

# NON-COMMUTATIVE PIERI OPERATORS ON POSETS

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ABSTRACT. We consider graded representations of the algebra  $NC$  of non-commutative symmetric functions on the  $\mathbb{Z}$ -linear span of a graded poset  $P$ . The matrix coefficients of such a representation give a Hopf morphism from a Hopf algebra  $\mathcal{HP}$  generated by the intervals of  $P$  to the Hopf algebra of quasi-symmetric functions. This provides a unified construction of quasi-symmetric generating functions from different branches of algebraic combinatorics, and this construction is useful for transferring techniques and ideas between these branches. In particular we show that the (Hopf) algebra of Billera and Liu related to Eulerian posets is dual to the peak (Hopf) algebra of Stembridge related to enriched  $P$ -partitions, and connect this to the combinatorics of the Schubert calculus for isotropic flag manifolds.

*G.-C. Rota inspired us to seek the algebraic structures underlying combinatorics.*

## 1. INTRODUCTION

The algebra  $\mathcal{Qsym}$  of quasi-symmetric functions was introduced by Gessel [14] as a source of generating functions for  $P$ -partitions [24]. Since then, quasi-symmetric functions have played an important role as generating functions [26, 27] in combinatorics. The relation of  $\mathcal{Qsym}$  to the more familiar algebra of symmetric functions was clarified by Gelfand *et. al.* [13] who defined the graded Hopf algebra  $NC$  of non-commutative symmetric functions and identified  $\mathcal{Qsym}$  as its Hopf dual.

Joni and Rota [17] made the fundamental observation that many discrete structures give rise to natural Hopf algebras whose coproducts encode the disassembly of these structures (see also [22]). A seminal link between these theories was shown by Ehrenborg [11], whose flag  $f$ -vector quasi-symmetric function of a graded poset gave a Hopf morphism from a Hopf algebra of graded posets to  $\mathcal{Qsym}$ . This theory was augmented in [4] where it was shown that the quasi-symmetric function associated to an edge-labelled poset similarly gives a Hopf morphism. That quasi-symmetric function generalised a quasi-symmetric function encoding the structure of the cohomology of a flag manifold as a module over the ring of symmetric functions [3, 6].

We extend and unify these results by means of a simple construction. Given a graded representation of  $NC$  on the  $\mathbb{Z}$ -linear span  $\mathbb{Z}P$  of a graded poset  $P$ , the matrix coefficients of such an action are linear maps on  $NC$  and hence quasi-symmetric

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functions. In Section 2 we show how this situation gives rise to a Hopf morphism as before. In Section 3, we extend this construction to an arbitrary oriented multigraph  $G$ . Sections 4, 6, and 7 give examples of this construction, including rank selection in posets, flag  $f$ -vectors of polytopes,  $P$ -partitions, Stanley symmetric functions, and the multiplication of Schubert classes in the cohomology of flag manifolds.

In Section 5, we discuss how properties of the combinatorial structure of  $G$  may be understood through the resulting quasi-symmetric function. This analysis allows us to relate work of Bayer, Billera, and Liu [1, 9] on Eulerian posets with work of Stembridge [28] on enriched  $P$ -partitions. More precisely, we show that the quotient of  $NC$  by the ideal of the generalised Dehn-Somerville relations is dual to the Hopf subalgebra of peak functions in  $Qsym$ . We also solve the conjecture of [2], showing that the shifted quasi-symmetric functions form a Hopf algebra. These functions were introduced by Billey and Haiman [10] to define Schubert polynomials for all types.

In Section 7, we show how a natural generating function for enumerating peaks in a labelled poset is the quasi-symmetric function for an enriched structure on the poset. Special cases of this combinatorics of peaks include Stembridge's Theory of enriched  $P$ -partitions [28], the Pieri-type formula for type  $B$  and  $C$  Schubert polynomials in [5], and Stanley symmetric functions of types  $B$ ,  $C$ , and  $D$ . These examples linking the diverse areas of Schubert calculus, combinatorics of polytopes and  $P$ -partitions illustrate how this theory transfers techniques and ideas between disparate areas of combinatorics.

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## 2. PIERI OPERATORS ON POSETS

Many interesting families of combinatorial constants can be understood as an enumeration of paths in a ranked partially ordered set (poset) satisfying certain conditions. One example of this is the Littlewood-Richardson rule in the theory of symmetric functions [19]. This rule describes the multiplicity  $c_{\mu,\nu}^\lambda$  of a Schur function  $S_\lambda$  in the product  $S_\mu S_\nu$  of two others. The constants  $c_{\mu,\nu}^\lambda$  can be seen as an enumeration of all paths in Young's lattice from  $\mu$  to  $\lambda$  satisfying some conditions imposed by  $\nu$ . We note that the constants  $c_{\mu,\nu}^\lambda$  are invariant under certain isomorphisms of intervals in Young's lattice. Namely  $c_{\mu,\nu}^\lambda = c_{\tau,\nu}^\pi$  whenever  $\lambda/\mu = \pi/\tau$ . The skew Schur functions  $S_{\lambda/\mu}$  are generating functions of these constants as we have

$$S_{\lambda/\mu} = \sum_{\nu} c_{\mu,\nu}^\lambda S_{\nu}.$$

We generalise the principles of this example, introducing families of algebraic operators to *select* paths in a given poset. Here, analogues of the Littlewood-Richardson constants count paths in the poset satisfying some conditions imposed by the family of operators. These enumerative combinatorial invariants of the poset are encoded by generating functions which generalise the skew Schur functions. We show that the association of this generating function to a poset induces a Hopf morphism to  $Qsym$ .

Let  $(P, <)$  be a graded poset with rank function  $rk: P \rightarrow \mathbb{Z}^+$  and let  $\mathbb{Z}P$  be the free graded  $\mathbb{Z}$ -module generated by the elements of  $P$ . For an integer  $k > 0$ , a (right) *Pieri operator* on  $P$  is a linear map  $\bar{h}_k: \mathbb{Z}P \rightarrow \mathbb{Z}P$  which respects the poset structure. By this we mean that for all  $x \in P$ , the support of  $x.\bar{h}_k \in \mathbb{Z}P$  consists only of elements  $y \in P$  such that  $x < y$  and  $rk(y) - rk(x) = k$ . We note that such an operator  $\bar{h}_k$  is of degree  $k$  on  $\mathbb{Z}P$ .

Gelfand *et. al.* [13] define the Hopf algebra  $NC$  of non-commutative symmetric functions to be the free associative algebra  $\mathbb{Z}\langle h_1, h_2, \dots \rangle$  with a generator  $h_k$  in each positive degree  $k$  and coproduct  $\Delta h_k = \sum_{i=0}^k h_i \otimes h_{k-i}$ , where  $h_0 = 1$ . It follows that given a family of Pieri operators  $\{\bar{h}_k\}_{k>0}$  on a poset  $P$ , the map  $h_k \mapsto \bar{h}_k$  turns  $\mathbb{Z}P$  into a graded (right)  $NC$ -module. Conversely, any graded right action of  $NC$  on  $\mathbb{Z}P$  which respects the poset structure of  $P$  gives a family of Pieri operators on  $P$ . When the context is clear, we may identify the generator  $h_k$  with the operator  $\bar{h}_k$ .

Given such a representation of  $NC$  on  $\mathbb{Z}P$  and  $x, y \in P$ , the association of  $\Psi \in NC$  to the coefficient of  $y$  in  $x.\Psi$  is a linear map on  $NC$ . These matrix coefficients are elements of the Hopf dual of  $NC$  which is the Hopf algebra  $\mathcal{Q}sym$  of quasi-symmetric functions [13]. These coefficients vanish unless  $x \leq y$ . This gives a collection of quasi-symmetric functions  $K_{[x,y]}$  associated to every interval  $[x, y]$  of  $P$ .

