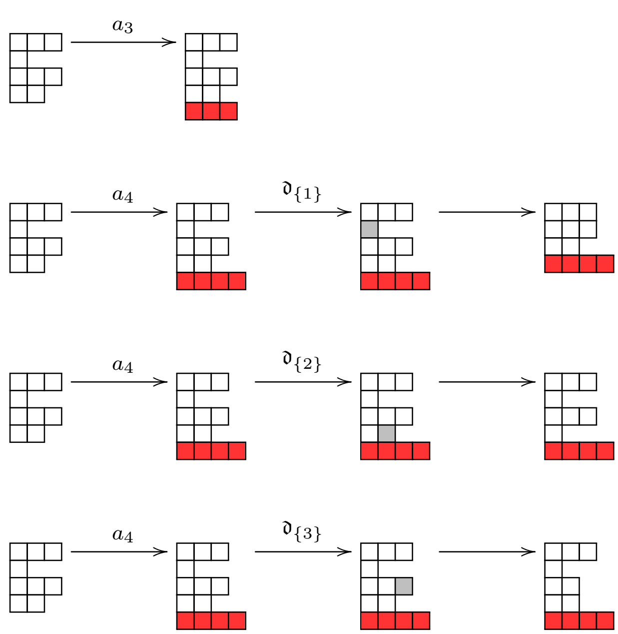
Now for a fixed set $J = \{j_1 < \cdots < j_n\}$, by Lemma 3.23, we have that

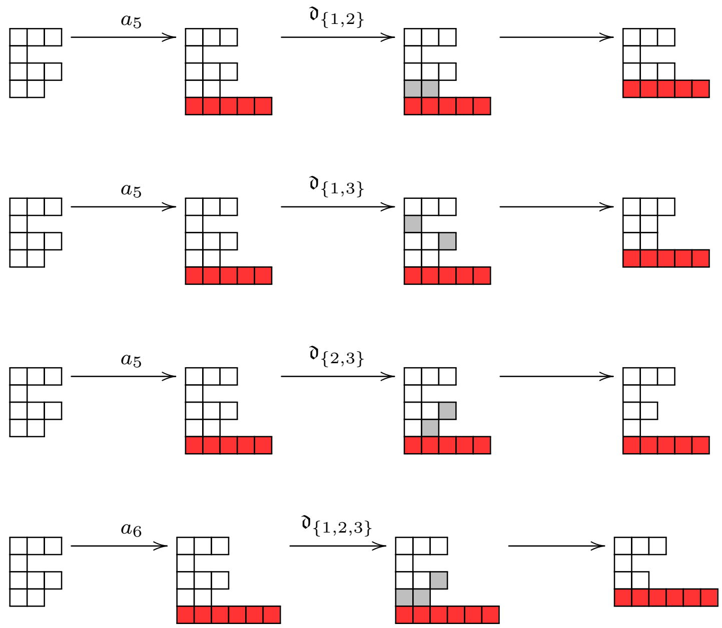
$$\mathfrak{u}_J = a_{j_n} \mathfrak{d}_{[j_n] \setminus J}$$

where the set $[j_n] \setminus J$ has cardinality $j_n - n$ and is clearly a subset of $[j_n - 1]$. Furthermore, given a product of the form on the right hand side of the equation above, it is easy to construct the corresponding $\mathfrak{u}_J$. Thus the equivalence between the right hand side of Equation (6.1) and the statement of the theorem follows. $\square$

Remark 6.2. The example below should convince the reader that the computations involved in Theorem 6.1 are less daunting than they might appear. Essentially, all compositions γ such that $\mathbf{s}_\gamma$ appears in the expansion $\mathbf{s}_\alpha \cdot \mathbf{s}_{(n)}$ are obtained by appending a row of a certain length i to α, and then removing an $(i-n)$-horizontal strip from the resulting composition so that the boxes are only removed from the first $(i-1)$ columns.

Example 6.3. *Consider the expansion of $\mathbf{s}_{(3,1,3,2)} \cdot \mathbf{s}_{(3)}$. From our description of the rule above, we know that diagrammatically we need to append a row of length i and then remove $(i-3)$ boxes from the first $(i-1)$ columns so that there is at most 1 box removed from any column, in all possible ways. Below we list all the ways. We denote the removed boxes in the lighter shade of grey, and the appended row in the darker shade of red.*





Thus we obtain the following expansion for $\mathbf{s}_{(3,1,3,2)} \cdot \mathbf{s}_{(3)}$*, suppressing commas and parentheses in compositions for ease of comprehension.*

$$\mathbf{s}_{3132} \cdot \mathbf{s}_3 = \mathbf{s}_{31323} + \mathbf{s}_{3324} + \mathbf{s}_{31314} + \mathbf{s}_{31224} + \mathbf{s}_{3135} + \mathbf{s}_{3225} + \mathbf{s}_{31215} + \mathbf{s}_{3126}$$

Before we state the other right Pieri rule, we will require some more notation. Let $M(\alpha)$ denote the multiset consisting of the parts of α. For example, if $\alpha = (3,1,2,2,4,5,3,3)$, then $M(\alpha) = \{1,2,2,3,3,3,4,5\}$ and $\{1,3,3,4\}$ is a submultiset of cardinality 4.

