QUASISYMMETRIC SCHUR FUNCTIONS

J. HAGLUND, K. LUOTO, S. MASON, AND S. VAN WILLIGENBURG

Abstract. We introduce a new basis for quasisymmetric functions, which arise from a specialization of nonsymmetric Macdonald polynomials to standard bases, also known as Demazure atoms. Our new basis is called the basis of quasisymmetric Schur functions, since the basis elements refine Schur functions in a natural way. We derive expansions for quasisymmetric Schur functions in terms of monomial and fundamental quasisymmetric functions, which give rise to quasisymmetric refinements of Kostka numbers and standard (reverse) tableaux. From here we derive a Pieri rule for quasisymmetric Schur functions that naturally refines the Pieri rule for Schur functions. After surveying combinatorial formulas for Macdonald polynomials, including an expansion of Macdonald polynomials into fundamental quasisymmetric functions, we show how some of our results can be extended to include the t parameter from Hall-Littlewood theory.

Contents

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E10, 33D52.

Key words and phrases. compositions, Kostka coefficients, nonsymmetric Macdonald polynomials, Pieri rule, quasisymmetric function, Schur function, tableaux.

The first author was supported in part by NSF grants DMS 0553619 and DMS 0901467. The third author was supported in part by NSF postdoctoral research fellowship DMS 0603351. The second and fourth authors were supported in part by the National Sciences and Engineering Research Council of Canada. The authors would like to thank the Banff International Research Station and the Centre de Recherches Mathématiques, where some of the research took place.

1. Introduction

Macdonald polynomials were originally introduced in 1988 [24, 25], as a solution to a problem involving Selberg's integral posed by Kadell [20]. They are q, t analogues of symmetric functions such that setting $q = t = 0$ in the Macdonald polynomial $P_\lambda(X; q, t)$, for λ a partition, yields the Schur function s_λ. Since their introduction they have arisen in further mathematical areas such as representation theory and quantum computation. For example, Cherednik [9] showed that nonsymmetric Macdonald polynomials are connected to the representation theory of double affine Hecke algebras, and setting $q = t^\alpha$, dividing by a power of $1 - t$ and letting $t \to 1$ yields Jack polynomials, which model bosonic variants of single component abelian and nonabelian fractional quantum Hall states [4]. The aforementioned nonsymmetric Macdonald polynomials, $E'_\alpha(X; q, t)$ where α is a weak composition, are a nonsymmetric refinement of the $P_\lambda(X; q, t)$. Setting $q = t = 0$ in an identity of Macdonald and Marshall expressing $P_\lambda(X; q, t)$ as a linear combination of modified versions of the E''s (see Section 7) implies that Schur functions can be decomposed into nonsymmetric functions $\mathcal{A}_\gamma$ for γ a weak composition. These functions were first studied in [22], where they were termed standard bases, however, to avoid confusion with other objects termed standard bases, we refer to them here as Demazure atoms since they decompose Demazure characters into their smallest parts. The definition we use also differs from that in [22] as our definition not only is arguably simpler than the one appearing there, but also is upward compatible with the new combinatorics appearing in the combinatorial formulae for Type A symmetric and nonsymmetric Macdonald polynomials [14, 15]. The equivalence of these two definitions is established in [31]. It should be stressed that Demazure *atoms* should not be confused with Demazure *characters*, which involve the combinatorial tool of crystal graphs. However, certain linear combinations of Demazure atoms form Demazure characters, and their relationship to each other and to nonsymmetric Macdonald polynomials can be found in [18, 19, 31].

Interpolating between symmetric functions and nonsymmetric functions are quasisymmetric functions. These were introduced as a source of generating functions for P-partitions [12] but since then, like Macdonald polynomials, they have impacted, and deepened the understanding of, other areas. For example in category theory they are a terminal object in the category of graded Hopf algebras equipped with a zeta function [1]; in lattice theory they induce Pieri rules analogous to those found in the algebra of symmetric functions [3]; in discrete geometry the quasisymmetric functions known as peak functions were found to be dual to the **cd**-index [6]; in symmetric function theory they identify equal ribbon Schur functions [8]; in representation theory they arise as characters of a degenerate quantum group [17, 21].

Therefore, a natural object to seek is a quasisymmetric function that interpolates between the nonsymmetric Schur functions, known as Demazure atoms, and Schur functions. Furthermore, since Demazure atoms exhibit many Schur function properties [30], a natural question to ask is which properties of Schur functions are exhibited by *quasisymmetric* Schur functions? In this paper we define quasisymmetric Schur functions and show they naturally lift well known combinatorial properties of symmetric functions indexed by partitions, to combinatorial properties of quasisymmetric functions indexed by compositions. More precisely, we show the following.

(1) The expression for Schur functions in terms of monomial symmetric functions refines to an expression for quasisymmetric Schur functions in terms of monomial quasisymmetric functions, giving rise to quasisymmetric Kostka coefficients.
(2) The expression for Schur functions in terms of fundamental quasisymmetric functions naturally refines to quasisymmetric Schur functions.
(3) The Pieri rule for multiplying a Schur function indexed by a row or a column with a generic Schur function refines to a rule for multiplying a quasisymmetric Schur function indexed by a row or a column with a generic quasisymmetric Schur function. Moreover, this rule is a new example of the construction studied in [3, 6], where the underlying poset involved is a poset of compositions.

The existence of such results introduces a plethora of research avenues to pursue concerning the quasisymmetric analogues of other symmetric function properties. For example, the latter result naturally raises the question of whether the Littlewood-Richardson rule for multiplying two generic Schur functions can be refined to quasisymmetric Schur functions. Such a refinement may not be easy to find as the classical Littlewood-Richardson rule produces nonnegative structure constants, whereas multiplying together two quasisymmetric Schur functions sometimes results in negative structure constants. The smallest such example exists at $n = 6$. However, in the sequel to this paper we successfully refine the Littlewood-Richardson rule by multiplying a generic Schur function and quasisymmetric Schur function [16].

More precisely, this paper is structured as follows. In Sections 2, 3, 4 we review the necessary, and sometimes nonstandard, background material regarding quasisymmetric and symmetric functions, and Demazure atoms. In Section 5 we introduce quasisymmetric Schur functions, and show in Proposition 5.5 that they form a $\mathbb{Z}$-basis for the algebra of quasisymmetric functions. Section 6 derives expansions for quasisymmetric Schur functions in terms of monomial and fundamental quasisymmetric functions in Theorems 6.1 and 6.2. In Section 6.2 we reinterpret these expansions as transition matrices to facilitate the expression of arbitrary quasisymmetric functions in terms of the quasisymmetric Schur function basis. Our main result of this section, however, is Theorem 6.3 in which we give a Pieri rule for quasisymmetric Schur functions. Finally, in Section 7 we show how to insert the parameter t into our model, defining new quasisymmetric functions which decompose Hall-Littlewood polynomials; contrast this result with an alternate decomposition obtained by letting $q = 0$ in a formula for Macdonald symmetric functions as a sum of Gessel's fundamental quasisymmetric functions occurring in [13], and discuss further avenues to pursue.

2. Quasisymmetric and symmetric functions

2.1. Compositions and partitions.

A *weak composition* $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ of n, often denoted $\gamma \vDash n$, is a list of nonnegative integers whose sum is n. We call the γ_i the *parts* of γ and n the *size* of γ, denoted $|\gamma|$. If γ_i appears n_i times we abbreviate this subsequence to $\gamma_i^{n_i}$. The *foundation* of γ is the set

$$\mathcal{F}o(\gamma) = \{i \mid \gamma_i > 0\}.$$

If every part of γ is positive then we call γ a *composition* and call $k := \ell(\gamma)$ the *length* of γ. Observe that every weak composition collapses to a composition $\alpha(\gamma)$, which is obtained by removing all $\gamma_i = 0$ from γ. If every part of γ is positive and satisfies $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$ we call γ a *partition* of n, denoted $\gamma \vdash n$. Observe that every weak composition γ determines a partition $\lambda(\gamma)$, which is obtained by reordering the positive parts of γ in weakly decreasing order.

Example.

$$\gamma = (3, 2, 0, 4, 2, 0),\ \mathcal{F}o = \{1, 2, 4, 5\},\ \alpha(\gamma) = (3, 2, 4, 2),\ \lambda(\gamma) = (4, 3, 2, 2).$$

Restricting our attention to compositions, there exist three partial orders in which we will be interested. First, given compositions α, β we say that α is a *coarsening* of β (or β is a *refinement* of α), denoted $\alpha \succeq \beta$, if we can obtain α by adding together adjacent parts of β, for example, $(3, 2, 4, 2) \succeq (3, 1, 1, 1, 2, 1, 2)$. Second, we say that α is *lexicographically greater* than β, denoted $\alpha >_{lex} \beta$, if $\alpha = (\alpha_1, \alpha_2, \ldots) \neq (\beta_1, \beta_2, \ldots) = \beta$ and the first i for which $\alpha_i \neq \beta_i$ satisfies $\alpha_i > \beta_i$. Third, we say $\alpha \blacktriangleright \beta$ if $\lambda(\alpha) >_{lex} \lambda(\beta)$ or $\lambda(\alpha) = \lambda(\beta)$ and $\alpha >_{lex} \beta$. For example, when $n = 4$ we have

$$(4) \blacktriangleright (3, 1) \blacktriangleright (1, 3) \blacktriangleright (2, 2) \blacktriangleright (2, 1, 1) \blacktriangleright (1, 2, 1) \blacktriangleright (1, 1, 2) \blacktriangleright (1, 1, 1, 1).$$

Additionally, to any composition $\beta = (\beta_1, \ldots, \beta_k)$ there is another closely related composition $\beta^* = (\beta_k, \ldots, \beta_1)$, called the *reversal* of β. Lastly, any composition $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \vDash n$ corresponds to a subset $S(\beta) \subseteq [n-1] = \{1, \ldots, n-1\}$ where

$$S(\beta) = \{\beta_1, \beta_1 + \beta_2, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{k-1}\}.$$

Similarly, any subset $S = \{i_1, i_2, \ldots, i_{k-1}\} \subseteq [n-1]$ corresponds to a composition $\beta(S) \vDash n$ where

$$\beta(S) = (i_1, i_2 - i_1, i_3 - i_2, \ldots, n - i_{k-1}).$$

2.2. **Quasisymmetric and symmetric function preliminaries.** A *quasisymmetric* function is a bounded degree formal power series $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for all k and $i_1 < i_2 < \cdots < i_k$ the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ is equal to the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ for all compositions $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. The set of all quasisymmetric functions forms a graded algebra $\mathcal{Q} = \mathcal{Q}_0 \oplus \mathcal{Q}_1 \cdots$.

Two natural bases for quasisymmetric functions are the monomial basis $\{M_\alpha\}$ and the fundamental basis $\{F_\alpha\}$ indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. The *monomial* basis consists of $M_0 = 1$ and all formal power series

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The *fundamental* basis consists of $F_0 = 1$ and all formal power series

$$F_\alpha = \sum_{\alpha \succeq \beta} M_\beta.$$

Furthermore, $\mathcal{Q}_n = \mathrm{span}_{\mathbb{Q}}\{\mathrm{M}_\alpha | \alpha \vDash \mathrm{n}\} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{F}_\alpha | \alpha \vDash \mathrm{n}\}$. We define the algebra of symmetric functions $\Lambda = \Lambda_0 \oplus \Lambda_1 \cdots$ as the subalgebra of $\mathcal{Q}$ spanned by the *monomial symmetric functions* $m_0 = 1$ and all formal power series

$$m_\lambda = \sum_{\alpha\,:\,\lambda(\alpha) = \lambda} M_\alpha, \quad \lambda \vdash n > 0.$$

Moreover, we have $\Lambda_n = \Lambda \cap \mathcal{Q}_n$.

Example.

$$F_{(1,2)} = M_{(1,2)} + M_{(1,1,1)}, \quad m_{(2,1)} = M_{(2,1)} + M_{(1,2)}.$$

Perhaps the most well known basis for Λ is the basis of Schur functions, $\{s_\lambda\}$, whose definition we devote the next section to.