Let  $\mathcal{H}P$  be the free  $\mathbb{Z}$ -module with basis given by cartesian products of intervals  $[x, y]$  of  $P$ , modulo identifying all singleton intervals  $[x, x]$  with the unit 1 and empty intervals with zero. Then  $\mathcal{H}P$  is a graded  $\mathbb{Z}$ -algebra whose product is the cartesian product of intervals, and whose grading is induced by the rank of an interval of  $P$ . It has a natural coalgebra structure induced by

$$\Delta A = \sum_{x \in A} [\hat{0}_A, x] \otimes [x, \hat{1}_A],$$

where  $A = [\hat{0}_A, \hat{1}_A]$  is an interval of  $P$  with minimal element  $\hat{0}_A$  and maximal element  $\hat{1}_A$ . Projection onto  $\mathbb{Z}$  of the degree 0 component of  $\mathcal{H}P$  is the counit. It follows that  $\mathcal{H}P$  is a bialgebra. It is graded, therefore by Proposition A.2, there is a unique antipode and  $\mathcal{H}P$  is a Hopf algebra.

**Theorem 2.1.** *For any graded poset  $P$ ,  $\mathcal{H}P$  is a Hopf algebra.*

Suppose we have a family of Pieri operators on a poset  $P$ . Since  $NC$  is a Hopf algebra, the action of  $NC \otimes NC$  on  $\mathbb{Z}P \otimes \mathbb{Z}P = \mathbb{Z}(P \times P)$  pulls back along the the coproduct  $\Delta$  to give an action of  $NC$  on  $\mathbb{Z}(P \times P)$ . We iterate this and use coassociativity to get an action of  $NC$  on the  $\mathbb{Z}$ -linear span of  $P^k$ . Since a product of intervals of  $P$  is an interval in such an iterated product of  $P$  with itself, we may extend the definition of  $K$  to the generators of  $\mathcal{H}P$  and then by linearity to  $\mathcal{H}P$  itself, obtaining a  $\mathbb{Z}$ -linear homogeneous map  $K: \mathcal{H}P \rightarrow \mathcal{Q}sym$ . Let  $\langle \cdot, \cdot \rangle$  be the bilinear form on  $\mathbb{Z}P$  induced by the Kronecker delta function on the elements of  $P$ .

**Theorem 2.2.** *The map  $K: \mathcal{H}P \rightarrow \mathcal{Q}sym$  is a morphism of Hopf algebras.*

*Proof.* We show that  $K$  respects product and coproduct, which suffices. For  $x, y \in P$  and  $\Psi \in NC$ , we have  $K_{[x,y]}(\Psi) = \langle x.\Psi, y \rangle$ . Thus for  $x \in P$  and  $\Psi \in NC$ ,

$$x.\Psi = \sum_y \langle x.\Psi, y \rangle y = \sum_y K_{[x,y]}(\Psi) y.$$

Let  $A = [\hat{0}_A, \hat{1}_A]$  and  $B = [\hat{0}_B, \hat{1}_B]$  be intervals of  $P^{k_1}$  and  $P^{k_2}$  respectively. For  $\Psi \in NC$ , using Sweedler notation for the coproduct

$$\Delta\Psi = \sum \Psi_a \otimes \Psi_b,$$

and the duality between the product of  $\mathcal{Q}sym$  and the coproduct of  $NC$ , we obtain

$$\begin{aligned} K_{A \times B}(\Psi) &= \langle (\hat{0}_A \otimes \hat{0}_B).\Psi, \hat{1}_A \otimes \hat{1}_B \rangle \\ &= \left\langle \sum \hat{0}_A.\Psi_a \otimes \hat{0}_B.\Psi_b, \hat{1}_A \otimes \hat{1}_B \right\rangle \\ &= \sum \langle \hat{0}_A.\Psi_a, \hat{1}_A \rangle \langle \hat{0}_B.\Psi_b, \hat{1}_B \rangle \\ &= \sum K_A(\Psi_a) K_B(\Psi_b) = (K_A \otimes K_B)(\Delta\Psi) = (K_A \cdot K_B)(\Psi). \end{aligned}$$

Let  $A = [\hat{0}_A, \hat{1}_A]$  be an interval of  $P^k$  and  $\Psi, \Phi \in NC$ . Using the duality between the coproduct of  $\mathcal{Q}sym$  and the product of  $NC$ , we have

$$\begin{aligned} (\Delta K_A)(\Psi \otimes \Phi) &= K_A(\Psi \cdot \Phi) = \langle (\hat{0}_A.\Psi).\Phi, \hat{1}_A \rangle \\ &= \left\langle \sum_y (K_{[\hat{0}_A,y]}(\Psi)y).\Phi, \hat{1}_A \right\rangle \\ &= \left\langle \sum_{x,y} K_{[\hat{0}_A,y]}(\Psi) K_{[y,x]}(\Phi)x, \hat{1}_A \right\rangle \\ &= \sum_{y \in A} K_{[\hat{0}_A,y]}(\Psi) K_{[y,\hat{1}_A]}(\Phi) = K_{\Delta A}(\Psi \otimes \Phi). \quad \blacksquare \end{aligned}$$

The map  $K$  is a generating function for the enumerative combinatorial invariants associated to the  $NC$ -structure of  $\mathbb{Z}P$ . Let  $\{a_\alpha\}$  be a graded basis of  $NC$  and let  $\{b_\alpha\}$  be the corresponding dual basis in  $\mathcal{Q}sym$ . Then

$$(2.1) \quad K_{[x,y]} = \sum_\alpha \langle x.\bar{a}_\alpha, y \rangle b_\alpha.$$

We interpret the coefficient of  $b_\alpha$  in  $K_{[x,y]}$  as the number of paths from  $x$  to  $y$  satisfying some condition imposed by  $a_\alpha$ .

We reformulate the Equation 2.1 in terms of the Cauchy element of Gelfand *et. al.* [13]. This element relates each graded basis of the Hopf algebra  $NC = \bigoplus_{n \geq 0} NC_n$  to its corresponding dual basis in the Hopf algebra  $\mathcal{Q}sym = \bigoplus_{n \geq 0} \mathcal{Q}sym_n$ . More precisely, let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  with  $\ell \geq 0$  be a sequence of positive integers. Such a sequence is a *composition* of  $n$ , denoted  $\alpha \models n$ , if  $n = \sum_{i=1}^\ell \alpha_i$ . By convention the empty sequence for  $\ell = 0$  is the unique composition of 0. The complete  $NC$ -functions  $\{S^\alpha = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell}\}_{\alpha \models n}$  and the ribbon  $NC$ -functions  $\{R_\alpha\}_{\alpha \models n}$  form

two bases of  $NC_n$ . Similarly the monomial quasi-symmetric functions  $\{M_\alpha\}_{\alpha \models n}$  and the complete quasi-symmetric functions  $\{F_\alpha\}_{\alpha \models n}$  form two bases of  $\mathcal{Qsym}_n$ . In the graded completion of  $\bigoplus_{n \geq 0} NC_n \otimes \mathcal{Qsym}_n$  we have the following *Cauchy element*:

$$(2.2) \quad \mathcal{C} := \sum_{\alpha} a_{\alpha} \otimes b_{\alpha} = \sum_{\alpha} R_{\alpha} \otimes F_{\alpha} = \sum_{\alpha} S^{\alpha} \otimes M_{\alpha},$$

where  $\{a_{\alpha}\}$  and  $\{b_{\alpha}\}$  is any pair of dual graded bases.