Theorem 6.4. *Let* α *be a composition and* n *be a positive integer. Then we have the following expansion*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum \mathbf{s}_{\beta \bullet \gamma}$$

where the sum runs over all submultisets $\{i_1 - 1 \leq \cdots \leq i_k - 1\}$ *of* $M(\alpha)$ *where* $0 \leq k \leq n$*, and*

$$\beta = \mathfrak{d}_{[i_1-1]} \cdots \mathfrak{d}_{[i_k-1]}(\alpha)$$
$$\gamma = (i_k, \ldots, i_1, 1^{n-k}).$$

Proof. The statement of Theorem 5.9 (namely, [36, Theorem 9.3]) says the following.

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum_{\substack{j_1 \leq \cdots \leq j_n \\ \mathfrak{u}_{j_1} \cdots \mathfrak{u}_{j_n}(\alpha) = \beta}} \mathbf{s}_\beta \tag{6.2}$$

Now note that given positive integers r, s such that $r \leq s$, we have the following equality.

$$\mathfrak{u}_r \mathfrak{u}_s = a_r a_s \mathfrak{d}_{[r-1]} \mathfrak{d}_{[s-1]} \tag{6.3}$$

This follows immediately from applying Lemma 3.17 repeatedly. From Equation (6.3), we obtain that

$$\mathfrak{u}_{j_1} \cdots \mathfrak{u}_{j_n} = a_{j_1} \cdots a_{j_n} \mathfrak{d}_{[j_1-1]} \cdots \mathfrak{d}_{[j_n-1]} \tag{6.4}$$

if $j_1 \leq \cdots \leq j_n$. Using this in Equation (6.2) gives us that

$$\mathbf{s}_\alpha \cdot \mathbf{s}_{(1^n)} = \sum_{a_{j_1} \cdots a_{j_n} \mathfrak{d}_{[j_1-1]} \cdots \mathfrak{d}_{[j_n-1]}(\alpha)=\beta} \mathbf{s}_\beta \tag{6.5}$$

where the sum runs over all sequences $j_1 \leq \cdots \leq j_n$. Amongst the numbers $j_1, \ldots, j_n$, let $i_1, \ldots, i_k$ denote the numbers that do not equal 1. We further assume that $i_1 \leq \cdots \leq i_k$. Clearly we have $0 \leq k \leq n$.

By assumption, we know that $\mathfrak{d}_{[0]}$ is the identity map. Hence we have that

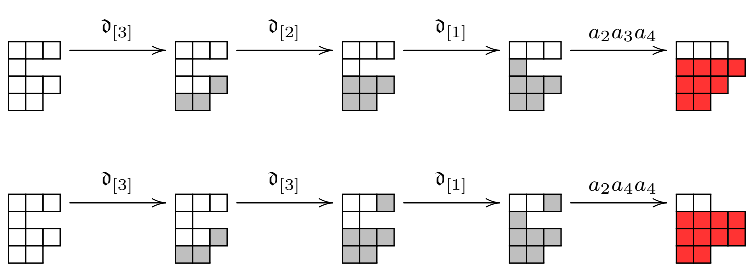
$$a_{j_1} \cdots a_{j_n} \mathfrak{d}_{[j_1-1]} \cdots \mathfrak{d}_{[j_n-1]} = (a_1)^k a_{i_1} \cdots a_{i_k} \mathfrak{d}_{[i_1-1]} \cdots \mathfrak{d}_{[i_k-1]}$$