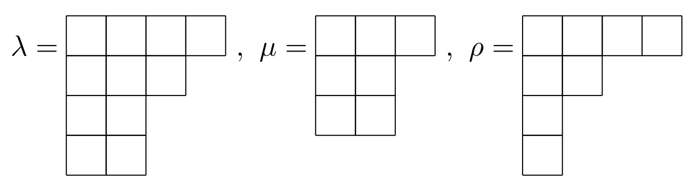
3. Schur functions

3.1. **Diagrams and reversetableaux.** Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, its corresponding (*Ferrers*) *diagram* is the array of left justified boxes or *cells* with λ_i cells in the i-th row *from the top*. We abuse notation by using λ to refer to both the partition λ and its corresponding diagram. We also describe cells by their row and column indices. Given two diagrams λ, μ, we say $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\mu)$, and if $\mu \subseteq \lambda$ then the *skew diagram* λ/μ is the array of cells contained in λ but not contained in μ. In terms of row and column indices

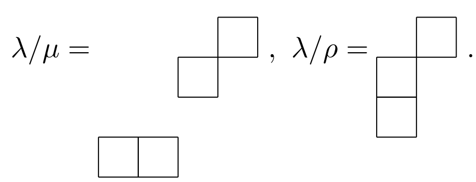
$$\lambda/\mu = \{(i,j) \mid (i,j) \in \lambda, (i,j) \notin \mu\}.$$

The number of cells in λ/μ is called the *size* and is denoted $|\lambda/\mu|$. Two types of skew diagram that will be of particular interest to us later are horizontal strips and vertical strips. We say a skew diagram is a *horizontal strip* if no two cells lie in the same column, and is a *vertical strip* if no two cells lie in the same row.

Example. *If*



then λ/μ is a horizontal strip and λ/ρ is a vertical strip:



Reversetableaux are formed from skew diagrams in the following way. Given a skew diagram λ/μ we define a *reversetableau* (or *reverse semistandard Young tableau*), T, of *shape* λ/μ to be a filling of the cells with positive integers such that

(1) the entries in the rows of T weakly decrease when read from left to right,
(2) the entries in the columns of T strictly decrease when read from top to bottom.

If $|\lambda/\mu| = n$ and the entries are such that each of $1, \ldots, n$ appears once and only once, then we call T a *standard reversetableau*. Classically, given a standard reversetableau, T, its *descent set* $D(T)$ is the set of all i such that $i+1$ appears in a higher row. However, by the definition of reversetableau it follows that $i+1$ can only appear

- strictly above and weakly right
- weakly below and strictly left

of i. Hence

$$D(T) = \text{the set of all } i \text{ that do } \textit{not} \text{ have } i+1 \text{ appearing strictly left of } i.$$

Example. *In the following reversetableau, to compute $D(T)$ note that 3 is not strictly left of 2.*

$$T = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad , \qquad D(T) = \{2\}.$$

The *weight* of a reversetableau, T, is the weak composition $w(T) = (w_1(T), w_2(T), \ldots)$ where $w_i(T) =$ the number of times i appears in T. The monomial associated with a reversetableau, T, is

$$x^T = x_1^{w_1(T)} x_2^{w_2(T)} \cdots .$$

For example, the monomial associated with any standard reversetableau, T, with n cells is $x^T = x_1 x_2 \cdots x_n$. We are now ready to define Schur functions.

3.2. **Schur function preliminaries.** There are many ways to define Schur functions, and we begin by defining them as generating functions for reversetableaux. For further details we refer the interested reader to [32, Chapter 7]. Let λ be a partition. Then the *Schur function* s_λ is

$$s_\lambda = \sum_T x^T$$

where the sum is over all reversetableaux, T, of shape λ. We now recall two further classical descriptions, which we include in order to compare with their quasisymmetric counterparts later. The first describes Schur functions in terms of monomial symmetric functions.

Proposition 3.1. *Let λ, μ be partitions. Then the Schur function s_λ is*

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

where $K_{\lambda\mu} =$ the number of reversetableaux of shape λ and weight μ^*.

The second description is in terms of fundamental quasisymmetric functions.

Proposition 3.2. *Let λ be a partition. Then the Schur function s_λ is*

$$s_\lambda = \sum_T F_{\beta(D(T))}$$

where the sum is over all standard reversetableaux, T, of shape λ. Equivalently,

$$s_\lambda = \sum_\beta d_{\lambda\beta} F_\beta$$

where $d_{\lambda\beta} =$ the number of standard reversetableaux, T, of shape λ such that $\beta(D(T)) = \beta$.

Example. *We compute*

$$\begin{aligned} s_{(2,1)} &= m_{(2,1)} + 2m_{(1,1,1)} \\ &= F_{(2,1)} + F_{(1,2)} \end{aligned}$$

from the reversetableaux

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \ , \ \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \ , \ \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \ .$$

To close this section we recall two classical products of Schur functions, collectively known as the Pieri rule, which we will later refine to a quasisymmetric setting.

Proposition 3.3 (Pieri rule for Schur functions)**.** *Let λ be a partition. Then*

$$s_{(n)}s_\lambda = \sum_\mu s_\mu$$

where the sum is taken over all partitions μ such that

(1) *$\delta = \mu/\lambda$ is a horizontal strip,*
(2) *$|\delta| = n$.*

Also,

$$s_{(1^n)}s_\lambda = \sum_\mu s_\mu$$

where the sum is taken over all partitions μ such that

(1) *$\epsilon = \mu/\lambda$ is a vertical strip,*
(2) *$|\epsilon| = n$.*

4. Demazure atoms

4.1. **Compositions and diagrams.** In this section we define an analogue of reversetableaux that arise naturally in the theory of nonsymmetric Macdonald polynomials. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a weak composition. Then its corresponding *augmented diagram*, $\widehat{dg}(\gamma)$, is the array of left justified cells with $\gamma_i + 1$ cells in the i-th row from the top. Furthermore, the cells of the leftmost column are filled with the integers $1, \ldots, n$ in increasing order from top to bottom, and this 0-th column is referred to as the *basement*.

Example.

$$\widehat{dg}(1,0,2) = \begin{array}{|c|c|c} \hline 1 & & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c}{} & \\ \hline 3 & & \multicolumn{1}{c|}{} \\ \hline \end{array} \quad .$$

Again we refer to cells by their row and column indices, with the basement being column 0. As with diagrams and reversetableaux we fill the remaining cells of an augmented diagram subject to certain conditions and create semistandard augmented fillings.

Given an augmented diagram $\widehat{dg}(\gamma)$, an *augmented filling*, σ, is an assignment of positive integer entries to the unfilled cells of $\widehat{dg}(\gamma)$. A pair of cells $a = (i, j)$ and $b = (i', j')$ are *attacking* if either $j = j'$ or ($j = j' + 1$ and $i > i'$). An augmented filling σ is *non-attacking* if $\sigma(a) \neq \sigma(b)$ whenever a and b are attacking cells.

Then three cells $\{a, b, c\} \in \widehat{dg}(\gamma)$ are called a *type A triple* if they are situated as follows

$$\begin{array}{cc} \boxed{c} & \boxed{a} \\ \\ & \boxed{b} \end{array}$$

where a and b are in the same column, possibly with cells between them, c is immediately left of a, and the length of the row containing a and c is greater than or equal to the length of the row containing b. We say that the cells a, b, c form a *type A inversion triple* if their entries, ordered from smallest to largest, form a counter-clockwise orientation. If two entries are equal, then the entry which appears first when the entries are read top to bottom, right to left, is considered smallest.

Similarly, three cells $\{a, b, c\} \in \widehat{dg}(\gamma)$ are a *type B triple* if they are situated as shown

$$\begin{array}{|c|cc} \hline a & \multicolumn{1}{|c}{} \\ \hline \\ \\ \hline b & \multicolumn{1}{|c|}{c} \\ \hline \end{array}$$

where a and b are in the same column, possibly the basement or with cells between them, c is immediately right of b, and the length of the row containing b and c is strictly greater than the length of the row containing a. We say that the cells a, b, c form a *type B inversion triple* if their entries, when ordered from smallest to largest, form a clockwise orientation. Again, if two entries are equal, then the entry which appears first when the entries are read top to bottom, right to left, is considered smallest.

Define a *semistandard augmented filling (SSAF)* of *shape* γ to be a non-attacking augmented filling of $\widehat{dg}(\gamma)$ such that the entries in each row are weakly decreasing when read from left to right (termed *no descents*), and every triple is an inversion triple of type A or B.

Remark. *Note that in [30] it was shown that the triple and no descent conditions guarantee the augmented filling will be non-attacking. However, we include the extra condition for use in later proofs.*

The weight of a SSAF, F, is the weak composition $w(F) = (w_1(F), w_2(F), \ldots)$ where $w_i(F) =$ (the number of times i appears in F) $-1 =$ the number of times i appears in F excluding entries in the basement. Again, the monomial associated with a SSAF, F, is

$$x^F = x_1^{w_1(F)} x_2^{w_2(F)} \cdots .$$

Example.

$$F = \begin{array}{|c|c|c|} \cline{1-2} 1 & \mathbf{1} & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \hline 3 & \mathbf{3} & \mathbf{3} \\ \hline \end{array}, \quad x^F = x_1 x_3^2.$$

A SSAF, F, of shape γ is a *standard augmented filling (SAF)* if for $|\gamma| = n$ we have $x^F = \prod_{i=1}^n x_i$, and F has *descent set*

$$\mathcal{D}(F) := \text{the set of all } i \text{ that do not have } i+1 \text{ appearing strictly left of } i$$
$$\text{(excluding entries in the basement)}.$$

Similarly, compositions give rise to composition tableaux. Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, its corresponding *composition diagram*, also denoted α, is the array of left justified cells with α_i cells in the i-th row from the top, and its cells are described by row and column indices.

Definition 4.1. *Given a composition diagram $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ with largest part m, we define a* composition tableau (ComT), *T, of* shape *α to be a filling of the cells of α with positive integers such that*

(1) *the entries in the rows of T weakly decrease when read from left to right,*

(2) *the entries in the leftmost column of T strictly increase when read from top to bottom.*
(3) Triple rule: *Supplement T by adding enough cells with zero valued entries to the end of each row so that the resulting supplemented tableau, $\hat{T}$, is of rectangular shape $\ell \times m$. Then for $1 \le i < j \le \ell, 2 \le k \le m$*

$$\left(\hat{T}(j,k) \neq 0 \text{ and } \hat{T}(j,k) \geq \hat{T}(i,k)\right) \Rightarrow \hat{T}(j,k) > \hat{T}(i,k-1).$$

In exact analogy with reversetableaux, the weight of a ComT, T, is the weak composition $w(T) = (w_1(T), w_2(T), \ldots)$ where $w_i(T) =$ the number of times i appears in T. The monomial associated with a ComT, T, is

$$x^T = x_1^{w_1(T)} x_2^{w_2(T)} \cdots .$$

Also, a ComT with n cells is *standard* if $x^T = \prod_{i=1}^n x_i$, and has *descent set* $\mathcal{D}(T) :=$ the set of all i that do not have $i+1$ appearing strictly left of i.

Example. *We use a standard composition tableau (ComT) to illustrate our definitions.*

$$T = \begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 \\ \cline{1-1} 8 & 7 & 2 \\ \cline{1-3}\end{array}\ ,\ \hat{T} = \begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 & 0 & 0 & 0 \\ \hline 8 & 7 & 2 & 0 \\ \hline\end{array}\ ,\ x^T = x_1x_2x_3x_4x_5x_6x_7x_8\ ,\ \mathcal{D}(T) = \{2,5,6\}.$$

It transpires that SSAFs and ComTs are closely related, and this relationship will be vital in simplifying subsequent proofs.