The right action of  $NC$  on  $\mathbb{Z}P$  extends linearly to an action of the graded completion of  $\bigoplus_{n \geq 0} NC_n \otimes \mathcal{Qsym}_n$  on the completion of  $\mathcal{Qsym} \otimes_{\mathbb{Z}} \mathbb{Z}P$ . The following theorem is simply a reformulation of Equation 2.1.

**Theorem 2.3.** *For any family of Pieri operators on a poset  $P$  and  $x, y \in P$ ,*

$$K_{[x,y]} = \langle x.\mathcal{C}, y \rangle.$$

Expanding the quasi-symmetric function  $K_{[x,y]}$  in a basis of  $\mathcal{Qsym}$  gives a family of enumerative combinatorial invariants for the given action of  $NC$  on the poset  $P$ . In this way, the functions  $K_{[x,y]}$  are seen to be analogues of the skew Schur functions presented at the beginning of this section.

### 3. PIERI OPERATORS ON GRAPHS

We extend this simple construction on graded posets to (locally finite) (locally finite) oriented multigraphs. Let  $G = (V, E)$  be a multigraph where  $V$  is the set of vertices and  $E$  is a function  $V \times V \rightarrow \mathbb{Z}^+$  such that

$$\sum_{y' \in V} E(x, y') \quad \text{and} \quad \sum_{x' \in V} E(x', y)$$

are both finite for all  $x \in V$ . The value  $E(x, y)$  identifies the number of arrows from  $x$  to  $y$  in  $G$ . The function  $E$  is the incidence matrix of the graph  $G$ , and  $E^r$  is the matrix product of  $r$  copies of  $E$ . Given  $x, y \in V$ , let  $[x, y]$  be the set of all paths from  $x$  to  $y$ . Consider a graded version of this set.

$$[x, y] = \bigcup_{r \geq 0} [x, y]^{(r)},$$

where the interval  $[x, y]^{(r)}$  is the set of all paths of length  $r$  from  $x$  to  $y$ . Note that  $|[x, y]^{(r)}| = E^r(x, y)$  is finite.

Let  $\mathbb{Z}G$  denote the free  $\mathbb{Z}$ -module generated by  $V$ . Here, a *Pieri operator* is a linear map  $\bar{h}_k: \mathbb{Z}G \rightarrow \mathbb{Z}G$  where for all  $x \in V$ , the support of  $x.\bar{h}_k \in \mathbb{Z}G$  consists of elements  $y \in V$  such that  $E^k(x, y) > 0$ . As before, a family  $\{\bar{h}_k\}_{k > 0}$  of Pieri operators induces on  $\mathbb{Z}G$  the structure of an  $NC$ -module. We thus obtain a collection of linear maps  $K_{[x,y]^{(r)}}: NC \rightarrow \mathbb{Z}$  given by  $\Psi \mapsto \langle x.\Psi^{(r)}, y \rangle$  where  $\Psi^{(r)}$  is the  $r$ th-homogeneous component of  $\Psi$ , and thus quasi-symmetric functions  $K_{[x,y]^{(r)}} \in \mathcal{Qsym}_r$ .

We define a Hopf algebra  $\mathcal{H}G$  associated to  $G$ . Define the product of two intervals by  $[x, y]^{(r)} \times [u, v]^{(s)} := [(x, u), (y, v)]^{(r+s)}$ , an interval in  $G \times G$ . Let  $\mathcal{H}G$  be the free  $\mathbb{Z}$ -module with basis given by products of intervals  $[x, y]^{(r)}$  in  $G$ , modulo identifying all intervals  $[x, x]^{(0)}$  with the unit 1 and all empty intervals  $[x, y]^{(r)}$  with zero. If we

let  $r$  be the degree of an element  $[x, y]^{(r)}$ , then  $\mathcal{H}G$  is a graded  $\mathbb{Z}$ -algebra with product  $\times$ . The algebra  $\mathcal{H}G$  has a natural coalgebra structure induced by

$$\Delta[x, y]^{(r)} = \sum_{s=0}^r \sum_{z \in V} [x, z]^{(s)} \otimes [z, y]^{(r-s)}.$$

The counit is again the projection onto  $\mathbb{Z}$  of the degree 0 component. Since the bialgebra  $\mathcal{H}G$  is graded, we have the following theorem.

**Theorem 3.1.** *For any oriented multigraph  $G$ ,  $\mathcal{H}G$  is a Hopf algebra.*

Suppose we have a family of Pieri operators on a graph  $G$ . As in Section 2, we have an action of  $NC$  on the  $\mathbb{Z}$ -linear span of  $G^k$ , for any positive integer  $k$ . Since the generators of  $\mathcal{H}G$  are sets of the form  $[w, z]^{(t)}$  in  $G^k$ , we may define the quasi-symmetric function  $K$  on each of these generators of  $\mathcal{H}G$ , and then extend by linearity to  $\mathcal{H}G$  itself, obtaining a  $\mathbb{Z}$ -linear graded map  $K : \mathcal{H}G \rightarrow \mathcal{Qsym}$ . We leave to the reader the straightforward extension of Theorem 2.2.

**Theorem 3.2.** *The map  $K : \mathcal{H}G \rightarrow \mathcal{Qsym}$  is a morphism of Hopf algebras.*

To extend Theorem 2.3, we decompose the Cauchy element  $\mathcal{C}$  into its homogeneous components:  $\mathcal{C} = \sum_{r \geq 0} \mathcal{C}_r$ . For example, we can use  $\mathcal{C}_r = \sum_{\alpha \models r} S^\alpha \otimes M_\alpha$ . The following is immediate.

**Theorem 3.3.** *For any family of Pieri operators on a graph  $G$  and  $x, y \in G$ ,*

$$K_{[x, y]^{(r)}} = \langle x \cdot \mathcal{C}_r, y \rangle.$$

**Remark 3.4.** These constructions generalise those of Section 2. Given a ranked poset  $P$  we associate to it the incidence graph  $G_P = (P, E)$  where  $E(x, y) = 1$  if  $y$  covers  $x$  and  $E(x, y) = 0$  otherwise. The intervals  $[x, y]^{(r)}$  are empty unless  $r = rk(y) - rk(x)$  in which case  $[x, y]^{(r)}$  is equal to the saturated chains in  $[x, y]$ . In such a case we omit the superscript  $(r)$  and arrive again at the results of Section 2.

#### 4. THREE SIMPLE EXAMPLES

We give three simple examples to illustrate our theory.

**Example 4.1.** *Simple path enumeration.* Given a graph  $G$ , define the Pieri operator  $\bar{h}_k : \mathbb{Z}G \rightarrow \mathbb{Z}G$  by

$$x \cdot \bar{h}_k = \sum_{y \in G} E^k(x, y)y.$$

This action of  $NC$  satisfies  $\bar{h}_a \bar{h}_b = \bar{h}_{a+b}$  for all  $a, b \in \mathbb{Z}^+$ . From this we deduce that  $K_{[x, y]^{(r)}} = E^r(x, y) \sum_{\alpha \models r} M_\alpha$ . Thus  $K$  simply enumerates all paths of length  $r$  from  $x$  to  $y$ . When  $G$  is a graded poset  $P$ ,  $E^r(x, y) = 0$  unless  $r = rk(y) - rk(x)$  and  $x \leq y$  in  $P$ . In this case,  $E^r(x, y)$  counts the saturated chains in the interval  $[x, y]$ .

**Example 4.2.** *Skew Schur functions.* Let  $(P, <)$  be Young's lattice of partitions. For  $\mu \in P$ , define  $\mu \cdot \bar{h}_k$  to be the sum of all partitions  $\lambda$  such that  $\lambda/\mu$  is a row strip and  $|\lambda| - |\mu| = k$ . This family of Pieri operators lifts the action of the algebra  $\Lambda$  of symmetric functions on itself. It follows that  $K_{[\mu, \lambda]}$  is the skew Schur function  $S_{\lambda/\mu}$ .