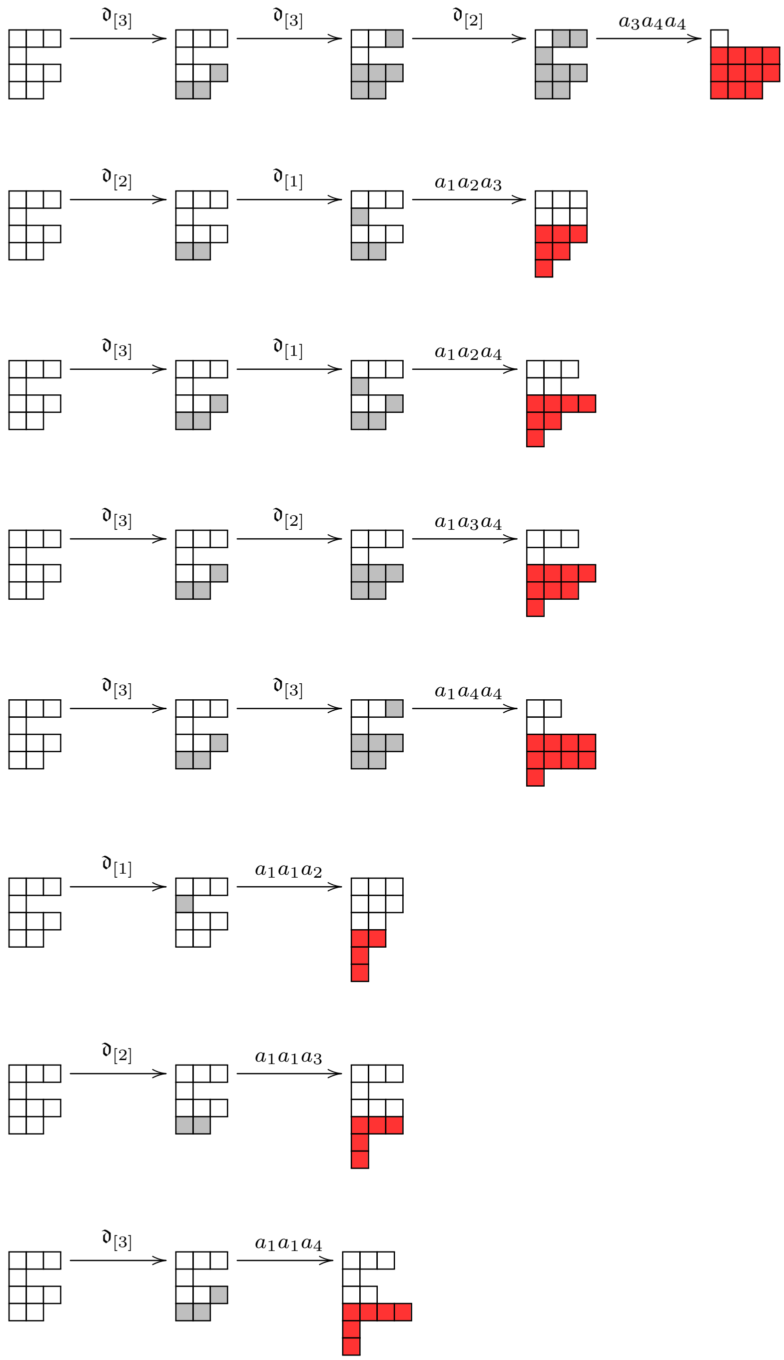
where $(a_1)^k$ denotes the operator a_1 applied k times, that is, the operator that appends k parts equalling 1 at the end of a composition. Observe now that if r is a positive integer, then $\mathfrak{d}_{[r]}(\alpha)$ is a valid composition (that is, it is not zero) if and only if α has a part equalling r. Furthermore, if α does indeed have a part equalling r, then it is straightforward to show that $M(\mathfrak{d}_{[r]}(\alpha))$ is $M(\alpha)$ with exactly one instance of r removed. Therefore, we deduce that $\mathfrak{d}_{[i_1-1]} \cdots \mathfrak{d}_{[i_k-1]}(\alpha)$ is a valid composition for a sequence of positive integers $i_1-1 \leq \cdots \leq i_k-1$ if and only if the multiset $\{i_1-1, \ldots, i_k-1\}$ is a submultiset of $M(\alpha)$. Taking this into account in Equation (6.5) finishes the proof. $\square$

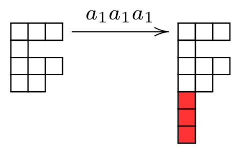
Example 6.5. *Consider the expansion of* $\mathbf{s}_{(3,1,3,2)} \cdot \mathbf{s}_{(1,1,1)}$. *We begin by listing the submultisets of cardinality* ≤ 3 *of* $M(\alpha)$.

$$\{1,2,3\}, \{1,3,3\}, \{2,3,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,3\}, \{1\}, \{2\}, \{3\}, \emptyset$$

For each of these multisets respectively, we will compute the term that arises in the expansion of $\mathbf{s}_{(3,1,3,2)} \cdot \mathbf{s}_{(1,1,1)}$ *We denote the removed boxes in the lighter shade of grey, and the appended rows in the darker shade of red.*




$\mathfrak{d}_{[3]}$ $\mathfrak{d}_{[3]}$ $\mathfrak{d}_{[2]}$ $a_3a_4a_4$
$\mathfrak{d}_{[2]}$ $\mathfrak{d}_{[1]}$ $a_1a_2a_3$
$\mathfrak{d}_{[3]}$ $\mathfrak{d}_{[1]}$ $a_1a_2a_4$
$\mathfrak{d}_{[3]}$ $\mathfrak{d}_{[2]}$ $a_1a_3a_4$
$\mathfrak{d}_{[3]}$ $\mathfrak{d}_{[3]}$ $a_1a_4a_4$
$\mathfrak{d}_{[1]}$ $a_1a_1a_2$
$\mathfrak{d}_{[2]}$ $a_1a_1a_3$
$\mathfrak{d}_{[3]}$ $a_1a_1a_4$



Thus we obtain the following expansion for $\mathbf{s}_{(3,1,3,2)} \cdot \mathbf{s}_{(1,1,1)}$*, suppressing commas and parentheses in compositions for ease of comprehension.*

$$\begin{aligned}\mathbf{s}_{3132} \cdot \mathbf{s}_{111} = \mathbf{s}_{3432} + \mathbf{s}_{2442} + \mathbf{s}_{1443} + \mathbf{s}_{33321} + \mathbf{s}_{32421} + \mathbf{s}_{31431} \\ + \mathbf{s}_{21441} + \mathbf{s}_{332211} + \mathbf{s}_{313311} + \mathbf{s}_{312411} + \mathbf{s}_{3132111}\end{aligned}$$

Remark 6.6. While it is not needed here, it can be proven that the operators $\mathfrak{d}_{[i]}$ and $\mathfrak{d}_{[j]}$ in fact commute. Thus the removing of horizontal strips that we performed in the above example could have been done in any order. Notice also that the lighter grey shaded boxes in the above example always correspond to a Young diagram (in English notation).