Lemma 4.2. *There exists a natural weight preserving bijection between the set of ComTs of shape α and the set of SSAFs of shape γ where $\alpha(\gamma) = \alpha$.*

Example. *The following pair consisting of a ComT and SSAF illustrates the natural bijection between them.*

$$\begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 \\ \cline{1-1} 8 & 7 & 2 \\ \cline{1-3}\end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|c|}\hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \hline 5 & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{1} \\ \hline 6 & \mathbf{6} \\ \cline{1-2} 7 \\ \hline 8 & \mathbf{8} & \mathbf{7} & \mathbf{2} \\ \cline{1-4}\end{array} \quad .$$

Proof. The mapping that is claimed to be a bijection is clear: given a SSAF, eliminate the basement and any zero parts. For the inverse mapping, given a ComT, let c be the largest element in the first column. Allocate a bare basement with c rows. Place each row of the original ComT to the immediate right of the basement entry that matches the largest row entry. We need to show that

(1) the resulting potential ComT satisfies the three rules above,
(2) taking a ComT and applying the inverse operation results in a SSAF.

For the first direction, assume that F is a SSAF of shape γ, and that σ is the resulting potential ComT of shape $\mu = \alpha(\gamma)$ with ℓ rows and m columns and maximum entry n. We first note that σ satisfies Rule 1. Showing that Rule 2 is satisfied is equivalent to showing that column 1 of F (the column adjacent to the basement) is strictly increasing top to bottom. Since F is non-attacking, we have that all the entries in each column (in particular, column 1) are distinct. Note that an entry in column 1 of the SSAF F having value i resides in the cell $(i,1)$. This follows immediately since F has no descents and is non-attacking. Thus it follows that the entries in column 1 of F are strictly increasing, and so σ satisfies Rule 2.

To show that σ satisfies Rule 3, suppose to the contrary that there exists a triple of indices $\hat{\imath}, \hat{\jmath}, k$ such that $1 \le \hat{\imath} < \hat{\jmath} \le \ell$, $2 \le k \le m$ such that $\widehat{\sigma}(\hat{\jmath},k) \neq 0$, $\widehat{\sigma}(\hat{\jmath},k) \ge \widehat{\sigma}(\hat{\imath},k)$, and $\widehat{\sigma}(\hat{\jmath},k) \le \widehat{\sigma}(\hat{\imath},k-1)$. Without loss of generality, we may assume that k is minimal over all such triples of indices. Let i, j be the rows of F corresponding to the respective rows $\hat{\imath}, \hat{\jmath}$ of σ. Note that $i < j$. We consider two cases.

Case: $\gamma_i \ge \gamma_j$. In this case, the cell (i,k) of F is nonempty, and by supposition $F(i,k) < F(j,k) \le F(i,k-1)$. But then the cells $(i,k),(j,k)$, $(i,k-1)$ form a non-inversion type A triple, contradicting the given that F is a SSAF.

Case: $\gamma_i < \gamma_j$. Since F has no descents, $F(j,k) \le F(j,k-1)$, and by supposition $F(j,k) \le F(i,k-1)$. The cells $(i,k-1),(j,k-1),(j,k)$ form a type B triple, which must be an inversion triple, so it follows that $F(i,k-1) > F(j,k-1)$. Since $i < j$, and since the first column of F is strictly increasing, $F(i,1) < F(j,1)$. Hence there must exist some k', $1 \le k' < k-1$ such that $F(i,k') < F(j,k')$ and $F(i,k'+1) > F(j,k'+1)$. Since F has no descents, $F(i,k'+1) \le F(i,k')$ and $F(j,k'+1) \le F(j,k')$, which also implies $F(j,k'+1) \le F(i,k')$. However, then we see that the cells $(i,k'),(j,k'),(j,k'+1)$ form a non-inversion type B triple, contradicting the given that F is a SSAF.

Thus in both cases we have a contradiction. It follows that there is no such triple of indices $\hat{\imath}, \hat{\jmath}, k$, hence σ satisfies Rule 3 as well as Rules 1 and 2, and hence is a ComT.

For the second direction, assume that σ is a ComT, say of shape μ, and let F be obtained by the inverse mapping described above, which must necessarily be of some shape γ, with $\mu = \alpha(\gamma)$. Since σ satisfies Rule 1, F has no descents. Since σ satisfies Rule 2, the first column of F is strictly increasing top to bottom, and in fact by construction, if cell $(i,1)$ of F is not empty, then $F(i,1) = i$. In conjunction with this, since σ satisfies Rule 3, we have that F must be non-attacking.

Suppose the cells $(i,k),(j,k),(i,k-1)$, $i < j$ form a type A triple in F. If $k = 1$, then $F(i,k) = F(i,k-1) = i < j = F(j,k)$, and so the triple is an inversion triple. Otherwise $k \ge 2$, and since σ satisfies Rules 3 and 1, we have that either $F(j,k) < F(i,k) \le F(i,k-1)$ or $F(i,k) \le F(i,k-1) < F(j,k)$, and in both cases the triple is an inversion triple. Thus all type A triples are inversion triples.

Suppose the cells $(i,k),(j,k),(j,k+1)$, $i < j$ form a type B triple in F. Then $\gamma_i < \gamma_j$. If $k = 0$, then $F(i,k) = i < j = F(j,k+1) = F(j,k)$, and so the triple is an inversion triple. Otherwise $k \ge 1$. Suppose the triple is not an inversion triple. This can only happen if $F(j,k+1) \le F(i,k) < F(j,k)$. Let $\hat{\imath}, \hat{\jmath}$ be the rows of σ corresponding respectively to the rows i, j of F. Then $\widehat{\sigma}(\hat{\jmath},k+1) \le \widehat{\sigma}(\hat{\imath},k)$, and Rule 3 then implies that $\widehat{\sigma}(\hat{\imath},k+1) > \widehat{\sigma}(\hat{\jmath},k+1)$. Since $\gamma_i < \gamma_j$, we have $\widehat{\sigma}(\hat{\imath},\gamma_j) = 0 < \widehat{\sigma}(\hat{\jmath},\gamma_j)$. There must then exist some k', $k+1 \le k' < \gamma_j$ such that $\widehat{\sigma}(\hat{\imath},k') > \widehat{\sigma}(\hat{\jmath},k')$ and $\widehat{\sigma}(\hat{\imath},k'+1) < \widehat{\sigma}(\hat{\jmath},k'+1)$. But then we have $\widehat{\sigma}(\hat{\imath},k'+1) < \widehat{\sigma}(\hat{\jmath},k'+1) \le \widehat{\sigma}(\hat{\jmath},k') < \widehat{\sigma}(\hat{\imath},k')$, violating Rule 3. Thus the triple must be an inversion triple, and so all type B triples are inversion triples.

We have that F is a non-attacking augmented filling with no descents and in which all type A and type B triples are inversion triples, i.e. F is a SSAF. ☐

4.2. **Demazure atom preliminaries.** Demazure atoms are formal power series $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$, which can be defined combinatorially as follows.

Definition 4.3. *Let γ be a weak composition. Then the* Demazure atom, $\mathcal{A}_\gamma$ *is*

$$\mathcal{A}_\gamma = \sum_F x^F$$

where the sum is over all SSAFs, F, of shape γ. Equivalently,

$$\mathcal{A}_\gamma = \sum_F x^F$$

where the sum is over all ComTs, F, of shape $\alpha(\gamma)$ and first column entries $\mathcal{F}o(\gamma)$.

Note the second definition follows immediately from Lemma 4.2.

Example. *We compute*

$$\mathcal{A}_{(1,0,2)} = x_1x_2x_3 + x_1x_3^2$$

from the SSAFs

$$\begin{array}{|c|c|c} \hline 1 & \mathbf{1} & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & \mathbf{3} & \multicolumn{1}{c|}{\mathbf{2}} \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c} \hline 1 & \mathbf{1} & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & \mathbf{3} & \multicolumn{1}{c|}{\mathbf{3}} \\ \hline \end{array}$$

or, equivalently, the ComTs

$$\begin{array}{|c|c} \hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad , \quad \begin{array}{|c|c} \hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \; .$$

It transpires that Demazure atoms can be used to describe Schur functions [22, 30].

Proposition 4.4. *Let λ be a partition. Then the Schur function is*

$$s_\lambda = \sum_{\gamma \,:\, \lambda(\gamma)=\lambda} \mathcal{A}_\gamma$$

where the sum is over all weak compositions γ.

4.3. **Bijection between reversetableaux and SSAFs.** We conclude this section by recalling the bijection ρ^{-1} from reversetableaux to SSAFs [30], which we describe algorithmically.

Given a reversetableau, T, we create a $SSAF$, $\rho^{-1}(T) = F$, as follows.

(1) If the maximum entry in T is n then allocate a basement with n rows.
(2) Taking the entries in T in the first column from top to bottom, place them in column $k = 1$ to the right of the basement in the uppermost or highest row i of F in which cell (i, k) of F is empty (that is, not yet filled from some earlier column entry of T)
   - such that the cell $(i, k-1)$ to the immediate left is filled, possibly a basement cell if $k = 1$, and

- ∘ such that the placement results in no descent.

(3) Repeat with the entries in T in the column k, from top to bottom, placing them in the column k to the right of the basement for $k = 2, 3, \ldots$.

Eliminating the basement and zero parts from $\rho^{-1}(T)$ yields a bijection between reversetableaux and ComTs, which we also refer to as ρ^{-1}.

Example. *If* $T = \begin{array}{|c|c|c|c|}\hline 8 & 7 & 3 & 1 \\ \hline 6 & 4 & 2 & \\ \cline{1-3} 5 & & & \\ \cline{1-1}\end{array}$ *then*

$$\rho^{-1}(T) \quad = \quad \begin{array}{|c|cccc} \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \\ 5 & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{1} \\ 6 & \mathbf{6} & & & \\ 7 & & & & \\ 8 & \mathbf{8} & \mathbf{7} & \mathbf{2} & \\ \hline \end{array} \quad \equiv \quad \begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 6 & & & \\ \hline 8 & 7 & 2 & \\ \hline \end{array}\ .$$

5. Quasisymmetric Schur functions

We now define our main objects of study and derive some elementary properties about them.

Definition 5.1. *Let* α *be a composition. Then the* quasisymmetric Schur function *is*

$$\mathcal{S}_\alpha = \sum_{\gamma\,:\,\alpha(\gamma)=\alpha} \mathcal{A}_\gamma$$

where the sum is over all weak compositions γ.

Example. *Restricting ourselves to three variables we compute*

$$\begin{array}{rcl} \mathcal{S}_{(1,2)} & = & \mathcal{A}_{(1,2,0)} + \mathcal{A}_{(1,0,2)} + \mathcal{A}_{(0,1,2)} \\ & = & x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2 \end{array}$$

where the summands arise from all SSAFs of shape $(1,2,0), (1,0,2)$ *and* $(0,1,2)$*:*

$$\begin{array}{|c|cc} \hline 1 & \mathbf{1} & \\ 2 & \mathbf{2} & \mathbf{2} \\ 3 & & \\ \hline \end{array} \qquad \begin{array}{|c|cc} \hline 1 & \mathbf{1} & \\ 2 & & \\ 3 & \mathbf{3} & \mathbf{2} \\ \hline \end{array} \quad \begin{array}{|c|cc} \hline 1 & \mathbf{1} & \\ 2 & & \\ 3 & \mathbf{3} & \mathbf{3} \\ \hline \end{array} \qquad \begin{array}{|c|cc} \hline 1 & & \\ 2 & \mathbf{2} & \\ 3 & \mathbf{3} & \mathbf{3} \\ \hline \end{array}$$

or, equivalently, from ComTs

$$\begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 2 & \\ \hline 3 & 3 \\ \hline \end{array}\ .$$

As illustrated by this example, we shall see later that the functions are indeed quasisymmetric, but first we focus on their connection to Schur functions.

Recall from Proposition 4.4 that the Schur function s_λ decomposes into the sum of all $\mathcal{A}_\gamma$ such that $\lambda(\gamma)=\lambda$. Hence by Definition 5.1 we obtain the decomposition of the Schur function in terms of quasisymmetric functions

$$s_\lambda = \sum_{\alpha\,:\,\lambda(\alpha)=\lambda} \mathcal{S}_\alpha,$$

which immediately evokes the definition of *monomial* symmetric functions in terms of *monomial* quasisymmetric functions

$$m_\lambda = \sum_{\alpha\,:\,\lambda(\alpha)=\lambda} M_\alpha.$$

Thus, the parallel construction justifies the use of the word Schur. We also prove the functions are quasisymmetric by describing quasisymmetric Schur functions in terms of fundamental quasisymmetric functions.