**Example 4.3.** *Rank selection Pieri operators and flag  $f$ -vectors.* Given any ranked poset  $P$ , consider the Pieri operator obtained by setting  $x.\bar{h}_k$  equal to the sum of all  $y > x$  such that  $rk(y) - rk(x) = k$ . In this case,  $\langle x.\bar{S}^\alpha, y \rangle$  counts all chains in the rank-selected poset obtained from  $[x, y]$  with ranks given by  $\alpha$ , and  $K$  is Ehrenborg's flag  $f$ -vector quasi-symmetric generating function [11].

### 5. STRUCTURE FROM HOPF SUBALGEBRAS

Suppose, for a family of posets, we have a class of enumerative combinatorial invariants which possesses some additional structure. In many situations, the associated families of Pieri operators satisfy some relations, and the resulting actions of  $NC$  are carried by a Hopf quotient of  $NC$ . Equivalently, the images of  $K$  lie in the dual of this quotient, a Hopf subalgebra of  $\mathcal{Qsym}$ .

More precisely, let an action of  $NC$  on  $\mathbb{Z}G$  be given by a homomorphism  $\phi$  from  $NC$  to the linear endomorphism ring  $\text{End}(\mathbb{Z}G)$ . Let  $\mathcal{I}$  be an ideal generated by some relations satisfied by the Pieri operators. When  $\mathcal{I}$  is a Hopf ideal, so that we have  $\Delta(\mathcal{I}) \subset \mathcal{I} \otimes NC + NC \otimes \mathcal{I}$ , we have the commuting diagram

$$\begin{array}{ccc}
 NC & \xrightarrow{\phi} & \text{End}(\mathbb{Z}G) \\
 \downarrow & \nearrow \phi^* & \\
 NC/\mathcal{I} & & 
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ccc}
 \mathcal{H}G & \xrightarrow{K} & \mathcal{Qsym} \\
 & \searrow K & \cup \\
 & & (NC/\mathcal{I})^*
 \end{array}$$

as the functions  $K_{[w,z](t)}$  are characters of representations  $\phi^{\otimes k}$  on  $\mathbb{Z}G^k$ .

In particular, Equation 2.1 has a more specialised form. Given a basis  $\{c_\lambda\}$  of  $NC/\mathcal{I}$  and  $\{d_\lambda\}$  its dual basis inside  $\mathcal{Qsym}$ , we have

$$(5.1) \quad K_{[x,y](r)} = \sum_{\lambda} \langle x.c_\lambda, y \rangle d_\lambda$$

where the sum is only over the index set of a basis for  $NC/\mathcal{I}$ . Here the numbers  $\langle x.c_\lambda, y \rangle$  are specialisations of the enumerative invariants in Equation 2.1.

We illustrate these principles in a series of examples which introduce certain classes of Pieri operators defined by quotients of  $NC$ .

**Example 5.1.** *Simple path enumeration.* In Example 4.1, the ideal  $\mathcal{I}$  is generated by  $h_{a+b} - h_a h_b$  for all  $a, b > 0$ . This is not a Hopf ideal since

$$\begin{aligned}
 \Delta(h_2 - h_1 h_1) &= h_2 \otimes 1 + h_1 \otimes h_1 + 1 \otimes h_2 - (h_1 \otimes 1 + 1 \otimes h_1)^2 \\
 &= (h_2 - h_1 h_1) \otimes 1 + 1 \otimes (h_2 - h_1 h_1) - h_1 \otimes h_1,
 \end{aligned}$$

which is not contained in  $\mathcal{I} \otimes NC + NC \otimes \mathcal{I}$ .

**Example 5.2.** *Symmetric Pieri operators.* A family of Pieri operators is *symmetric* if  $\bar{h}_a \bar{h}_b = \bar{h}_b \bar{h}_a$  for all  $a, b > 0$ . In this case,  $NC/\mathcal{I} \cong \mathbb{Z}[h_1, h_2, \dots]$ , which is the self-dual Hopf algebra  $\Lambda$  of symmetric functions (see [19, 27]), and thus  $\mathcal{I}$  is a Hopf ideal. Symmetric Pieri operators satisfy  $x.S^\alpha = x.S^\beta$  whenever  $\alpha$  and  $\beta$  determine the same

partition, and hence by Equation 2.1 we can write  $K_{[x,y]}$  in the form

$$\sum_{\lambda \vdash r} A^\lambda \sum_{\lambda(\alpha)=\lambda} M_\alpha$$

where  $r$  is the rank of the interval  $[x, y]$ ,  $\lambda(\alpha)$  is the partition determined by  $\alpha$ , and  $A^\lambda$  is some constant. By definition,  $\sum_{\lambda(\alpha)=\lambda} M_\alpha$  is the symmetric function  $m_\lambda$ , and so we see again that the image of  $K$  lies in  $\Lambda$ . Symmetric Pieri operators can be found in Example 4.2 and in Sections 6 and 7.

It is interesting to use other known dual bases of  $\Lambda$  in Equation 5.1, in particular, its self-dual basis  $\{S_\lambda\}$  of Schur functions.

**Example 5.3.** *Flag  $f$ -vectors of Eulerian posets.* Consider Example 4.3 when the given ranked poset  $P$  is Eulerian. The flag  $f$ -vectors of Eulerian posets satisfy the *linear* generalised Dehn-Sommerville or Bayer-Billera relations [1]. Billera and Liu [9, Proposition 3.3] show that the ideal of relations satisfied by such Pieri operators are generated by the (even) Euler relations

$$(5.2) \quad \sum_{i+j=2n} (-1)^i \bar{h}_i \bar{h}_j = 2\bar{h}_{2n} + \sum_{i=1}^{2n-1} (-1)^i \bar{h}_i \bar{h}_{2n-i} = 0,$$

where  $n$  is a positive integer. As in [9], let  $\mathcal{I}$  be the ideal of  $NC$  generated by

$$X_{2n} := \sum_{i+j=2n} (-1)^i h_i h_j = 2h_{2n} + \sum_{i=1}^{2n-1} (-1)^i h_i h_{2n-i}.$$

Then we have the following algebra isomorphism

$$\mathbb{Q} \otimes NC / \mathcal{I} \cong \mathbb{Q}\langle y_1, y_3, y_5, \dots \rangle.$$

where  $y_i$  has degree  $i$ . We identify the dual  $(NC/\mathcal{I})^*$  inside  $\mathcal{Q}_{sym} \otimes_{\mathbb{Q}}$  to be the peak Hopf algebra  $\Pi$  introduced by Stembridge [28] in his study of enriched P-partitions. This shows that  $\mathcal{I}$  is a Hopf ideal over  $\mathbb{Q}$ .

**Theorem 5.4.**  $(\mathbb{Q} \otimes NC / \mathcal{I})^* \cong \mathbb{Q} \otimes \Pi$ .

Let us clarify some notation for the proof of Theorem 5.4. Given compositions  $\alpha, \beta \models m$ , write  $\beta \preceq \alpha$  if  $\beta$  is a refinement of  $\alpha$  and let  $\beta^*$  be the refinement of  $\beta$  obtained by replacing all components  $\beta_i > 1$  of  $\beta$  for  $i > 1$  with  $[1, \beta_i - 1]$ . Given a composition  $\alpha \models m$  such that  $\alpha_1 > 1$  if  $n > 1$ , the Billey-Haiman shifted quasi-symmetric functions [10] are shown [2] to have the formula

$$(5.3) \quad \theta_\alpha = \sum_{\substack{\beta \models m \\ \beta^* \preceq \alpha}} 2^{k(\beta)} M_\beta,$$

where  $k(\beta)$  is the number of components of  $\beta$ .