7. Generalized skew Pieri rules

7.1. **Quasisymmetric skew Pieri rules.** We now turn our attention to proving skew Pieri rules for skew quasisymmetric Schur functions. The statement of the rules is in the spirit of the Pieri rules for skew shapes of Assaf and McNamara [1] and is no coincidence as we recover their rules as a special case in Corollary 7.7. However first we prove a crucial observation.

Observation 7.1. *Let* α, β *be compositions. Then* $\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha = \mathcal{S}_{\alpha/\!\!/\beta}$.

Proof. By definition, [3, Definition 2.19], we have that

$$\Delta(\mathcal{S}_\alpha) = \sum_\gamma \mathcal{S}_{\alpha/\!\!/\gamma} \otimes \mathcal{S}_\gamma \tag{7.1}$$

where the sum is over all compositions γ. Thus using Equations (5.3) and (7.1) we obtain

$$\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha = \sum_\gamma \langle \mathcal{S}_\gamma, \mathbf{s}_\beta \rangle \mathcal{S}_{\alpha/\!\!/\gamma} \tag{7.2}$$

where the sum is over all compositions γ. Since $\langle \mathcal{S}_\gamma, \mathbf{s}_\beta \rangle$ equals 1 if $\beta = \gamma$ and 0 otherwise, the claim follows. ☐

Remark 7.2. The observation above does not tell us when $\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha = \mathcal{S}_{\alpha/\!\!/\beta}$ equals 0. However, by the definition of $\alpha/\!\!/\beta$ this is precisely when α and β satisfy $\beta \not<_c \alpha$. Consequently in the theorem below the *nonzero* contribution will only be from those α^+ and β^- that satisfy $\beta^- <_c \alpha^+$. As always if any parts of size 0 arise during computation, then they are ignored.

Theorem 7.3. *Let* α, β *be compositions and* n *be a positive integer. Then*

$$\mathcal{S}_{\alpha/\!\!/\beta} \cdot \mathcal{S}_{(n)} = \sum_{i+j=n} (-1)^j \mathcal{S}_{\alpha^+/\!\!/\beta^-}$$

where α^+ is a composition such that α can be obtained by removing an i-horizontal strip from it, and β^- is a composition such that it can be obtained by removing a j-vertical strip from β.

Similarly,

$$\mathcal{S}_{\alpha/\!\!/\beta} \cdot \mathcal{S}_{(1^n)} = \sum_{i+j=n} (-1)^j \mathcal{S}_{\alpha^+/\!\!/\beta^-}$$

where α^+ is a composition such that α can be obtained by removing an i-vertical strip from it, and β^- is a composition such that it can be obtained by removing a j-horizontal strip from β.

Proof. For the first part of the theorem, our aim is to calculate $\mathcal{S}_{\alpha/\!\!/\beta} \cdot \mathcal{S}_{(n)}$, which in light of Observation 7.1, is the same as calculating $(\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha) \cdot \mathcal{S}_{(n)}$.

Taking $a = \mathbf{s}_\beta$, $g = \mathcal{S}_\alpha$ and $h = \mathcal{S}_{(n)}$ in Lemma 5.11 gives the LHS as $(\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha) \cdot \mathcal{S}_{(n)}$. For the RHS observe that $\mathcal{S}_{(n)} = F_{(n)}$ and the coproduct on fundamental quasisymmetric functions is given by [14]

$$\Delta(F_{(n)}) \quad = \quad \sum_{i+j=n} F_{(i)} \otimes F_{(j)}. \tag{7.3}$$

Substituting these in yields

$$\sum_{i+j=n} (S(F_{(j)}) \rightharpoonup \mathbf{s}_\beta) \rightharpoonup (\mathcal{S}_\alpha \cdot F_{(i)}). \tag{7.4}$$

Now, by the action of the antipode on the basis of fundamental quasisymmetric functions, we have that $S(F_{(j)}) = (-1)^j F_{(1^j)}$ [8, 24]. This reduces (7.4) to

$$\sum_{i+j=n} ((-1)^j F_{(1^j)} \rightharpoonup \mathbf{s}_\beta) \rightharpoonup (\mathcal{S}_\alpha \cdot F_{(i)}). \tag{7.5}$$

We will first deal with the task of evaluating $F_{(1^j)} \rightharpoonup \mathbf{s}_\beta$. We need to invoke Equation (5.4) and thus we need $\Delta(\mathbf{s}_\beta)$. Assume that

$$\Delta(\mathbf{s}_\beta) = \sum_{\gamma,\delta} b^\beta_{\gamma,\delta} \mathbf{s}_\gamma \otimes \mathbf{s}_\delta \tag{7.6}$$

where the sum is over all compositions γ, δ. Thus Equation (5.4) yields

$$F_{(1^j)} \rightharpoonup \mathbf{s}_\beta \quad = \quad \sum_{\gamma,\delta} b^\beta_{\gamma,\delta} \langle F_{(1^j)}, \mathbf{s}_\delta \rangle \mathbf{s}_\gamma. \tag{7.7}$$