Proposition 5.2. *Let α be a composition. Then*

$$\mathcal{S}_\alpha = \sum_T F_{\beta(D(T))}$$

where the sum is over all standard reversetableaux, T, of shape $\lambda(\alpha)$ that map under ρ^{-1} to a SSAF of shape γ satisfying $\alpha(\gamma)=\alpha$ (or, equivalently, under ρ^{-1} to a ComT of shape α).

Proof. To prove this we need to show

(1) for each T satisfying the conditions stated, $F_{\beta(D(T))}$ is a summand of $\mathcal{S}_\alpha$ appearing exactly once,
(2) the coefficient of each monomial appearing in $\mathcal{S}_\alpha$ is equal to the sum of its coefficients in each of the $F_{\beta(D(T))}$s in which it appears.

To show the first point note that $F_{\beta(D(T))}$ is a sum of monomials that arise from reversetableaux, which standardize to T. Furthermore, any reversetableau $\tilde{T}$ that standardizes to T, denoted $std(\tilde{T})=T$, maps under ρ^{-1} to a SSAF $\rho^{-1}(\tilde{T})$ that standardizes to $\rho^{-1}(T)$ [30]. That is, if we say for a SSAF, F, that its standardization is $\rho^{-1}(std(\rho(F)))$ then $\rho^{-1}(T)=\rho^{-1}(std(\rho(\rho^{-1}(\tilde{T}))))$, where ρ is the inverse of ρ^{-1}. Thus, if T is of shape $\lambda(\alpha)$ and $\rho^{-1}(T)$ is of shape γ such that $\alpha(\gamma)=\alpha$, then $F_{\beta(D(T))}$ is a summand of $\mathcal{S}_\alpha$ appearing exactly once.

To show the second point, observe if given a SSAF $\rho^{-1}(\tilde{T})$ of shape γ such that $\alpha(\gamma)=\alpha$, which contributes a monomial towards $\mathcal{S}_\alpha$ and also standardizes to $\rho^{-1}(T)$, then under ρ this maps bijectively to $\tilde{T}$ that standardizes to T of shape $\lambda(\alpha)$. Computing $D(T)$ then yields which fundamental quasisymmetric function the monomial belongs to. ☐

A combinatorially more straightforward description in terms of the F_α is given in the next section, and hence we delay giving an example until then. We will now show that, in fact, the set of all quasisymmetric Schur functions forms a basis for $\mathcal{Q}$. Before we do this, we will work towards two lemmas.

For a composition α, let T_α be the unique standard reversetableau of shape $\lambda(\alpha)$ and $\beta(D(T))=\alpha$. To see that T_α exists, construct the left justified array of cells of shape α^*, which has $1,\dots,\alpha_1$ in the bottom row and

$$\alpha_1+\cdots+\alpha_{i-1}+1,\dots,\alpha_1+\cdots+\alpha_i$$

in the i-th row from bottom appearing in decreasing order when read from left to right. Then move every cell as far north as possible to form T_α. To see that T_α is unique, note that the number of descents in T_α is one less than the number of rows in T_α and so all entries in the first column except n must be all i such that $i \in D(T_\alpha)$. This and the fact that T_α must be a reversetableau yield uniqueness.

Example. *If* $\alpha = (1, 3, 2)$ *then we construct* $\begin{array}{|c|c|c|} \hline 6 & 5 & \\ \hline 4 & 3 & 2 \\ \hline 1 & & \\ \hline \end{array}$ *and* $T_\alpha = \begin{array}{|c|c|c|} \hline 6 & 5 & 2 \\ \hline 4 & 3 & \\ \hline 1 & & \\ \hline \end{array}$.

The following lemma is straightforward to verify using the algorithm for ρ^{-1}.

Lemma 5.3. *For a composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \vDash n$*, performing* ρ^{-1} *on* T_α *yields the SSAF with basement* $1, \dots, n$ *and row* α_1 *containing* $1, \dots, \alpha_1$*, row* $\alpha_1 + \alpha_2$ *containing* $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$ *etc. Equivalently, performing* ρ^{-1} *on* T_α *yields the ComT with row* 1 *containing* $1, \dots, \alpha_1$*, row* 2 *containing* $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$ *etc.*

Lemma 5.4. F_α *will always be a summand of* $\mathcal{S}_\alpha$ *with coefficient 1.*

Proof. This follows immediately from the existence and uniqueness of T_α, Proposition 5.2 and Lemma 5.3. □

We are now ready to prove that quasisymmetric Schur functions form a basis for $\mathcal{Q}$.

Proposition 5.5. *The set* $\{\mathcal{S}_\alpha | \alpha \vDash n\}$ *forms a* $\mathbb{Z}$*-basis for* $\mathcal{Q}$*.*

Proof. For a fixed n and $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)}) \vDash n$ consider the summand F_δ appearing in $\mathcal{S}_\alpha$. By Proposition 5.2 it follows that $\lambda(\alpha) \geq_{lex} \lambda(\delta)$ because if not then the first i when $\lambda(\delta)_i > \lambda(\alpha)_i$ will yield a row in any diagram $\lambda(\alpha)$ that cannot be filled to create a standard reversetableau, T, satisfying $D(T) = S(\delta)$. If $\lambda(\alpha) = \lambda(\delta)$ then by Lemma 5.3 and the uniqueness of T_α we know the coefficient of F_δ will be 0 unless $\alpha = \delta$.

Let M be the matrix whose rows and columns are indexed by $\alpha \vDash n$ ordered by $\blacktriangleright$ and entry $M_{\alpha\delta}$ is the coefficient of F_δ in $\mathcal{S}_\alpha$. By the above argument and Lemma 5.4 we have that M is upper unitriangular, and the result follows. □

6. Properties of quasisymmetric Schur functions

A natural question to ask about quasisymmetric Schur functions is how many properties of Schur functions refine to *quasisymmetric* Schur functions? In this regard there are many avenues to pursue. In this section we provide the expansion of a quasisymmetric Schur function in terms of monomial symmetric functions, and a more explicit expression in terms of fundamental quasisymmetric functions. Our main result of this section, however, is to show that quasisymmetric Schur functions exhibit a Pieri rule that naturally refines the original Pieri rule for Schur functions.

To appreciate these quasisymmetric refinements we invite the reader to compare the classical Schur function properties of Propositions 3.1, 3.2 and 3.3 with the quasisymmetric Schur function properties of Theorems 6.1, 6.2 and 6.3, respectively.

Theorem 6.1. *Let* α, β *be compositions. Then*

$$\mathcal{S}_\alpha = \sum_\beta K_{\alpha\beta} M_\beta$$

where $K_{\alpha\beta}$ = the number of SSAFs of shape γ satisfying $\alpha(\gamma) = \alpha$ and weight β (or, equivalently, $K_{\alpha\beta}$ = the number of ComTs of shape α and weight β).

Proof. We know

$$\mathcal{S}_\alpha = \sum_{\gamma\,:\,\alpha(\gamma)=\alpha} \mathcal{A}_\gamma = \sum x^F = \sum_\beta c_{\alpha\beta} M_\beta$$

where the middle sum is over all SSAFs F of shape γ where $\alpha(\gamma) = \alpha$. The leading term of any M_β appearing in the last sum is $x_1^{\beta_1} x_2^{\beta_2} \cdots x_\ell^{\beta_\ell}$, and the number of times it will appear is, by the middle equality, the number of SSAFs of shape γ where $\alpha(\gamma) = \alpha$ and weight β. Hence $c_{\alpha\beta} = K_{\alpha\beta}$ and the result follows. $\square$

Theorem 6.2. *Let α, β be compositions. Then*

$$\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ = the number of SAFs T of shape γ satisfying $\alpha(\gamma) = \alpha$ and $\beta(\mathcal{D}(T)) = \beta$ (or, equivalently, $d_{\alpha\beta}$ = the number of standard ComTs T of shape α and $\beta(\mathcal{D}(T)) = \beta$).

Proof. Since $\mathcal{S}_\alpha = \sum_T F_{\beta(D(T))}$ where the sum is over all standard reversetableaux, T, of shape $\lambda(\alpha)$ that map under ρ^{-1} to a SSAF of shape γ that satisfies $\alpha(\gamma) = \alpha$, and since ρ^{-1} maps the entries appearing in column j of T to column j of $\rho^{-1}(T)$, the result follows. $\square$

Example. *We compute*

$$\begin{aligned} \mathcal{S}_{(1,2)} &= M_{(1,2)} + M_{(1,1,1)} \\ &= F_{(1,2)} \end{aligned}$$

from the SSAFs

$$\begin{array}{|c|c|c|} \hline 1 & \mathbf{1} & \\ \hline 2 & \mathbf{2} & \mathbf{2} \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline 1 & \mathbf{1} & \\ \hline 2 & & \\ \hline 3 & \mathbf{3} & \mathbf{2} \\ \hline \end{array}$$

or, equivalently, the ComTs

$$\begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 2 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 3 & 2 \\ \hline \end{array}$$

for the first equality, and just the latter SAF or standard ComT for the second equality.

Remark. *For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ [27] defines the* symmetric *function indexed by a composition, known as a* ribbon Schur *function, r_α. The relationship between ribbon Schur functions and similarly indexed quasisymmetric Schur functions is straightforward to deduce as follows.*

By the Littlewood-Richardson rule, say [32, Chapter 7], we have

$$r_\alpha = \sum c_{\alpha\lambda} s_\lambda$$

where $c_{\alpha\lambda}$ is the number of Littlewood-Richardson fillings of the connected skew diagram containing no 2×2 skew diagram that has α_1 cells in the top row, α_2 cells in the second row etc. Since

$s_\lambda = \sum_{\alpha\,:\,\lambda(\alpha)=\lambda} \mathcal{S}_\alpha$ *it immediately follows that*

$$r_\alpha = \sum c_{\alpha\lambda(\beta)} \mathcal{S}_\beta$$

where $c_{\alpha\lambda(\beta)}$ *is as above.*

In [8, Theorem 4.1] necessary and sufficient conditions for equality of ribbon Schur functions were determined. Meanwhile, in [34, Theorem 2.2] necessary and sufficient conditions for uniqueness of Littlewood-Richardson fillings were proved. Combining these results with the above, we conclude that the simple relationship between Schur functions and quasisymmetric Schur functions is only achieved again with $r_{(u,1^v)}$ *or* $r_{(1^v,u)}$*, that is*

$$r_{(u,1^v)} = r_{(1^v,u)} = \sum_{\lambda(\alpha)=(u,1^v)} \mathcal{S}_\alpha.$$

Thus concludes our remark.

We now come to our Pieri rule for quasisymmetric Schur functions, whose proof we delay until the next subsection, and whose statement requires the following definitions.

Remark. *In practice the following* rem_s *operator subtracts 1 from the rightmost part of size* s *in a composition, or returns the empty composition. Meanwhile the* $row_{\{s_1<\cdots<s_j\}}$ *operator subtracts 1 from the rightmost part of size* $s_j, s_{j-1}, \ldots$ *recursively. Similarly, the* $col_{\{m_1\leq\cdots\leq m_j\}}$ *operator subtracts 1 from the rightmost part of size* $m_1, m_2, \ldots$ *recursively.*

Example. *If* $\alpha = (1,2,3)$ *then*

$$row_{\{2,3\}}(\alpha) = rem_2(rem_3((1,2,3))) = rem_2((1,2,2)) = (1,2,1)$$

and

$$col_{\{2,3\}}(\alpha) = rem_3(rem_2((1,2,3))) = rem_3((1,1,3)) = (1,1,2).$$

Now we define these three operators formally. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition whose largest part is m, and let $s \in [m]$. If there exists $1 \leq i \leq k$ such that $s = \alpha_i$ and $s \neq \alpha_j$ for all $j > i$, then define

$$rem_s(\alpha) = (\alpha_1, \ldots, \alpha_{i-1}, (s-1), \alpha_{i+1}, \ldots, \alpha_k),$$

otherwise define $rem_s(\alpha)$ to be the empty composition. Let $S = \{s_1 < \cdots < s_j\}$. Then define

$$row_S(\alpha) = rem_{s_1}(\ldots(rem_{s_{j-1}}(rem_{s_j}(\alpha)))\ldots).$$

Similarly let $M = \{m_1 \leq \cdots \leq m_j\}$. Then define

$$col_M(\alpha) = rem_{m_j}(\ldots(rem_{m_2}(rem_{m_1}(\alpha)))\ldots).$$

We collapse $row_S(\alpha)$ or $col_M(\alpha)$ to obtain a composition if needs be.