If  $\alpha$  is a composition with all components greater than 1, except perhaps the last, then  $\theta_\alpha$  a *peak* function. In [28] Stembridge shows that the linear span  $\Pi$  of the peak functions is a subalgebra of  $\mathcal{Q}_{sym}$ . In fact,  $\Pi$  is a Hopf subalgebra of  $\mathcal{Q}_{sym}$  [2].



Recall that in the identification of  $\mathcal{Q}sym$  as the graded linear dual of  $NC$ , the families  $\{M_\alpha\}$  and  $\{S^\alpha\}$  are dual bases. That is,  $M_\alpha(S^\beta) = 1$  if  $\alpha = \beta$  and 0 otherwise. Given any two compositions  $\eta = [\eta_1, \eta_2, \dots]$  and  $\epsilon = [\epsilon_1, \epsilon_2, \dots]$ , let  $\eta \cdot \epsilon$  be the concatenation  $[\eta_1, \eta_2, \dots, \epsilon_1, \epsilon_2, \dots]$ .

**Lemma 5.5.** *The peak algebra  $\Pi$  annihilates the ideal  $\mathcal{I}$ .*

*Proof.* We show that a peak function  $\theta_\alpha$  annihilates any function of the form  $S^\beta X_{2n} S^\gamma$ . Since  $M^\eta(S^\epsilon) = 0$  unless  $\eta = \epsilon$ , it follows that we need only study those summands  $2^{k(\delta)} M^\delta$  in  $\theta_\alpha$  such that either  $\delta = \beta \cdot 2n \cdot \gamma$  or else  $\delta = \beta \cdot i \cdot (2n - i) \cdot \gamma$ . Now if  $2^{k(\beta \cdot 2n \cdot \gamma)} M^{\beta \cdot 2n \cdot \gamma}$  is a summand of  $\theta_\alpha$ , then it follows that all summands of the form  $2^{k(\beta \cdot i \cdot (2n - i) \cdot \gamma)} M^{\beta \cdot i \cdot (2n - i) \cdot \gamma}$  will also belong to  $\theta_\alpha$ . By the Euler relations 5.2, it follows immediately that  $S^\beta X_{2n} S^\gamma$  is annihilated by  $\theta_\alpha$ .

We are left to consider the case where  $2^{k(\beta \cdot i \cdot (2n - i) \cdot \gamma)} M^{\beta \cdot i \cdot (2n - i) \cdot \gamma}$  is a summand of  $\theta_\alpha$  but not  $2^{k(\beta \cdot 2n \cdot \gamma)} M^{\beta \cdot 2n \cdot \gamma}$ . Observe that from the definition 5.3 we must have  $n > 1$  since if  $2^{k(\beta \cdot 1 \cdot 1 \cdot \gamma)} M^{\beta \cdot 1 \cdot 1 \cdot \gamma}$  is a summand of  $\theta_\alpha$  then  $2^{k(\beta \cdot 2 \cdot \gamma)} M^{\beta \cdot 2 \cdot \gamma}$  will be too as  $(\beta \cdot 1 \cdot 1 \cdot \gamma)^* = (\beta \cdot 2 \cdot \gamma)^*$ . Suppose  $\beta \models m$  and let  $j$  be such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_{j-1} \leq m < \alpha_1 + \alpha_2 + \dots + \alpha_j.$$

Then  $M^{\beta \cdot i \cdot (2n - i) \cdot \gamma}$  is in the support of  $\theta_\alpha$  if and only if  $\alpha_1 + \alpha_2 + \dots + \alpha_j$  is  $m + i$  or  $m + i + 1$ . If it is  $m + i$ , then for  $i \neq 1$

$$\theta_\alpha(S^\beta X_{2n} S^\gamma) = \theta_\alpha((-1)^i S^{\beta \cdot i \cdot (2n - i) \cdot \gamma} + (-1)^{(i-1)} S^{\beta \cdot i - 1 \cdot (2n - i + 1) \cdot \gamma}) = 0.$$

If  $i = 1$ , then  $(\beta \cdot 1 \cdot (2n - 1) \cdot \gamma)^* \preceq \alpha$ . Since  $\theta_\alpha$  is a peak function and  $n > 1$ , we must have  $\alpha_{j+1} > 1$ . This implies that  $2n - 1 \leq \alpha_{j+1}$ , hence  $(\beta \cdot 2n \cdot \gamma)^* \preceq \alpha$  and  $2^{k(\beta \cdot 2n \cdot \gamma)} M^{\beta \cdot 2n \cdot \gamma}$  is a summand of  $\theta_\alpha$ , which contradicts our assumption.

A similar argument for  $m + i + 1$  completes the proof of the lemma.  $\blacksquare$

*Proof of Theorem 5.4.* By Lemma 5.5,  $\mathbb{Q} \otimes \Pi \subseteq (\mathbb{Q} \otimes NC / \mathcal{I})^* \cong (\mathbb{Q} \langle y_1, y_3, y_5, \dots \rangle)^*$ . This containment is an equality since the dimension of the  $i$ th homogeneous component of both  $\Pi$  [28] and  $\mathbb{Q} \langle y_1, y_3, y_5, \dots \rangle$  [9] is the  $i$ th Fibonacci number.  $\blacksquare$

**Definition 5.6.** Pieri operators are symmetric if the image of  $K$  lies within the algebra  $\Lambda$  of symmetric functions. Similarly, Pieri operators are *Eulerian* if the image of  $K$  lies within  $\Pi_{\mathbb{Q}}$ , the  $\mathbb{Q}$ -span of  $\Pi$ . This occurs if there is some scalar multiple  $\alpha_k \bar{h}_k$  of each Pieri operator such that the  $\alpha_k \bar{h}_k$  satisfy the Euler relations 5.2.

We solve the conjecture presented in [2] related to the general functions  $\theta_\alpha$  introduced by Billey and Haiman [10]. Let  $\Xi$  be the  $\mathbb{Q}$ -linear span of all the  $\theta_\alpha$ .

**Theorem 5.7.** *The space  $\Xi$  is a Hopf subalgebra of  $\mathbb{Q} \otimes \mathcal{Q}sym$ . Moreover the set*

$$\mathcal{J} = \{\Psi \in NC \mid \theta(\Psi) = 0 \text{ for all } \theta \in \Xi\}$$

*is the principal ideal generated by  $X_2 = 2h_2 - h_1h_1$ .*

*Proof.* We first show that  $\mathcal{J}$  is an ideal and it is included in  $\mathcal{I} = \langle X_{2n} \rangle$ , the ideal generated by the Euler relations. By Theorem 3.2 of [2],  $\Xi$  is a coalgebra. Hence  $\Xi^* = NC / \mathcal{J}$  is an algebra, which shows that  $\mathcal{J}$  is an ideal. Since  $\Pi \subset \Xi$  we have

that  $\mathcal{J} \subset \mathcal{I}$ . Now it is straightforward to check that  $X_2 \in \mathcal{J}$ . Let  $\hat{\mathcal{J}} \subseteq \mathcal{J}$  be the principal ideal generated by  $X_2$ . Since  $\Delta(X_2) = 1 \otimes X_2 + X_2 \otimes 1$  we have that  $\hat{\mathcal{J}}$  is a Hopf ideal and  $NC/\hat{\mathcal{J}}$  is a Hopf algebra. Its dual  $(NC/\hat{\mathcal{J}})^*$  is a Hopf subalgebra of  $\mathcal{Q}_{sym}$  contained in  $\Xi$ . To conclude our argument, we show that the dimension of the homogeneous components of degree  $n$  in  $NC/\hat{\mathcal{J}}$  and  $\Xi$  are equal for all  $n$ . In  $NC/\hat{\mathcal{J}}$ , the homogeneous component of degree  $n$  has dimension given by the number of compositions of  $n$  that contain no part equal to 2. This satisfies the recurrence  $\pi_n = \pi_{n-1} + \pi_{n-2} + \pi_{n-4}$  with initial conditions  $\pi_1 = 1, \pi_2 = 1, \pi_3 = 2$  and  $\pi_4 = 4$ . This is exactly the recurrence of Theorem 4.3 in [2] given for calculating the dimension of  $\Xi$ . Hence  $(\mathbb{Q} \otimes NC/\hat{\mathcal{J}})^* = \Xi$  is a Hopf algebra and  $\hat{\mathcal{J}} = \mathcal{J}$ .  $\blacksquare$