Observing that $F_{(1^j)} = \mathcal{S}_{(1^j)}$ and that $\langle \mathcal{S}_{(1^j)}, \mathbf{s}_\delta \rangle$ equals 1 if $\delta = (1^j)$ and equals 0 otherwise, we obtain

$$F_{(1^j)} \rightharpoonup \mathbf{s}_\beta \quad = \quad \sum_{\gamma} b^\beta_{\gamma,(1^j)} \mathbf{s}_\gamma. \tag{7.8}$$

Since $\langle \mathcal{S}_\gamma \otimes \mathcal{S}_\delta, \Delta(\mathbf{s}_\beta)\rangle = \langle \mathcal{S}_\gamma \cdot \mathcal{S}_\delta, \mathbf{s}_\beta\rangle = b^{\beta}_{\gamma,\delta}$, we get that

$$\langle \mathcal{S}_\gamma \cdot \mathcal{S}_{(1^j)}, \mathbf{s}_\beta\rangle \quad = \quad b^{\beta}_{\gamma,(1^j)}. \tag{7.9}$$

The Pieri rules for quasisymmetric Schur functions in Theorem 5.7 state that $b^{\beta}_{\gamma,(1^j)}$ is 1 if there exists a weakly decreasing sequence $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_j$ such that $\mathfrak{d}_{\ell_1} \cdots \mathfrak{d}_{\ell_j}(\beta) = \gamma$, and is 0 otherwise. Thus this reduces Equation (7.8) to

$$F_{(1^j)} \rightharpoonup \mathbf{s}_\beta \quad = \sum_{\substack{\mathfrak{d}_{\ell_1}\cdots\mathfrak{d}_{\ell_j}(\beta)=\gamma \\ \ell_1\geq\cdots\geq\ell_j}} \mathbf{s}_\gamma. \tag{7.10}$$

The same Pieri rules also imply that

$$\mathcal{S}_\alpha \cdot F_{(i)} \quad = \sum_{\substack{\mathfrak{d}_{r_1}\cdots\mathfrak{d}_{r_i}(\varepsilon)=\alpha \\ r_1<\cdots<r_i}} \mathcal{S}_\varepsilon. \tag{7.11}$$

Using Equations (7.10) and (7.11) in (7.5), we get

$$\sum_{i+j=n} ((-1)^j F_{(1^j)} \rightharpoonup \mathbf{s}_\beta) \rightharpoonup (\mathcal{S}_\alpha \cdot F_{(i)}) \quad = \quad \sum_{i+j=n} \left((-1)^j \sum_{\substack{\mathfrak{d}_{\ell_1}\cdots\mathfrak{d}_{\ell_j}(\beta)=\gamma \\ \ell_1\geq\cdots\geq\ell_j}} \mathbf{s}_\gamma \right) \rightharpoonup \left(\sum_{\substack{\mathfrak{d}_{r_1}\cdots\mathfrak{d}_{r_i}(\varepsilon)=\alpha \\ r_1<\cdots<r_i}} \mathcal{S}_\varepsilon \right). \tag{7.12}$$

Using Observation 7.1, we obtain that

$$\sum_{i+j=n} ((-1)^j F_{(1^j)} \rightharpoonup \mathbf{s}_\beta) \rightharpoonup (\mathcal{S}_\alpha \cdot F_{(i)}) \quad = \quad \sum_{i+j=n} \left(\sum_{\substack{\mathfrak{d}_{\ell_1}\cdots\mathfrak{d}_{\ell_j}(\beta)=\gamma \\ \ell_1\geq\cdots\geq\ell_j \\ \mathfrak{d}_{r_1}\cdots\mathfrak{d}_{r_i}(\varepsilon)=\alpha \\ r_1<\cdots<r_i}} (-1)^j \mathcal{S}_{\varepsilon/\!\!/\gamma} \right). \tag{7.13}$$

Thus

$$\mathcal{S}_{\alpha/\!\!/\beta} \cdot \mathcal{S}_{(n)} \quad = \quad \sum_{i+j=n} \left(\sum_{\substack{\mathfrak{d}_{\ell_1}\cdots\mathfrak{d}_{\ell_j}(\beta)=\gamma \\ \ell_1\geq\cdots\geq\ell_j \\ \mathfrak{d}_{r_1}\cdots\mathfrak{d}_{r_i}(\varepsilon)=\alpha \\ r_1<\cdots<r_i}} (-1)^j \mathcal{S}_{\varepsilon/\!\!/\gamma} \right). \tag{7.14}$$

The first part of the theorem now follows from the definitions of i-horizontal strip and j-vertical strip.