For any horizontal strip δ we denote by $S(\delta)$ the set of columns its skew diagram occupies, and for any vertical strip ϵ we denote by $M(\epsilon)$ the multiset of columns its skew diagram occupies, where multiplicities for a column are given by the number of cells in that column. We are now ready to state our refined Pieri rule.

Theorem 6.3 (Pieri rule for quasisymmetric Schur functions)**.** *Let α be a composition. Then*

$$\mathcal{S}_{(n)}\mathcal{S}_\alpha = \sum_\beta \mathcal{S}_\beta$$

where the sum is taken over all compositions β such that

(1) $\delta = \lambda(\beta)/\lambda(\alpha)$ *is a horizontal strip,*
(2) $|\delta| = n$,
(3) $row_{S(\delta)}(\beta) = \alpha$.

Also,

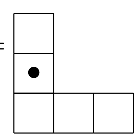
$$\mathcal{S}_{(1^n)}\mathcal{S}_\alpha = \sum_\beta \mathcal{S}_\beta$$

where the sum is taken over all compositions β such that

(1) $\epsilon = \lambda(\beta)/\lambda(\alpha)$ *is a vertical strip,*
(2) $|\epsilon| = n$,
(3) $col_{M(\epsilon)}(\beta) = \alpha$.

For a more visual interpretation of Theorem 6.3 we use composition diagrams in place of compositions in the next example. Then rem_s is the operation that removes the rightmost cell from the lowest row of length s.

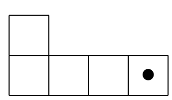
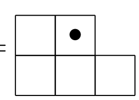
Example. *If we place $\bullet$ in the cell to be removed then*

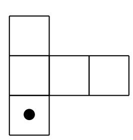
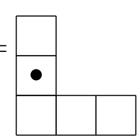
$rem_1((1,1,3)) =$  $= (1,3).$

If we wish to compute $\mathcal{S}_{(1)}\mathcal{S}_{(1,3)}$ then we consider the four skew diagrams

$$(4,1)/(3,1),\ (3,2)/(3,1),\ (3,1,1)/(3,1),\ (3,1,1)/(3,1)\ (again)$$

with horizontal strips containing one cell in column $4, 2, 1, 1$ respectively. Then

$row_{\{4\}}((1,4)) =$  $row_{\{2\}}((2,3)) =$ 

$row_{\{1\}}((1,3,1)) =$  $row_{\{1\}}((1,1,3)) =$ 

and hence

$$\mathcal{S}_{(1)}\mathcal{S}_{(1,3)} = \mathcal{S}_{(1,4)} + \mathcal{S}_{(2,3)} + \mathcal{S}_{(1,3,1)} + \mathcal{S}_{(1,1,3)}.$$

Classically, the Pieri rule gives rise to Young's lattice on partitions in the following way. Let λ, μ be partitions, then λ covers μ in Young's lattice if the coefficient of s_λ in $s_{(1)}s_\mu$ is 1. Therefore, Theorem 6.3 analogously gives rise to a poset on compositions: Let α, β be compositions, then β covers α if the coefficient of $\mathcal{S}_\beta$ in $\mathcal{S}_{(1)}\mathcal{S}_\alpha$ is 1. It would be interesting to see what properties of Young's lattice are exhibited by this new poset, which differs from the poset of compositions in [2], and contains Young's lattice as a subposet.

6.1. **Proof of the Pieri rule for quasisymmetric Schur functions.** In order to prove our Pieri rule we require three known combinatorial constructs, which we recall here in terms of reversetableaux for convenience.

The first construct is *Schensted insertion*, which inserts a positive integer k_1 into a reversetableau T, denoted $T \leftarrow k_1$ by

1. if k_1 is less than or equal to the last entry in row 1, place it there, else
2. find the leftmost entry in that row strictly smaller than k_1, say k_2, then
3. replace k_2 by k_1, that is k_1 *bumps* k_2.
4. Repeat the previous steps with k_2 and row 2, k_3 and row 3, etc.

The set of cells whose values are modified by the insertion, including the final cell added, is called the *insertion path*, and the final cell is called the *new cell*.

Example.

$$\begin{array}{|c|c|c|c|}\hline 7&5&4&2\\\hline 6&4&3\\\cline{1-3} 3&2&2\\\cline{1-3} 1&1\\\cline{1-2}\end{array} \leftarrow 5 \quad = \quad \begin{array}{|c|c|c|c|}\hline 7&5&\boldsymbol{5}&2\\\hline 6&4&\boldsymbol{4}\\\cline{1-3} 3&\boldsymbol{3}&2\\\cline{1-3} \boldsymbol{2}&1\\\cline{1-2} \boldsymbol{1}\\\cline{1-1}\end{array}$$

where the bold italic cells indicate the insertion path.

Insertion paths have the useful property encompassed in the next lemma, commonly known as the *row bumping lemma.*

Lemma 6.4 (Row bumping lemma)**.** *Let T be a reversetableau. Consider two successive insertions $(T \leftarrow x) \leftarrow x'$, giving rise to two insertion paths R and R', with respective new cells B and B'.*

1. *If $x \geq x'$, then R is strictly left of R', and B is strictly left of and weakly below B'.*
2. *If $x < x'$, then R' is weakly left of R, and B' is weakly left of and strictly below B.*

The second combinatorial construct we require is the *plactic monoid*, which can be described as the monoid whose elements consist of all reversetableaux. To describe the product, recall the *row reading word* of a reversetableau, T, is the sequence of the entries of the cells of T read from left to right, and bottom to top. It is denoted $w_{row}(T)$. For example, $w_{row}(T) = 113226437542$ for the original reversetableau in the previous example. Then, given reversetableaux T and U, the plactic monoid product is

$$T \cdot U = ((T \leftarrow w_1) \leftarrow w_2) \cdots \leftarrow w_n$$

where $w_{row}(U) = w_1 w_2 \cdots w_n$. The empty reversetableau is the monoid identity.

The group ring of the plactic monoid, R, is called the *reversetableaux ring* and $S_\lambda \in R$ is

$$S_\lambda = \sum T$$

where the sum is over all reversetableaux, T, of shape λ.

There exists a surjective homomorphism

$$\begin{array}{rcccl} \varepsilon & : & R & \longrightarrow & \mathbb{Z}[[x_1, x_2, \ldots]] \\ & & T & \mapsto & x^T \end{array}$$

that importantly satisfies

$$\varepsilon(S_\lambda) = s_\lambda.$$

The third, and last, construct is an analogy to Schensted insertion for a SSAF, or *skyline insertion*. We state it here for ComTs since ComTs will be used in the remaining proofs. However, it can be found in its original form in [30, Procedure 3.3].

Suppose we start with a ComT F whose longest row has length r. To insert a positive integer k_1, the result being denoted $k_1 \to F$, scan column positions starting with the top position in column $j = r + 1$.

(1) If the current position is empty and at the end of a row of length $j - 1$, and k_1 is weakly less than the last entry in the row, then place k_1 in this empty position and stop. Otherwise, if the position is nonempty and contains $k_2 < k_1$ and k_1 is weakly less than the entry to the immediate left of k_2, let k_1 bump k_2, i.e. swap k_2 and k_1.
(2) Using the possibly new k_i value, continue scanning successive positions in the column top to bottom as in the previous step, bumping whenever possible, and then continue scanning at the top of the next column to the left. (Decrement j.)
(3) If an element is bumped into the first column, then create a new row containing one cell to contain the element, placing the row such that the first column is strictly increasing top to bottom, and stop.

The set of cells whose values are modified by the insertion, including the final cell added, is called the *insertion sequence*, and the final cell is called the *new cell*. The row in which the new cell is added is called the *row augmented by the insertion*, and we note that the number of cells, or length of the row, increases by one.

Example.

$5 \to$

1	1		
3	2	2	2
6	5	4	
7	4	3	

$=$

1	1		
2			
3	***3***	2	2
6	5	***5***	
7	4	***4***	

where the bold italic cells indicate the insertion sequence.

Schensted and skyline insertion commute in the following sense [30, Proposition 3.1].

Proposition 6.5. *If ρ is the inverse map of ρ^{-1} and F is a ComT then*

$$\rho(k \to F) = (\rho(F) \leftarrow k).$$

We are now ready to prove the Pieri rule for quasisymmetric Schur functions after we prove

Lemma 6.6. *Let D be a ComT, k a positive integer, and $D' = k \to D$ with row i of D' being the row augmented by the insertion. Then for all rows $r > i$ of D'*

$$\text{length of row } i \neq \text{ length of row } r.$$

Proof. Assume that the lemma is false, that is, that there exists a row $r > i$ of D' having the same length as the augmented row i, say length equal to j. Note that in this case, the r-th row of D' is the same as the r-th row of D, except in the case that the augmented row i is a new row of length

1, in which case the $(r+1)$-th row of D' is the same as the r-th row of D. In the algorithm for inserting a new element k into a ComT D, consider the value x that was bumped from column $j+1$ into column j.

This bumped value x must be larger than $D(r,j)$, for otherwise either x was the value of the variable k compared against $D(r,j)$ during the pass of the algorithm over column (j+1), in which case the value x would have been placed in the vacant position $D(r,j+1)$, or x was bumped from position $D(s,j+1)$ for some row $s > r$, in which case D would have violated the triple rule for ComTs (namely $D(r,j) \geq x = D(s,j+1) > D(r,j+1)$), a contradiction in either case.

Now if $j=1$, then x was simply inserted into D' as the new row i of length one. However, since the first column is strictly increasing top to bottom, x would have been inserted as a new row *after* (higher row number than) $D(r,j)$, i.e. $i > r$, contrary to supposition. So we can assume that $j > 1$.

Recall that the entries in any given column are all distinct. We must have $D'(r,j) > D'(i,j-1)$, for otherwise we would have had a triple rule violation in D (namely $D(i,j-1) \geq D(r,j) > D(i,j) =$ empty). This then would require that $D'(r,j) > D'(i,j)$ as well. Now consider the portion of the insertion sequence that lies in column j, say in rows $\{i_0,\ldots,i_t = i\}$, whose first value, scanning top to bottom, is $x = D'(i_0,j)$ and whose last value is $D'(i,j) = D'(i_t,j)$. Since $x > D(r,j) = D'(r,j) > D'(i,j)$, and since the entries in the insertion path are decreasing top to bottom, there must be some index $0 \leq \ell < t$ such that $D'(i_\ell,j) > D'(r,j) > D'(i_{\ell+1},j)$. This would imply $D'(i_\ell,j-1) > D'(r,j)$ as well. However, since $D(i_k,j) = D'(i_{k+1},j)$ for all $0 \leq k < t$, and $D(h,k) = D'(h,k)$ for all $k < j$, this would imply a triple rule violation in D, namely $D(i_\ell,j-1) > D(r,j) > D(i_\ell,j)$.

Thus in all cases we have a contradiction. □

We note that as Schensted insertion for reversetableaux is reversible (invertible), so the analogous insertion into ComTs is reversible. In particular, given a ComT D of shape α and a given positive integer $\ell = \alpha_i$ for some i, where we assume that i is the largest index such that $\ell = \alpha_i$, then one can *uninsert* an element k from D to obtain a ComT D' such that $D = k \to D'$ and the shape of D' is $(\alpha_1,\ldots,\alpha_{i-1},\ell-1,\alpha_{i+1},\ldots)$, that is, the shape obtained from D by removing the last square from row i.