## 6. DESCENT PIERI OPERATORS

**Definition 6.1.** An (edge)-labelled poset is a graded poset  $P$  whose covers (edges of its Hasse diagram) are labelled with integers. To enumerate chains according to the descents in their sequence of (edge) labels, we use the *descent Pieri operator*:

$$x.\bar{h}_k := \sum_{\omega} \text{end}(\omega),$$

the sum over all chains  $\omega$  of length  $k$  starting at  $x$

$$\omega : x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} x_k =: \text{end}(\omega)$$

with no descents, that is  $b_1 \leq b_2 \leq \cdots \leq b_k$ . The resulting quasi-symmetric function  $K_P$  was studied in [4], where (with some effort), it was shown to give a Hopf morphism from a reduced incidence Hopf algebra to  $\mathcal{Q}_{sym}$ . We may likewise have edge-labelled graphs, and define descent Pieri operators in that context.

To a subset  $S = \{j_1 < j_2 < \cdots, j_k\}$  of  $[n-1]$ , we associate the composition  $(j_1, j_2 - j_1, \dots, n - j_k)$ . Given a saturated chain  $\omega$  in  $P$  with labels  $b_1, b_2, \dots, b_n$ , let  $D(\omega)$  be the *descent composition* of  $\omega$ , the composition associated to the descent set  $\{i \mid b_i > b_{i+1}\}$  of  $\omega$ . Then (Equation 4 of [4]) we have

$$(6.1) \quad K_{[x,y]} = \sum F_{D(\omega)},$$

the sum over all saturated chains  $\omega$  in the interval  $[x, y]$ , where  $F_{\alpha}$  is the complete (or fundamental) quasi-symmetric function.

If we label a cover  $\mu \lessdot \lambda$  in Young's lattice consistently by either the column or content of the box in  $\lambda/\mu$ , then the descent Pieri operator coincides with the Pieri operator of Example 4.2.

**Example 6.2.** *k-Bruhat order and skew Schubert functions.* The Pieri-type formula for the classical flag manifold [18, 23] suggests a symmetric Pieri operator on a suborder of the Bruhat order on the symmetric group, which encodes the structure of the cohomology of the flag manifold as a module over the ring of symmetric polynomials. Let  $\mathcal{S}_n$  denote the symmetric group on  $n$  elements and let  $\ell(w)$  be the length of a permutation  $w$  in this Coxeter group.

We define the  $k$ -Bruhat order  $<_k$  by its covers. Given permutations  $u, w \in \mathcal{S}_n$ , we say that  $u <_k w$  if  $\ell(u) + 1 = \ell(w)$  and  $u^{-1}w = (i, j)$ , where  $(i, j)$  is a reflection with  $i \leq k < j$ . When  $u <_k w$ , we write  $wu^{-1} = (a, b)$  with  $a < b$  and label the cover  $u <_k w$  in the  $k$ -Bruhat order with the integer  $b$ .

The descent Pieri operators on this labelled poset are symmetric as  $\bar{h}_m$  models the action of the Schur polynomial  $h_m(x_1, \dots, x_k)$  on the basis of Schubert classes (indexed by  $\mathcal{S}_n$ ) in the cohomology of the flag manifold  $SL(n, \mathbb{C})/B$ . We also have

$$K_{[u,w]} = \sum_{\lambda} c_{u,(\lambda,k)}^w S_{\lambda}$$

where  $c_{u,(\lambda,k)}^w$  is the coefficient of the Schubert polynomial  $\mathfrak{S}_w$  in the product  $\mathfrak{S}_u \cdot S_{\lambda}(x_1, \dots, x_k)$ . This is the skew Schubert function  $S_{wu^{-1}}$  of [6]. Geometry shows these coefficients  $c_{u,(\lambda,k)}^w$  are non-negative. It is an important open problem to give a combinatorial or algebraic proof of this fact.

**Example 6.3.** *The weak order on  $\mathcal{S}_n$  and Stanley symmetric functions.* The weak order on the symmetric group  $\mathcal{S}_n$  is the labelled poset whose covers are  $w < w(i, i+1)$ , with label  $i$  if  $\ell(w) + 1 = \ell(w(i, i+1))$ . In [4], it is shown that the descent Pieri operators on this labelled poset are symmetric and  $K_{[u,w]}$  is the Stanley symmetric function or stable Schubert polynomial  $F_{wu^{-1}}$ , introduced by Stanley to study reduced decompositions of chains in the weak order on  $\mathcal{S}_n$  [25].

**Example 6.4.** *Non-commutative Schur functions of Fomin and Greene.* Fomin and Greene have a theory of *combinatorial representations* of certain non-commutative Schur functions [12]. These are a different non-commutative version of symmetric functions than  $NC$ . Using the Cauchy element in their algebra, they obtain symmetric functions  $F_{y/x}$  which include Schur functions, Stanley symmetric functions, stable Grothendieck polynomials, and others. A combinatorial representation gives rise to an edge-labelled directed graph so that the functions  $F_{y/x}$  of Fomin-Greene are the functions  $K_{[x,y]}$  coming from the descent Pieri operators on this structure.

Let  $FG_n$  be the quotient of the free associative algebra  $\mathbb{Z}\langle u_1, u_2, \dots, u_n \rangle$  by the two-sided ideal generated by the following relations

$$(6.2) \quad \begin{aligned} u_i u_k u_j &= u_k u_i u_j, & i \leq j < k & \quad |i - k| \geq 2 \\ u_j u_i u_k &= u_j u_k u_i, & i < j \leq k & \quad |i - k| \geq 2 \\ (u_i + u_{i+1})u_{i+1}u_i &= u_{i+1}u_i(u_i + u_{i+1}). \end{aligned}$$

In  $FG_n \times \mathbb{Z}[z_1, z_2, \dots, z_m]$  define the non-commutative Cauchy element to be

$$\psi := \prod_{i=1}^m \prod_{j=1}^n (1 + z_i u_j).$$

Let  $R$  be any set whose cardinality is at most countable, and let  $\mathbb{Z}R$  be the free abelian additive group with basis consisting of the elements of  $R$ . A representation of  $FG_n$  on  $\mathbb{Z}R$  is *combinatorial* if for all  $x \in R$ , we have  $x \cdot u_i \in R \cup \{0\}$ . Given a combinatorial representation of  $FG_n$  on  $\mathbb{Z}R$  and  $x, y \in R$ , set

$$F_{y/x} := \langle x \cdot \psi, y \rangle.$$

We define an edge-labelled directed multigraph  $\mathfrak{R}$  with vertex set  $R$  for which  $F_{y/x}$  is the quasi-symmetric function coming from the descent Pieri operator on that structure. We construct  $\mathfrak{R}$  by drawing an edge with label  $-i$  from  $x$  to  $x.u_i$

$$x \xrightarrow{-i} x.u_i$$

if  $x.u_i \neq 0$ . Considering the descent Pieri operators on  $\mathfrak{R}$ , we have the following.

**Theorem 6.5.** *For every  $x, y \in R$ ,  $F_{y/x} = K_{[x,y]}(z_1, \dots, z_m, 0)$ .*

**Remark 6.6.** We identify the generators  $z_i$  in  $\mathbb{Z}[z_1, \dots, z_m]$  with those in the algebra  $\mathcal{Qsym}$  generated by the indeterminates  $z_1, z_2, \dots$ .