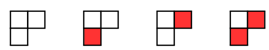
For the second part of the theorem we use the same method as the first part, but this time calculate

$$(\mathbf{s}_\beta \rightharpoonup \mathcal{S}_\alpha) \cdot \mathcal{S}_{(1^n)}.$$

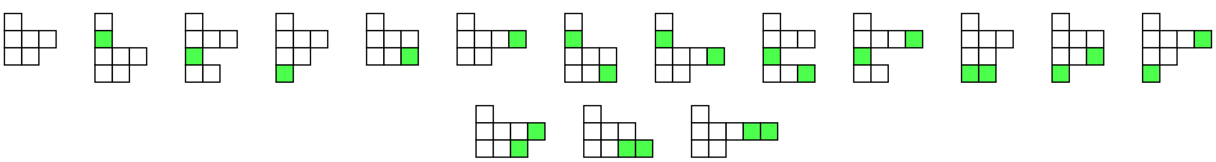
$\square$

Remark 7.4. Notice that as opposed to the classical case where one can apply the ω involution to obtain the corresponding Pieri rule, we can not do this here. This is because the image of the skew quasisymmetric Schur functions under the ω involution is not yet known explicitly. Notice that the ω map applied to quasisymmetric Schur functions results in the row strict quasisymmetric functions of Mason and Remmel [26].

Example 7.5. *Let us compute $\mathcal{S}_{(1,3,2)/\!\!/(2,1)} \cdot \mathcal{S}_{(2)}$. We first need to compute all compositions γ that can be obtained by removing a vertical strip of size at most 2 from $\beta = (2,1)$. These compositions correspond to the white boxes in the diagrams below, while the boxes in the darker shade of red correspond to the vertical strips that are removed from β.*



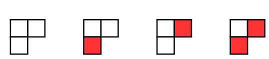
Next we need to compute all compositions ε such that a horizontal strip of size at most 2 can be removed from it so as to obtain α. We list these εs below with the boxes in the lighter shade of green corresponding to horizontal strips that need to be removed to obtain α.



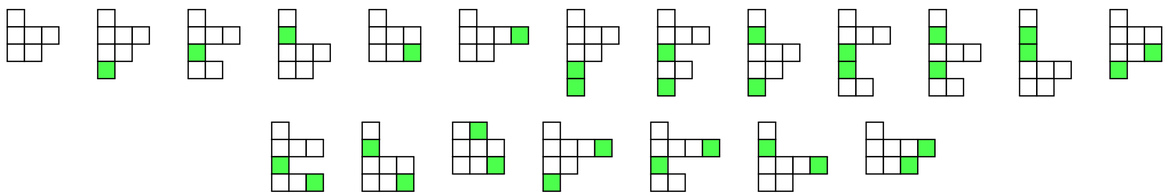
Now to compute $\mathcal{S}_{(1,3,2)/\!\!/(2,1)} \cdot \mathcal{S}_{(2)}$, our result tells us that for every pair of compositions in the above lists (ε, γ) such that (the number of green boxes in ε)+(the number of red boxes in γ)=2, and $\gamma <_c \varepsilon$ we have a term $\mathcal{S}_{\varepsilon/\!\!/\gamma}$ with a sign $(-1)^{\text{number of red boxes}}$. Hence we have the following expansion, suppressing commas and parentheses in compositions for ease of comprehension.

$$\begin{aligned}\mathcal{S}_{132/\!\!/21} \cdot \mathcal{S}_2 =& \mathcal{S}_{132/\!\!/1} - \mathcal{S}_{1132/\!\!/2} - \mathcal{S}_{1132/\!\!/11} - \mathcal{S}_{1312/\!\!/2} - \mathcal{S}_{1321/\!\!/11} - \mathcal{S}_{133/\!\!/2} - \mathcal{S}_{133/\!\!/11} \\ & - \mathcal{S}_{142/\!\!/2} - \mathcal{S}_{142/\!\!/11} + \mathcal{S}_{1133/\!\!/21} + \mathcal{S}_{1142/\!\!/21} + \mathcal{S}_{1322/\!\!/21} + \mathcal{S}_{1331/\!\!/21} \\ & + \mathcal{S}_{1421/\!\!/21} + \mathcal{S}_{143/\!\!/21} + \mathcal{S}_{152/\!\!/21}\end{aligned}$$

Example 7.6. *Let us compute $\mathcal{S}_{(1,3,2)/\!\!/(2,1)} \cdot \mathcal{S}_{(1,1)}$. We first need to compute all compositions γ that can be obtained by removing a horizontal strip of size at most 2 from $\beta = (2,1)$. These compositions correspond to the white boxes in the diagrams below, while the boxes in the darker shade of red correspond to the horizontal strips that are removed from β.*



Next we need to compute all compositions ε such that a vertical strip of size at most 2 can be removed from it so as to obtain α. We list these εs below with the boxes in the lighter shade of green corresponding to vertical strips that need to be removed to obtain α.



Now to compute $\mathcal{S}_{(1,3,2)/\!\!/(2,1)} \cdot \mathcal{S}_{(1,1)}$, our result tells us that for every pair of compositions in the above lists (ε, γ) such (that the number of green boxes in ε)+(the number of red boxes in γ)=2 and $\gamma <_c \varepsilon$, we have a term $\mathcal{S}_{\varepsilon/\!\!/\gamma}$ with a sign $(-1)^{\text{number of red boxes}}$. Hence we have the following expansion, suppressing commas and parentheses in compositions for ease of comprehension.