Proof. (of Theorem 6.3) We start with the first formula. We consider S_n to be the sum of reversetableaux in the reversetableaux ring R of shape (n), and H_α to be the sum of reversetableaux in R of shape $\lambda(\alpha)$ which map to a ComT of shape α under the mapping ρ^{-1}. We consider a typical term $U \cdot V$ of the product $H_\alpha \cdot S_n$, where U is one of the reversetableau terms of H_α and V is one of the reversetableau terms of S_n. Suppose the reversetableau $C = U \cdot V$, and ComT $D = \rho^{-1}(C)$, where the shape of D is β. Note that U has shape $\lambda(\alpha)$ and C has shape $\lambda(\beta)$. The set of n new cells added to U in the product $U \cdot V$ to form C forms the skew reversetableau C/U of shape $\lambda(\beta)/\lambda(\alpha)$. Now V will be of the form

$$V = \boxed{x_1 \mid x_2 \mid \cdots \mid x_n}$$

where $x_1 \geq x_2 \geq \cdots \geq x_n$. Thus

$$C = U \leftarrow x_1 \leftarrow x_2 \leftarrow \cdots \leftarrow x_n.$$

By Lemma 6.4, C/U is a horizontal strip with n cells, and over the successive insertions, the cells in this horizontal strip are added to U from left to right, say in columns $j_1 < j_2 < \cdots < j_n$. Suppose

ComT $E = \rho^{-1}(U)$, which by assumption has shape α. Under the map ρ^{-1} and insertion for ComTs, the corresponding new cells added to E to form D are added to columns $j_1, j_2, \dots, j_n$ in the same order by Proposition 6.5. By Lemma 6.6, each time a new cell is added, the augmented row in which it appears is the last row in the new diagram of that length. That is, assuming that $\alpha_i = j_1 - 1$ and $\alpha_k \neq j_1 - 1$ for all $k > i$, then the shape of $(E \leftarrow x_1)$ is $\alpha' = (\alpha_1, \dots, \alpha_{i-1}, j_1, \alpha_{i+1}, \dots, \alpha_{\ell(\alpha)})$, that is $\alpha = rem_{j_1}(\alpha')$. The pattern continues, that is, if the shape of $(E \leftarrow x_1 \leftarrow x_2)$ is α'', then $\alpha' = rem_{j_2}(\alpha'')$, etc., and by induction we have

$$\alpha = rem_{j_1}(\dots(rem_{j_{n-1}}(rem_{j_n}(\beta)))\dots) = row_J(\beta)$$

where $J = \{j_1, \dots, j_n\}$. Thus $C = U \cdot V$ is a term (reversetableau summand) of H_β where β is one of the summand indices specified by the formula.

Conversely, suppose the reversetableau C is a term of H_β where β is one of the summand indices on the right hand side of the formula. By definition, $\lambda(\beta)/\lambda(\alpha)$ is a horizontal strip with n cells, say in columns $j_1 < j_2 < \dots < j_n$. Since insertion is reversible, we can perform uninsertion on C, removing the cells of the horizontal strip starting with the last column j_n and working left. Uninserting the bottom cell from column j_n yields an element x_n, then uninserting the bottom cell from column j_{n-1} yields an element x_{n-1}, etc.

$$(\dots((C \xrightarrow{j_n} x_n) \xrightarrow{j_{n-1}} x_{n-1})\dots) \xrightarrow{j_1} x_1$$

Let U be the reversetableau resulting from uninserting the n cells. Now Lemma 6.4 implies that $x_1 \geq x_2 \geq \dots \geq x_n$, and so we may set V to be the reversetableau of shape (n) having these entries, and $C = U \cdot V$. Let $D = \rho^{-1}(C)$, and $E = \rho^{-1}(U)$. By Lemma 6.6 and Proposition 6.5, under the mapping ρ^{-1}, each successive cell removed from D to obtain E is removed from the last row of the ComT whose length is the column index of the cell being removed, that is the shape of E is $\alpha = row_J(\beta)$, where $J = \{j_1, \dots, j_n\}$. Thus $C = U \cdot V$ is a term in the product $H_\alpha \cdot S_n$ from the left hand side. Moreover, since we are able to uniquely determine U and V from C, C appears exactly once on each side of the formula. This proves the first formula through applying the map ε.

The proof of the second formula, involving vertical strips, is very much analogous to the first, making use of the second case of the row bumping lemma. □

6.2. **Transition matrices.** From Theorems 6.1 and 6.2 and the proof of Proposition 5.5 we are able to describe the transition matrices between quasisymmetric Schur functions and monomial or fundamental quasisymmetric functions.

Proposition 6.7. *Let A be the matrix whose rows and columns are indexed by $\alpha \vDash n$ ordered by $\blacktriangleright$ and entry $A_{\alpha\beta}$ is the coefficient of M_β in $\mathcal{S}_\alpha$. Then $A_{\alpha\beta}$ is the number of ComTs of shape α and weight β. Furthermore $A_{\alpha\beta} = 0$ if $\alpha \blacktriangleright \beta$ and $A_{\alpha\alpha} = 1$.*

Proof. The first statement follows from Theorem 6.1. The second statement follows from the second paragraph of the proof of Proposition 5.5 and the fact that $F_\alpha = \sum_{\alpha \succeq \beta} M_\beta$. □

Proposition 6.8. *Let A be the matrix whose rows and columns are indexed by $\alpha \vDash n$ ordered by $\blacktriangleright$ and entry $A_{\alpha\beta}$ is the coefficient of F_β in $\mathcal{S}_\alpha$. Then $A_{\alpha\beta}$ is the the number of standard ComTs T of shape α and $\beta(\mathcal{D}(T)) = \beta$. Furthermore $A_{\alpha\beta} = 0$ if $\alpha \blacktriangleright \beta$ and $A_{\alpha\alpha} = 1$.*

Proof. The first statement follows from Theorem 6.2, while the second statement follows from the second paragraph of the proof of Proposition 5.5. □

The transition matrix from quasisymmetric Schur functions to monomial or fundamental quasisymmetric functions is therefore upper unitriangular by Propositions 6.7 and 6.8. Consequently, to expand any quasisymmetric function in terms of the quasisymmetric Schur basis, simply invert the appropriate matrix depending on whether the initial quasisymmetric function is given in the monomial or fundamental basis.

Another straightforward application of Proposition 6.7 and Proposition 6.8 yields the following.

Corollary 6.9. *Let α be a composition. Then $\mathcal{S}_\alpha = M_\alpha$ if and only if $\alpha = (1^f)$. Similarly, $\mathcal{S}_\alpha = F_\alpha$ if and only if $\alpha = (m, 1^{e_1}, 2, 1^{e_2}, \dots, 2, 1^f)$ where m, f, e_i are nonnegative integers such that $m \neq 1$, $f \geq 0$, and $e_i \geq 1$ for all i.*

Example. *By Corollary 6.9 we know that the transition matrix between $\{\mathcal{S}_\alpha\}_{\alpha \vDash n}$ and $\{F_\alpha\}_{\alpha \vDash n}$ is the identity matrix for $n = 1, 2, 3$. For $n = 4$ we get*

$$\begin{matrix} (4) \\ (3,1) \\ (1,3) \\ (2,2) \\ (2,1,1) \\ (1,2,1) \\ (1,1,2) \\ (1,1,1,1) \end{matrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

where $\cdot$ denotes 0 and the rows are indexed by quasisymmetric Schur functions. We can hence conclude that our basis differs from those appearing in [7, 23, 33].

7. Further avenues

As indicated in the introduction, there are many further avenues to pursue, and in our conclusion we discuss three of them here.

7.1. **A quasisymmetric refinement of the Littlewood-Richardson rule.**

The Pieri rule generalizes to the celebrated Littlewood-Richardson rule, say [32, Chapter 7], for expanding the product of two generic Schur functions in terms of Schur functions

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ are positive integers that can be computed combinatorially, given partitions λ, μ, ν. The combinatorial computation requires enumerating all reversetableaux of shape λ/μ and weight ν subject to one further condition known as the lattice condition.

Since Theorem 6.3 refines the classical Pieri rule, it is natural to ask whether expanding the product of two generic quasisymmetric Schur functions in terms of quasisymmetric Schur functions refines the classical Littlewood-Richardson rule simply. Such a refinement does not presently seem simple, as expanding the product of two generic quasisymmetric Schur functions in terms of quasisymmetric Schur functions often results in negative structure constants. The smallest example is

$$\begin{aligned}\mathcal{S}_{(2,1)}\mathcal{S}_{(2,1)} &= \mathcal{S}_{(4,2)} + \mathcal{S}_{(4,1,1)} + 2\mathcal{S}_{(3,2,1)} + \mathcal{S}_{(3,1,2)} + 2\mathcal{S}_{(2,3,1)} \\ &+ \mathcal{S}_{(1,3,2)} + \mathcal{S}_{(3,1,1,1)} + \mathcal{S}_{(2,2,2)} + \mathcal{S}_{(2,2,1,1)} + \mathcal{S}_{(2,1,2,1)} \\ &- \mathcal{S}_{(1,4,1)} - \mathcal{S}_{(1,3,1,1)} - \mathcal{S}_{(1,1,3,1)} - \mathcal{S}_{(1,2,2,1)}.\end{aligned}$$

However, a product that does naturally refine the classical Littlewood-Richardson rule is the product of a generic *Schur* polynomial with a generic quasisymmetric Schur polynomial expanded in terms of quasisymmetric Schur polynomials. More precisely, in the sequel to this paper [16] we prove that

$$s_\lambda(x_1,\ldots,x_n)\mathcal{S}_\alpha(x_1,\ldots,x_n) = \sum_\beta C^\beta_{\alpha\lambda}\mathcal{S}_\beta(x_1,\ldots,x_n)$$

where the $C^\beta_{\alpha\lambda}$ are positive integers whose computation requires enumerating all ComTs of shape β with α removed from the top left corner, with weight the parts of λ taken in reverse order, and subject to a lattice-type condition. In addition, we show that similar combinatorial rules exist for the product of a generic Schur polynomial and a Demazure atom and generic Schur polynomial and a Demazure character when expanded as a linear combination of Demazure atoms and characters, respectively. Moreover, we recover the classical Littlewood-Richardson rule as a special case of this latter result, when we restrict Demazure characters to Schur polynomials.

7.2. **Skew quasisymmetric functions and duality.** In the classical theory of symmetric functions, say [25, Chapter 1], there exists the *Hall inner product* $\langle\cdot,\cdot\rangle$, which pairs dual graded bases in the self-dual Hopf algebra Λ. This inner product reveals that the Schur functions form an orthonormal basis of Λ, that is

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$$

where λ,μ are partitions and $\delta_{\lambda\mu} = 1$ if $\lambda=\mu$ and 0 otherwise. Equivalently, the *Cauchy formula* states that

$$\sum_\lambda s_\lambda(x_1,\ldots)s_\lambda(y_1,\ldots) = \prod_{i,j}(1-x_iy_j)^{-1}.$$

One might wonder how such notions extend to the Hopf algebra of quasisymmetric functions $\mathcal{Q} = \bigoplus_{n\geq 0}\mathcal{Q}_n$ and its dual, the algebra of *noncommutative symmetric functions*, $NSym = \bigoplus_{n\geq 0} NSym_n$, introduced in [11].

In [11, Section 6], following the work of [12] and [28], a pairing between dual graded bases of $\mathcal{Q}$ and $NSym$ was introduced as an analogue to the Hall inner product. This pairing yielded

$$\langle F_\alpha, R_\beta\rangle = \delta_{\alpha\beta}$$

where α,β are compositions and R_β is the noncommutative ribbon Schur function whose commutative image is the ribbon Schur function r_β. Also, [11] introduced the equivalent *Cauchy element* in the graded completion of $\bigoplus_{n\geq 0} NSym_n\otimes\mathcal{Q}_n$ as

$$\mathcal{C} := \sum_\alpha R_\alpha\otimes F_\alpha = \sum_\alpha a_\alpha\otimes b_\alpha$$

where $\{a_\alpha\}$ and $\{b_\alpha\}$ is any pair of dual graded bases, as a means to describe the dual bases of the various bases of $NSym$.