Before we prove Theorem 6.5, we recall some results from [12]. Define

$$e_k(\mathbf{u}) := \sum_{i_1 > i_2 > \dots > i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$$

and for a partition  $\lambda = (\lambda_1, \dots, \lambda_m)$  set  $e_\lambda(\mathbf{u}) := e_{\lambda_1}(\mathbf{u}) \cdots e_{\lambda_m}(\mathbf{u})$ .

**Proposition 6.7** (Fomin-Greene).

- (1) *For any positive integers  $a, b$ , we have  $e_a(\mathbf{u})e_b(\mathbf{u}) = e_b(\mathbf{u})e_a(\mathbf{u})$ .*
- (2)  $\psi = \sum_\lambda m_\lambda(z) e_\lambda(\mathbf{u})$ .

Here,  $m_\lambda(z) := m_\lambda(z_1, \dots, z_m)$  is the monomial symmetric polynomial.

*Proof of Theorem 6.5.* Observe that for  $x \in R$ ,

$$x.e_k(\mathbf{u}) = \sum_{\substack{x \xrightarrow{-i_1} \dots \xrightarrow{-i_k} y \\ i_1 > \dots > i_k}} y = x.\bar{h}_k.$$

From this, and Proposition 6.7 it follows that the Pieri operators are symmetric, that is  $\bar{h}_a \bar{h}_b = \bar{h}_b \bar{h}_a$  for all  $a, b \in \mathbb{Z}^+$ . Hence, as in Example 5.2, we have  $x.S^\alpha = x.S^\beta$  whenever  $\alpha$  and  $\beta$  are two compositions that determine the same partition.

Then

$$\begin{aligned} F_{y/x} &= \langle x.\psi, y \rangle = \sum_\lambda m_\lambda(z) \langle x.e_\lambda(\mathbf{u}), y \rangle \\ &= \sum_\lambda m_\lambda(z) \langle x.S^\lambda, y \rangle \\ &= \langle x.\sum_\alpha M_\alpha(z) S^\alpha, y \rangle \\ &= \sum_r \langle x.\sum_{\alpha \models r} M_\alpha(z) S^\alpha, y \rangle \\ &= \sum_r K_{[x,y]^{(r)}}(z_1, \dots, z_m, 0) \end{aligned}$$

by Theorem 3.3, which by definition is equal to  $K_{[x,y]}(z_1, \dots, z_m, 0)$ .  $\blacksquare$

**Example 6.8.** *P-Partitions.* Let  $P$  be a poset and consider any (vertex) labelling  $\gamma: P \rightarrow \mathbb{N}$  of  $P$ . A  $(P, \gamma)$ -partition is an order preserving function  $f: P \rightarrow \mathbb{N}$  such that if  $x < y$  and  $\gamma(x) > \gamma(y)$ , then  $f(x) < f(y)$ . It is sufficient to check these conditions for covers  $x \lessdot y$  in  $P$ .

Let  $\mathcal{A}(P, \gamma)$  be the set of all  $(P, \gamma)$ -partitions. The *weight enumerator*  $\Gamma(P, \gamma)$  of the labelled poset  $(P, \gamma)$  is

$$\Gamma(P, \gamma) := \sum_{f \in \mathcal{A}(P, \gamma)} \prod_{x \in P} z_{f(x)}.$$

This is obviously quasi-symmetric.

Properties of this weight enumerator are tied up with Stanley's Fundamental Theorem of  $P$ -partitions [24]. Let  $\mathcal{L}(P, \gamma)$  be the set of all linear extensions of  $P$ . A linear extension  $w$  of  $P$  lists the elements of  $P$  in order  $w_1, w_2, \dots, w_n$ , with  $w_i < w_j$  (in  $P$ ) implying  $i < j$ . Here  $n = |P|$ . Let  $D(w, \gamma)$  be the composition of  $n$  associated to the set of descents in the sequence of integers  $\gamma(w_1), \gamma(w_2), \dots, \gamma(w_n)$ .

For a linear ordering  $w$  of  $P$ , the set  $\mathcal{A}(w, \gamma)$  may be identified with the set of all weakly increasing functions  $f: [n] \rightarrow \mathbb{N}$  where if  $\gamma(w_i) > \gamma(w_{i+1})$  then  $f(w_i) < f(w_{i+1})$ . Thus  $\Gamma(w, \gamma)$  is Gessel's fundamental quasi-symmetric function [14]  $F_{D(w, \gamma)}$ .

The Fundamental Theorem of  $P$ -partitions notes that

$$\mathcal{A}(P, \gamma) = \coprod_{w \in \mathcal{L}(P, \gamma)} \mathcal{A}(w, \gamma).$$

This implies that

$$(6.3) \quad \Gamma(P, \gamma) = \sum_{w \in \mathcal{L}(P, \gamma)} \Gamma(w, \gamma) = \sum_{w \in \mathcal{L}(P, \gamma)} F_{D(w, \gamma)}.$$

We show that  $\Gamma(P, \gamma)$  is given by descent Pieri operators on the (graded) poset  $\mathcal{IP}$  of lower order ideals of  $P$  with (edge) labelling induced from the vertex labelling of  $P$ . A subset  $I \subset P$  is a *lower order ideal* of  $P$  if whenever  $x \in I$  and  $y < x$ , then  $y \in I$ . The set  $\mathcal{IP}$  of lower order ideals of  $P$  is ordered by inclusion. We label a cover  $I \lessdot J$  in  $\mathcal{IP}$  with  $\gamma(x)$ , where  $x$  is the unique element  $x \in J \setminus I$ . Then  $\mathcal{L}(P, \gamma)$  is in bijection with the maximal chains of  $\mathcal{IP}$ . Using the descent Pieri operators for this structure, Equation 6.1 shows that every maximal chain of  $\mathcal{IP}$  contributes the summand  $F_{D(w, \gamma)}$  to  $K_{\mathcal{IP}}$  where  $w$  is the linear extension of that chain. Thus

$$K_{\mathcal{IP}} = \Gamma(P, \gamma).$$

The Hopf structure of  $\mathcal{H}(\mathcal{IP})$  was studied by Malvenuto in [20].

**Example 6.9.** *Quantum cohomology of Grassmannian, fusion coefficients, and the Hecke algebra at a root of unity.* Let  $m, p$  be positive integers and let  $\mathcal{C}_{p, m}$  be the set of sequences  $\alpha: 0 < \alpha_1 < \dots < \alpha_p$  which also satisfy  $\alpha_p - \alpha_1 < m + p$ . We order this set of sequences by componentwise comparison to obtain a ranked poset. Given a cover  $\alpha \lessdot \beta$ , there is a unique index  $i$  with  $\alpha_i + 1 = \beta_i$  and  $\alpha_j = \beta_j$  for  $i \neq j$ . We label such a cover with  $\beta_i$ .

The elements of the poset  $\mathcal{C}_{m,p}$  may alternately be described by pairs  $(a, \lambda)$ , where  $a$  is a positive integer and  $\lambda$  is a partition with  $\lambda_{p+1} = 0$  and  $\lambda_1 \leq m$ . We obtain  $(a, \lambda)$  from the sequence  $\alpha$  by

$$\begin{aligned} \{\lambda_1 + p, \dots, \lambda_p + 1\} &\equiv \{\alpha_1, \dots, \alpha_p\} \bmod (m + p), \\ a \cdot (m + p) &= \sum_{i=1}^p \alpha_i - \lambda_i - i. \end{aligned}$$

We may likewise pass from the indexing scheme  $(a, \lambda)$  to sequences  $\alpha$ , as this association is invertible (see [21]).