$$\begin{aligned}\mathcal{S}_{132/\!\!/21} \cdot \mathcal{S}_{11} =& \mathcal{S}_{132/\!\!/1} - \mathcal{S}_{1132/\!\!/2} - \mathcal{S}_{1132/\!\!/11} - \mathcal{S}_{1312/\!\!/2} - \mathcal{S}_{1321/\!\!/11} - \mathcal{S}_{133/\!\!/2} - \mathcal{S}_{133/\!\!/11} \\ &- \mathcal{S}_{142/\!\!/2} - \mathcal{S}_{142/\!\!/11} + \mathcal{S}_{13121/\!\!/21} + \mathcal{S}_{11321/\!\!/21} + \mathcal{S}_{11132/\!\!/21} + \mathcal{S}_{1331/\!\!/21} \\ &+ \mathcal{S}_{1133/\!\!/21} + \mathcal{S}_{233/\!\!/21} + \mathcal{S}_{1421/\!\!/21} + \mathcal{S}_{1142/\!\!/21} + \mathcal{S}_{143/\!\!/21}\end{aligned}$$

Since skew Schur functions can be written as a sum of skew quasisymmetric Schur functions [4, Lemma 2.23] one might ask whether we can recover the Pieri rules for skew shapes of Assaf and McNamara by expanding a skew Schur function as a sum of skew quasisymmetric Schur functions, applying our quasisymmetric skew Pieri rules and then collecting suitable terms. However, a much simpler proof exists.

We know that the skew Schur function for partitions λ, μ (where, as usual, $\ell(\lambda) > \ell(\mu)$) can be given by

$$s_{\lambda/\mu} = \mathcal{S}_{\lambda+1^{\ell(\lambda)}/\!\!/\mu+1^{\ell(\lambda)}} \tag{7.15}$$

where $\lambda + 1^{\ell(\lambda)} = (\lambda_1 + 1, \ldots, \lambda_{\ell(\lambda)} + 1)$, and $\mu + 1^{\ell(\lambda)} = (\mu_1 + 1, \ldots, \mu_{\ell(\mu)} + 1, 1^{\ell(\lambda)-\ell(\mu)})$ [3, Section 5.1]. Plus it follows immediately that $s_{(n)} = \mathcal{S}_{(n)}$ and $s_{(1^n)} = \mathcal{S}_{(1^n)}$ by the relationship between Schur functions and quasisymmetric Schur functions [16, Section 5]. Then as a corollary of our skew Pieri rules we recover the skew Pieri rules of Assaf and McNamara as follows.

Corollary 7.7. [1, Theorem 3.2] *Let λ, μ be partitions where $\ell(\lambda) > \ell(\mu)$ and n be a positive integer. Then*

$$s_{\lambda/\mu} \cdot s_{(n)} = \sum_{i+j=n} (-1)^j s_{\lambda^+/\mu^-}$$

where λ^+ is a partition such that the boxes of λ^+ not in λ are i boxes such that no two lie in the same column, and μ^- is a partition such that the boxes of μ not in μ^- are j boxes such that no two lie in the same row.

Similarly,

$$s_{\lambda/\mu} \cdot s_{(1^n)} = \sum_{i+j=n} (-1)^j s_{\lambda^+/\mu^-}$$

where λ^+ is a partition such that the boxes of λ^+ not in λ are i boxes such that no two lie in the same row, and μ^- is a partition such that the boxes of μ not in μ^- are j boxes such that no two lie in the same column.

Proof. For ease of notation, let $N = \ell(\lambda)$. Then consider the product $\mathcal{S}_{\lambda+1^N/\!\!/\mu+1^N} \cdot \mathcal{S}_{(n)}$ (respectively, $\mathcal{S}_{\lambda+1^N/\!\!/\mu+1^N} \cdot \mathcal{S}_{(1^n)}$) where λ, μ are partitions and $\ell(\lambda) > \ell(\mu)$. By the paragraph preceding the corollary, this is equivalent to what we are trying to compute.

First note that if

$$\mathfrak{d}_1\mathfrak{d}_{r_2} \cdots \mathfrak{d}_{r_i}(\alpha') = \lambda + 1^N$$

where $1 < r_2 < \cdots < r_i$ (respectively, $\mathfrak{d}_{r_1} \cdots \mathfrak{d}_{r_{i-q}}\mathfrak{d}_1^q(\alpha') = \lambda + 1^N$ where $q \geq 0$ and $r_1 \geq \cdots \geq r_{i-q} > 1$) and

$$\mathfrak{d}_{\ell_1} \cdots \mathfrak{d}_{\ell_{j-p}}\mathfrak{d}_1^p(\mu + 1^N) = \beta'$$

where $p \geq 0$ and $\ell_1 \geq \cdots \geq \ell_{j-p} > 1$ (respectively, $\mathfrak{d}_1\mathfrak{d}_{\ell_2} \cdots \mathfrak{d}_{\ell_j}(\mu + 1^N) = \beta'$ where $1 < \ell_2 < \cdots < \ell_j$) plus

$$\mathfrak{d}_{r_2} \cdots \mathfrak{d}_{r_i}(\alpha'') = \lambda + 1^N$$

(respectively, $\mathfrak{d}_{r_1} \cdots \mathfrak{d}_{r_{i-q}}\mathfrak{d}_1^{q+1}(\alpha'') = \lambda + 1^N$) and

$$\mathfrak{d}_{\ell_1} \cdots \mathfrak{d}_{\ell_{j-p}}\mathfrak{d}_1^{p+1}(\mu + 1^N) = \beta''$$