Conversely, we can ask what can be deduced about the dual basis of quasisymmetric Schur functions $\{\mathcal{S}^*_\alpha\}$ from quasisymmetric Schur functions themselves? For this we need *skew* quasisymmetric Schur functions, and this question is fully addressed in [5].

7.3. **Quasisymmetric Hall-Littlewood and Macdonald polynomial decompositions.** In view of the fact that the Demazure atoms and characters can be obtained by setting $q = t = 0$ in various versions of Macdonald polynomials, a natural question to ask is whether q and/or t parameters can be inserted in a natural way into the construction of quasisymmetric Schur functions. In this section we show how the t parameter can easily be added to some of our constructions, resulting in a decomposition of the Hall-Littlewood polynomial into quasisymmetric functions. We contrast this with an alternate decomposition obtained from a result in [13, Appendix A], and discuss obstacles preventing the insertion of an additional q parameter into our model. Throughout this section we let X_n denote the ordered sequence of variables $x_1, \dots, x_n$.

Let γ be a weak composition into n parts, and $s \in \gamma$, i.e. s a cell or *square* of the diagram of γ. Let row(s), col(s), West(s), and East(s) denote the row containing s, the column containing s, the square of $\widehat{dg}(\gamma)$ in row(s) immediately left of s, and the square of γ in row(s) immediately right of s (if it exists), respectively. Furthermore let leg(s) be the number of squares in row(s), but to the right of s, and arm(s) the number of squares of γ in the same column as s, below s, and in a row not longer than row(s), plus the number of squares of $\widehat{dg}(\gamma)$ in the column just left (which may be in the basement) of col(s), in a row above s, and also in a row strictly shorter than row(s). For a filling τ of γ, we let $\tau(s)$ denote the entry of τ in s.

Example. *On the left, the leg lengths, and on the right, the arm lengths, for the squares of the augmented diagram (with unmarked basement)* $(1, 0, 3, 2, 3)$.

	0		
	2	1	0
	1	0	
	2	1	0

	0		
	4	3	1
	2	1	
	3	2	1

We let $E'_\gamma(X_n; q, t)$ denote the nonsymmetric Macdonald polynomial introduced by Macdonald in [26] and studied by Cherednik [10], and

$$E_\gamma(X_n; q, t) = E'_{\gamma^*}(x_n, \dots, x_2, x_1; 1/q, 1/t)$$

the modified version of the E' appearing in work of Marshall [29], where again $\gamma^* = (\gamma_n, \dots, \gamma_1)$. Furthermore let $\mathcal{E}'$ and $\mathcal{E}$ be the integral forms of the E's and Es, respectively, defined via

$$\mathcal{E}'_\gamma(X_n; q, t) = \prod_{s \in \gamma^*} (1 - q^{\text{leg}(s)+1} t^{\text{arm}(s)+1})\, E'_\gamma(X_n; q, t) \tag{7.1}$$

$$\mathcal{E}_\gamma(X_n; q, t) = \prod_{s \in \gamma} (1 - q^{\text{leg}(s)+1} t^{\text{arm}(s)+1})\, E_\gamma(X_n; q, t). \tag{7.2}$$

For μ a partition, we let $P_\mu(X_n; q, t)$ denote the symmetric Macdonald polynomial [25, Chapter 7] and $J_\mu(X_n; q, t)$ its integral form [25, p. 352],

$$J_\mu(X_n; q, t) = \prod_{s\in\mu}(1 - q^{\mathrm{leg}(s)} t^{\mathrm{arm}(s)+1}) P_\mu(X_n; q, t) \tag{7.3}$$

in our notation. (They are called integral forms since the coefficients of monomials in them are in $\mathbb{Z}[q, t]$, while those in the E's, Es, and Ps are in $\mathbb{Q}[q, t]$.) We note that in E'_γ, E_γ and P_μ, the leading coefficient of x^γ, x^γ, and x^μ, respectively, is one where $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots$.

In [15] the following combinatorial formula for $\mathcal{E}_\gamma(X_n; q, t)$ is obtained;

$$\begin{aligned} \mathcal{E}_\gamma(X_n; q, t) = &\sum_{\substack{\text{non-attacking fillings } \tau \text{ of } \gamma \\ b_i = i}} x^\tau q^{\mathrm{maj}(\tau,\gamma)} t^{\mathrm{coinv}(\tau,\gamma)} \\ &\times \prod_{\substack{s\in\gamma \\ \tau(s)=\tau(\mathrm{West}(s))}} (1 - q^{\mathrm{leg}(s)+1} t^{\mathrm{arm}(s)+1}) \prod_{\substack{s\in\gamma \\ \tau(s)\neq\tau(\mathrm{West}(s))}} (1 - t), \end{aligned} \tag{7.4}$$

where $\mathrm{coinv}(\tau, \gamma)$ is the number of triples of the filling which are not inversion triples (i.e. are *coinversion triples*), and $\mathrm{maj}(\tau, \gamma)$ is the sum of $\mathrm{leg}(s) + 1$, over all $s \in \gamma$ where $\tau(\mathrm{West}(s))$ is smaller than $\tau(s)$ (i.e. a "descent"). By $b_i = i$ we mean the square in the i-th row of the basement contains i, for $1 \le i \le n$. As usual, basement squares can be included in triples.

A nice feature of (7.4) is that if we change the basement to $b_i = n - i + 1$, replace γ by γ^*, and sum over non-attacking fillings as above, we get a formula for $\mathcal{E}'_\gamma(X_n; q, t)$, while if we sum over non-attacking fillings with basement $b_i = n + 1$ for all i, we get a formula for $J_\mu(X_n; q, t)$, where $\mu = \lambda(\gamma)$. Letting $q = t = 0$ in these results give formulas for Demazure atoms ($\mathcal{E}_\gamma(X_n; 0, 0)$), Demazure characters, ($\mathcal{E}'_\gamma(X_n; 0, 0)$) and Schur functions ($J_\mu(X_n; 0, 0)$).

Macdonald obtained an expression for P_μ as a linear combination of the E'_γ. Expressed in terms of the E's, this takes the form [29], [15, Eq. (72)]

$$P_\mu(X_n; q, t) = \prod_{s\in\mu}(1 - q^{\mathrm{leg}(s)+1} t^{\mathrm{arm}(s)}) \sum_{\substack{\gamma \\ \lambda(\gamma)=\mu}} \frac{E_\gamma(X_n; q, t)}{\prod_{s\in\gamma}(1 - q^{\mathrm{leg}(s)+1} t^{\mathrm{arm}(s)})}. \tag{7.5}$$

By setting $q = 0$ in this formula we get

$$P_\mu(X_n; t) = \sum_{\substack{\gamma \\ \lambda(\gamma)=\mu}} E_\gamma(X_n; 0, t) \tag{7.6}$$

where $P_\mu(X_n; t) = P_\mu(X_n; 0, t)$ is the Hall-Littlewood polynomial [25, p. 208].

It is natural to refer to the function

$$E_\gamma(X_n; 0, t) = \mathcal{E}_\gamma(X_n; 0, t) \tag{7.7}$$

as a nonsymmetric Hall-Littlewood polynomial, and we denote this function by $E_\gamma(x_1, \ldots, x_n; t)$. From (7.4) we have the explicit formula

$$E_\gamma(X_n; t) = \sum_{\substack{\text{non-attacking fillings } \tau \text{ of } \gamma \\ b_i = i,\ \mathrm{maj}(\tau,\gamma)=0}} x^\tau t^{\mathrm{coinv}(\tau,\gamma)} \prod_{\substack{s\in\gamma \\ \tau(s)\neq\tau(\mathrm{West}(s))}} (1 - t). \tag{7.8}$$

For a given composition α, let $\mathcal{L}_\alpha(X_n;t)$ be the polynomial obtained by summing $E_\gamma(X_n;t)$ over all compositions γ for which $\alpha(\gamma)=\alpha$,

$$\mathcal{L}_\alpha(X_n;t) = \sum_{\gamma:\alpha(\gamma)=\alpha} \sum_{\substack{\text{non-attacking fillings } \tau \text{ of } \gamma \\ b_i=i,\ \mathrm{maj}(\tau,\gamma)=0}} x^\tau t^{\mathrm{coinv}(\tau,\gamma)} \prod_{\substack{s\in\gamma \\ \tau(s)\neq\tau(\mathrm{West}(s))}} (1-t). \tag{7.9}$$

Since the quasisymmetric Schur functions $\mathcal{S}_\alpha$ are obtained by summing the specialization of $E_\gamma(x;q,t)$ to $q=t=0$ over all compositions which collapse to α, we have $\mathcal{L}_\alpha(X_n;0)=\mathcal{S}_\alpha$. We now show that the $\mathcal{L}_\alpha$ are quasisymmetric.

Proposition 7.1. *The polynomials $\mathcal{L}_\alpha$ are quasisymmetric in $x_1,\ldots,x_n$.*

Proof. Note $\mathcal{L}_\alpha$ is quasisymmetric if and only if the monomial $x_{j_1}^{a_1}x_{j_2}^{a_2}\cdots x_{j_k}^{a_k}$ where $j_1<j_2<\cdots<j_k$ has the same coefficient as $x_{i_1}^{a_1}x_{i_2}^{a_2}\cdots x_{i_k}^{a_k}$ for any other sequence $i_1<i_2<\cdots<i_k$. We prove this by exhibiting a coinv-preserving bijection between descentless fillings σ of a weak composition γ which collapses to α, containing the multiset of entries $\{i_1^{a_1},{i_2}^{a_2},\ldots,{i_k}^{a_k}\}$, and descentless fillings σ' of a (possibly different) weak composition which also collapses to α, containing the multiset of entries $\{j_1^{a_1},{j_2}^{a_2},\ldots,{j_k}^{a_k}\}$. Our bijection will also preserve the number of squares s of γ where $\sigma(s)\neq\sigma(\mathrm{West}(s))$, and hence will preserve the power of t and $1-t$ multiplying x^σ in (7.8).

It is straightforward to check that in a descentless, non-attacking filling with basement $b_i=i$, if the entry in the first column of a given row is j, then the given row must be the j-th row. Let F be such a filling, of a weak composition γ with $\alpha(\gamma)=\alpha$, whose entries are given by the multiset $\{i_1^{a_1},{i_2}^{a_2},\ldots,{i_k}^{a_k}\}$. Simply replace each entry i_s by the entry j_s for all s from 1 to k and slide the rows so that the r^{th} row is the row whose first column-entry is r. Note that this preserves the order of the nonzero rows since their relative order (given by the entries in the leftmost columns) is not affected by the replacement of i_s by j_s. This also implies that the rows remain weakly decreasing. The relative orders of the entries in the triples are preserved, so the number of inversion triples is preserved. Also, squares s where $\sigma(s)\neq\sigma(\mathrm{West}(s))$ are mapped to other such squares, and similarly for squares with $\sigma(s)=\sigma(\mathrm{West}(s))$. To invert this map, simply replace j_s by i_s. □

Proposition 7.1 together with (7.6) imply that

$$P_\mu(X_n;t) = \sum_{\substack{\alpha \\ \lambda(\alpha)=\mu}} \mathcal{L}_\alpha(X_n;t) \tag{7.10}$$

is a decomposition of the Hall-Littlewood polynomial into quasisymmetric functions. We mention that Hivert [17] has introduced other quasisymmetric functions $G_\alpha(X_n;t)$ that he calls quasisymmetric Hall-Littlewood functions, which he defines via difference operators. Hivert obtains expansions for the G_α in terms of the fundamental quasisymmetric functions F_β, and also in terms of the monomial quasisymmetric functions M_β, and shows the G_α satisfy the interesting relations $G_\alpha(X_n;0)=F_\alpha(X_n)$ and $G_\alpha(X_n;1)=M_\alpha(X_n)$. On the other hand, $\mathcal{L}_\alpha(X_n;0)=\mathcal{S}_\alpha(X_n)$. Furthermore, when $t=1$ the only fillings σ defining $\mathcal{L}_\alpha$ in (7.8) which survive are those for which there are no squares s with $\sigma(s)\neq\sigma(\mathrm{West}(s))$, i.e. those σ which are constant across rows. Such σ have no coinversions, and it follows that the coefficient of x^α in $\mathcal{L}_\alpha(X_n;1)$ equals 1, and thus $\mathcal{L}_\alpha(X_n;1)=M_\alpha(X_n)$. For means of comparison,

$$G_{13}(X_n;t) = M_{13} + (1-q^2)M_{121} + (1-q^2)M_{112} + (1-2q^2+q^4)M_{1111}$$

while

$$\mathcal{L}_{13}(X_n;t) = M_{13} + (1-q)M_{22} + (1-q)M_{211} + (1-q)M_{121} + (2-2q)M_{112} + (2+q)(1-q)^2 M_{1111}$$

where we drop the brackets around the compositions for brevity.