(respectively, $\mathfrak{d}_{\ell_2} \cdots \mathfrak{d}_{\ell_j}(\mu + 1^N) = \beta''$) then $\beta' <_c \alpha'$ if and only if $\beta'' <_c \alpha''$. This follows because $\ell(\lambda) > \ell(\mu)$ plus α' and α only differ by $\mathfrak{d}_1$, and β' and β only differ by $\mathfrak{d}_1$. Moreover, $\alpha' /\!\!/ \beta' = \alpha'' /\!\!/ \beta''$. Furthermore, by our skew Pieri rules, the summands $\mathcal{S}_{\alpha'/\!\!/\beta'}$ and $\mathcal{S}_{\alpha''/\!\!/\beta''}$ will be of opposite sign, and thus will cancel since $\alpha' /\!\!/ \beta' = \alpha'' /\!\!/ \beta''$. Consequently, any nonzero summand appearing in the product $\mathcal{S}_{\lambda+1^N/\!\!/\mu+1^N} \cdot \mathcal{S}_{(n)}$ (respectively, $\mathcal{S}_{\lambda+1^N/\!\!/\mu+1^N} \cdot \mathcal{S}_{(1^n)}$) is such that no box can be removed from the first column of $(\lambda + 1^N)^+$ to obtain $\lambda + 1^N$, nor from the first column of $\mu + 1^N$ to obtain $(\mu + 1^N)^-$.

Next observe that we can obtain $\lambda + 1^N$ by removing an i-horizontal (respectively, i-vertical) strip not containing a box in the first column from $(\lambda+1^N)^+$ if and only if $(\lambda+1^N)^+ = \lambda^+ + 1^N$ where λ^+ is a partition such that the boxes of λ^+ not in λ are i boxes such that no two lie in the same column (respectively, row).

Similarly, we can obtain $(\mu + 1^N)^-$ by removing a j-vertical (respectively, j-horizontal) strip not containing a box in the first column from $\mu + 1^N$ if and only if $(\mu + 1^N)^- = \mu^- + 1^N$ where μ^- is a partition such that the boxes of μ not in μ^- are j boxes such that no two lie in the same row (respectively, column). $\square$

7.2. **Noncommutative skew Pieri rules.** It is also natural to ask whether skew Pieri rules exist for the dual counterparts to skew quasisymmetric Schur functions and whether our methods are applicable in order to prove them. To answer this we first need to define these dual counterparts, namely skew noncommutative Schur functions.

Definition 7.8. *Given compositions α, β, the* skew noncommutative Schur function $\mathbf{s}_{\alpha/\beta}$ *is defined implicitly via the equation*

$$\Delta(\mathbf{s}_\alpha) \;=\; \sum_{\beta} \mathbf{s}_{\alpha/\beta} \otimes \mathbf{s}_\beta$$

where the sum ranges over all compositions β.

With this definition and using Equation (5.4) we can observe that

$$\mathcal{S}_\beta \rightharpoonup \mathbf{s}_\alpha = \mathbf{s}_{\alpha/\beta}$$

via a proof almost identical to that of Observation 7.1. We know from [22] that for $n \geq 1$ the coproduct on $\mathbf{s}_{(n)}$ and $\mathbf{s}_{(1^n)}$ is given by

$$\Delta(\mathbf{s}_{(n)}) = \sum_{i+j=n} \mathbf{s}_{(i)} \otimes \mathbf{s}_{(j)}$$

and

$$\Delta(\mathbf{s}_{(1^n)}) = \sum_{i+j=n} \mathbf{s}_{(1^i)} \otimes \mathbf{s}_{(1^j)}.$$

Plus the action of the antipode S on $\mathbf{s}_{(n)}$ and $\mathbf{s}_{(1^n)}$ is given by

$$S(\mathbf{s}_{(j)}) = (-1)^j \mathbf{s}_{(1^j)}$$

$$S(\mathbf{s}_{(1^j)}) = (-1)^j \mathbf{s}_{(j)}.$$

Using all the above in conjunction with the simplified right Pieri rules for noncommutative Schur functions from Section 6 yields our concluding theorem, whose proof is completely analogous to the proof of Theorem 7.3, and hence is omitted. As always if any parts of size 0 arise during computation, then they are ignored.

Theorem 7.9. *Let α, β be compositions and n be a positive integer. Then*

$$\mathbf{s}_{\alpha/\beta} \cdot \mathbf{s}_{(n)} = \sum_{i+j=n} (-1)^j \mathbf{s}_{\alpha^+/\beta^-}$$

where α^+ is a composition such that it can be obtained by adding an i-right horizontal strip to α, and β^- is a composition such that β can be obtained by adding a j-right vertical strip to it.

Similarly,

$$\mathbf{s}_{\alpha/\beta} \cdot \mathbf{s}_{(1^n)} = \sum_{i+j=n} (-1)^j \mathbf{s}_{\alpha^+/\beta^-}$$

where α^+ is a composition such that it can be obtained by adding an i-right vertical strip to α, and β^- is a composition such that β can be obtained by adding a j-right horizontal strip to it.

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