In [13] an explicit decomposition of $J_\mu(X_n;q,t)$ into the F_α is obtained. Since this formula has not appeared in a journal article before, we include a detailed description of it here, and contrast the $q=0$ case of it with the decomposition of $P_\mu(X_n;t)$ into the quasisymmetric functions $\mathcal{L}_\alpha$ above. Consider a standard filling τ of μ, with basement $(n+1,\ldots,n+1)$. Such a filling is automatically non-attacking, and τ can be identified with the permutation obtained by reading in the entries of τ, from top to bottom within columns, starting with the rightmost column and working right to left. Given a triple of τ (neccessarily of type A since μ is a partition) which doesn't involve any basement squares, we call the square containing the middle of the three entries (i.e. neither the largest nor the smallest) the "base" of the triple. If the triple involves a basement square, we call the square containing the smallest of the three entries the base of the triple.

Example. *A standard filling of* $(3,3,1)$*, with basement* $(8,8,8)$*.*

8	**5**	**6**	**1**
8	**2**	**7**	**4**
8	**3**		

For the filling above, the base square of the triple consisting of entries $5,6,7$ *contains the* 6*, the triple with entries* $1,4,6$ *has base containing the* 4*, the triple with entries* $2,3,8$ *has base containing the* 2*, and the base square of the triple consisting of entries* $3,5,8$ *contains the* 3*.*

Let $\mathrm{coinv}_s(\tau,\mu)$ be the number of coinversion triples, and $\mathrm{inv}_s(\tau,\mu)$ the number of inversion triples, where s is the base square. Also, if $\tau(\mathrm{East}(s)) > \tau(s)$ (so there is a descent at $\mathrm{East}(s)$), let $\mathrm{maj}_s(\tau,\mu) = \mathrm{leg}(s)$, else set $\mathrm{maj}_s(\tau,\mu)=0$. And, if $\tau(\mathrm{West}(s)) \geq \tau(s)$ (so there is no descent at s), let $\mathrm{nondes}_s(\tau,\mu) = \mathrm{leg}(s)+1$, else set $\mathrm{nondes}_s(\tau,\mu)=0$. Note that $\mathrm{coinv}(\tau,\mu) = \sum_{s\in\mu}\mathrm{coinv}_s(\tau,\mu)$, with similar statements for $\mathrm{inv}(\tau,\mu)$ and $\mathrm{maj}(\tau,\mu)$. Then we have [13, p. 133]

(7.11)
$$J_\mu(X_n;q,t) = \sum_{\tau\in S_n} F_{\beta(\{i:\tau^{-1}(i)>\tau^{-1}(i+1)\})}(X_n)\prod_{s\in\mu}\left(q^{\mathrm{inv}_s(\tau,\mu)}t^{\mathrm{nondes}_s(\tau,\mu)} - q^{\mathrm{coinv}_s(\tau,\mu)}t^{1+\mathrm{maj}_s(\tau,\mu)}\right),$$

which gives an expansion of J_μ into Gessel's fundamental quasi-symmetric functions. It is also shown in [13] that if τ is such that some entry j occurs in a column to the right of the j-th column, then the factor

$$\prod_{s\in\mu}\left(q^{\mathrm{inv}_s(\tau,\mu)}t^{\mathrm{nondes}_s(\tau,\mu)} - q^{\mathrm{coinv}_s(\tau,\mu)}t^{1+\mathrm{maj}_s(\tau,\mu)}\right) \tag{7.12}$$

in (7.11) is zero.

By letting $q=0$ in (7.11), we get a decomposition of the integral form Hall-Littlewood polynomial ($Q_\mu(X_n;t)$ in the notation of [25, p. 210]) into fundamental quasisymmetric functions. It is complicated, though, to work with the set of permutations over which (7.12) does not vanish when $q=0$, i.e. the set where every square s of μ satisfies either $\mathrm{coinv}_s(\tau,\mu)=0$ or $\mathrm{inv}_s(\tau,\mu)=0$, or both. Also, the formula for $\mathcal{L}_\alpha$ as a sum over non-attacking fillings is a positive formula in the sense that each coefficient of a monomial is a sum of terms of the form $t^*(1-t)^*$ for nonnegative integers $*$, while the monomial coefficients in the $q = 0$ case of formula (7.11) could involve terms of the form $\pm t^* \prod(1 - t^*)$. We also mention that we need to divide the $q = 0$ case of (7.11) by the product

$$\prod_{\substack{s \in \mu \\ \mathrm{leg}(s)=0}} (1 - t^{\mathrm{arm}(s)+1}) \tag{7.13}$$

to convert from the integral form $J_\mu(X_n; 0, t)$ to $P_\mu(X_n; t)$, and once we do the coefficient of a given monomial in the x's is a rational function in t, not clearly a polynomial.

One would naturally hope to insert a q parameter into the construction of $\mathcal{L}_\alpha(X_n; t)$, and end up with a decomposition of $J_\mu(X_n; q, t)$ into quasisymmetric functions, where the quasisymmetric extension of the $\mathcal{L}_\alpha(X_n; t)$ is a positive sum in the sense of the above paragraph, along the lines of the formula (7.4). The problem is that the bijective map from the proof of Proposition 7.1 does not apply as is to fillings with descents just right of basement squares. Thus at this time the authors do not see how to extend the construction of the $\mathcal{L}_\alpha$ in an elegant way to include the q parameter. Another interesting question we leave for future research is how to decompose the $\mathcal{L}_\alpha$ into fundamental quasisymmetric functions.

8. Acknowledgements

The authors would like to thank Ole Warnaar and the referees for helpful comments and suggestions.

References

[1] M. Aguiar, N. Bergeron and F. Sottile, Combinatorial Hopf algebras and generalized Dehn-Sommerville equations, *Compos. Math.* **142** (2006), 1-30.

[2] F. Bergeron, M. Bousquet-Melou and S. Dulucq, Standard paths in the composition poset, *Ann. Sci. Math. Québec* **19** (1995), 139–151.

[3] N. Bergeron, S. Mykytiuk, F. Sottile and S. van Willigenburg, Non-commutative Pieri operators on posets, *J. Combin. Theory Ser. A* **91** (2000), 84–110.

[4] B. Bernevig and F. Haldane, Model fractional quantum Hall states and Jack polynomials, *Phys. Rev. Lett.* **100** (2008), 246802.

[5] C. Bessenrodt, K. Luoto and S. van Willigenburg, Skew quasisymmetric Schur functions, in preparation.

[6] L. Billera, S. Hsiao and S. van Willigenburg, Peak quasisymmetric functions and Eulerian enumeration, *Adv. Math.* **176** (2003), 248–276.

[7] L. Billera, N. Jia and V. Reiner, A quasisymmetric function for matroids, *European J. Combin.*, to appear.

[8] L. Billera, H. Thomas and S. van Willigenburg, Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions, *Adv. Math.* **204** (2006), 204–240.

[9] I. Cherednik, Double affine Hecke algebras and Macdonald's conjectures, *Ann. Math.* **141** (1995), 191–216.

[10] I. Cherednik, Nonsymmetric Macdonald polynomials, *Int. Math. Res. Not.* **10** (1995), 483–515.

[11] I. Gel'fand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon, Noncommutative symmetric functions, *Adv. Math.* **112** (1995), 218–348.

[12] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, combinatorics and algebra (Boulder, Colo., 1983) 289–317, Contemp. Math., 34, Amer. Math. Soc., Providence, RI, 1984.

[13] J. Haglund, *The q,t-Catalan numbers and the space of diagonal harmonics (With an appendix on the combinatorics of Macdonald polynomials)*, American Mathematical Society, University Lecture Series, Volume 41, Providence, RI, USA, 2008.

[14] J. Haglund, M. Haiman and N. Loehr, A combinatorial formula for Macdonald polynomials, *J. Amer. Math. Soc.* **18** (2005), 735–761.

[15] J. Haglund, M. Haiman and N. Loehr, A combinatorial formula for nonsymmetric Macdonald polynomials, *Amer. J. of Math.* **103** (2008), 359–383.

[16] J. Haglund, K. Luoto, S. Mason and S. van Willigenburg, Refinements of the Littlewood-Richardson rule, ArXiv: 0908.3540.

[17] F. Hivert, Hecke algebras, difference operators and quasi-symmetric functions, *Adv. Math.* **155** (2000), 181–238.

[18] B. Ion, Nonsymmetric Macdonald polynomials and Demazure characters, *Duke Math. J.* **116** (2003), 299–318.

[19] B. Ion, Standard bases for affine parabolic modules and nonsymmetric Macdonald polynomials, *J. Algebra* **319** (2008), 3480–3517.

[20] K. Kadell, A proof of some analogues of Selberg's integral for $k = 1$, *SIAM J. Math. Anal.* **19** (1988), 944–968.

[21] D. Krob and J.-Y. Thibon, Noncommutative symmetric functions V: a degenerate version of $U_q(gl_N)$, *Int. J. Alg. Comput.* **9** (1999), 405–430.

[22] A. Lascoux and M.-P. Schützenberger, Keys and standard bases, 125–144, Tableaux and Invariant Theory, IMA Volumes in Mathematics and its Applications, vol. 19, 1990.

[23] K. Luoto, A matroid-friendly basis for the quasisymmetric functions, *J. Combin. Theory Ser. A* **115** (2008), 777–798.

[24] I. Macdonald, A new class of symmetric functions, *Sém. Lothar. Combin.* **372** (1988), 131–171.

[25] I. Macdonald, *Symmetric functions and Hall polynomials, 2nd Edition*, Oxford University Press, New York, USA, 1995.

[26] I. Macdonald, Affine Hecke algebras and orthogonal polynomials, *Astérisque* **237** (1996), 189–207, Séminaire Bourbaki 1994/95, Exp. no. 797.

[27] P. MacMahon, Combinatory Analysis, Dover Publications, 2004.

[28] C. Malvenuto and C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, *J. Algebra* **177** (1995), 967–982.

[29] D. Marshall, Symmetric and nonsymmetric Macdonald polynomials, *Ann. Comb.* **3** (1999), 385–415.

[30] S. Mason, A decomposition of Schur functions and an analogue of the Robinson-Schensted-Knuth algorithm, *Sém. Lothar. Combin.* **57** (2008), B57e.

[31] S. Mason, An explicit construction of Type A Demazure atoms, *J. Algebraic Combin.* **29** (2009), 295–313.

[32] R. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press, Cambridge, UK, 1999.

[33] R. Stanley, The descent set and connectivity set of a permutation, *J. Integer Seq.* **8** (2005), 05.3.8.

[34] S. van Willigenburg, Equality of Schur and skew Schur functions, *Ann. Comb.* **9** (2005), 355–362.

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA
E-mail address: jhaglund@math.upenn.edu

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: kwluoto@math.ubc.ca

Department of Mathematics, Davidson College, Davidson, NC 28035-7129, USA
E-mail address: sarahkmason@gmail.com

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: steph@math.ubc.ca