For  $x \in \mathcal{C}_{m,p}$  and  $0 < k \leq p$ , consider the Pieri operator

$$(6.4) \quad x.\bar{h}_k := \sum \text{end}(\omega),$$

the sum over all chains  $\omega$  of length  $k$  starting at  $x$ , that is

$$\omega : x \xrightarrow{b_1} x_1 \xrightarrow{b_2} \dots \xrightarrow{b_k} x_k =: \text{end}(\omega)$$

whose word has no descents  $b_1 \leq b_2 \leq \dots \leq b_k$  and also satisfies the restriction  $b_k - b_1 < m + p$ . Thus these operators  $\bar{h}_k$  are not an instance of one of the families previously introduced (rank-selection or descent).

These Pieri operators  $\bar{h}_k$  are symmetric as they model the Pieri formula in the quantum cohomology ring [7] of the Grassmannian of  $p$ -planes in  $\mathbb{C}^{m+p}$ . This commutative quantum cohomology ring has a basis  $q^a \sigma_\lambda$  for  $(a, \lambda) \in \mathcal{C}_{m,p}$ , and

$$q^a \sigma_\lambda \cdot \sigma_k = \sum_{(b, \mu)} q^b \sigma_\mu,$$

where the sum is over all indices  $(b, \mu)$  appearing in the product  $(a, \lambda).\bar{h}_k$  (6.4), when it is written in terms of pairs. Thus we have the following formula

$$K_{[(b, \mu), (a, \lambda)]} = \sum_{\nu} c_{\mu, \nu}^\lambda S_\nu,$$

where the sum is over all partitions  $\nu$  which partition  $rk(a, \lambda) - rk(b, \mu)$ , and have  $\nu_{p+1} = 0$  and  $m \geq \nu_1$ . Here,  $c_{\mu, \nu}^\lambda$  is the quantum Littlewood-Richardson coefficient [8], the coefficient of  $q^a \sigma_\lambda$  in the product  $q^b \sigma_\mu \cdot \sigma_\nu$ .

These Pieri operators also model the fusion product in the Verlinde algebra (see [8] for a discussion), and the Pieri formula in the representation rings of Hecke algebras at roots of unity [15]. Geometry and representation theory show that these coefficients  $c_{\mu, \nu}^\lambda$  are non-negative, but a combinatorial proof of this fact is lacking.

## 7. PEAK ENUMERATION AND EULERIAN PIERI OPERATORS

**Definition 7.1.** Let  $\omega$  be a labelled ordered chain, that is

$$\omega : x_0 \xrightarrow{b_1} x_1 \xrightarrow{b_2} \dots \xrightarrow{b_k} x_k.$$

We say that  $\omega$  has a *peak* at  $i$  if  $b_{i-1} \leq b_i > b_{i+1}$ . Let  $\Lambda(\omega)$  be the composition of  $k$  associated to the set of peaks of  $\omega$ . Let  $P$  be a labelled poset. To enumerate chains in intervals  $[x, y]$  of  $P$  according to their peaks, we use the peak enumerator

$$\tilde{K}_{[x,y]} := \sum_{\omega} \theta_{\Lambda(\omega)},$$

the sum over all saturated chains  $\omega$  in the interval  $[x, y]$ . We show this peak enumerator is the quasi-symmetric function  $K_{\delta[x,y]}$  associated to the descent Pieri operators on an enriched structure  $\delta P$  defined on the labelled poset  $P$ .

Given a labelled poset  $P$ , where (for simplicity) we assume that the labels  $b_i$  are positive integers, we define its *doubling*  $\delta P$  to be the labelled directed graph with vertex set  $P$ , where every edge  $x \xrightarrow{b} y$  of  $P$  is doubled, but with one label the negative of the original label

$$x \begin{array}{c} \xrightarrow{b} \\ \xrightarrow{-b} \end{array} y.$$

Such a graded poset whose Hasse diagram has some multiple edges, is called a *réseau*. To define descent Pieri operators on the réseau  $\delta P$ , we say that there is a descent at  $i$  if consecutive labels  $b_i, b_{i+1}$  satisfy either  $b_i > b_{i+1}$  or else  $b_i = b_{i+1} < 0$ .

**Theorem 7.2.** *Let  $P$  be any labelled poset and  $\delta P$  its doubled réseau. Then the modified descent Pieri operators on  $\delta P$  are Eulerian, and we have*

$$K_{\delta[x,y]} = \tilde{K}_{[x,y]} = \sum c_{x,\alpha}^y \theta_{\alpha}.$$

where the sum is only over peak compositions  $\alpha$ .

These combinatorial invariants  $c_{x,\alpha}^y$  of  $\delta P$  enumerate the chains of  $P$  whose peak sets have composition  $\alpha$ .

Before we prove Theorem 7.2, we make some definitions and prove two auxiliary lemmas. For a composition  $\alpha$  of  $n$ , let  $\alpha^+$  be the composition of  $n+1$  obtained from  $\alpha$  by increasing its last part by 1, and  $\alpha \cdot 1$  be the composition of  $n+1$  obtained by appending a part of size 1 to  $\alpha$ . Define linear maps  $\psi, \varphi: \mathcal{Qsym}_n \rightarrow \mathcal{Qsym}_{n+1}$  by

$$\begin{aligned} \psi(M_{\beta}) &:= M_{\beta^+} + 2M_{\beta \cdot 1}, \\ \varphi(M_{\beta}) &:= \delta_{1,\beta_l} M_{\beta^+} + 2M_{\beta \cdot 1}, \end{aligned}$$

where  $\beta_l$  is the last part of  $\beta$  and  $\delta_{1,\beta_l}$  is the Kronecker delta function. Using the relation  $F_{\beta} = \sum_{\alpha \preceq \beta} M_{\alpha}$  between the two bases of  $\mathcal{Qsym}$ , we see that

$$\psi(F_{\beta}) = F_{\beta^+} + F_{\beta \cdot 1}.$$

**Lemma 7.3.**  $\psi(\theta_{\alpha}) = \theta_{\alpha^+}$  and  $\varphi(\theta_{\alpha}) = \theta_{\alpha \cdot 1}$ .

*Proof.* The function  $\theta_{\alpha \cdot 1}$  is the sum of terms  $2^{k(\beta)} M_{\beta}$  for each  $\beta$  satisfying  $\beta^* \preceq \alpha \cdot 1$ . Suppose  $\beta^* \preceq \alpha \cdot 1$ . If  $\beta = \gamma \cdot 1$ , then  $\beta^* = \gamma^* \cdot 1$  and we have  $\gamma^* \preceq \alpha$ . Conversely, if  $\gamma^* \preceq \alpha$ , then  $\beta := \gamma \cdot 1$  satisfies  $\beta^* \preceq \alpha \cdot 1$ . Thus every summand  $2^{k(\gamma)} M_{\gamma}$  of  $\theta_{\alpha}$  contributes a summand  $2 \cdot 2^{k(\gamma)} M_{\gamma \cdot 1}$  of  $\theta_{\alpha \cdot 1}$ .

The other summands  $\beta$  have  $\beta = \gamma \cdot \beta_l$  with  $\beta_l > 1$ . Then  $\beta^* = \gamma^* \cdot 1 \cdot (\beta_l - 1)$ . If  $\beta^* \preceq \alpha$ , then we must have  $\beta_l = 2$ , so that  $\beta = \gamma \cdot 2$ . Then  $\beta^* = \gamma^* \cdot 1 \cdot 1 \preceq \alpha \cdot 1$ ,

which implies that  $(\gamma \cdot 1)^* \preceq \alpha$ . Conversely, if  $(\gamma \cdot 1)^* \preceq \alpha$ , then  $(\gamma \cdot 2)^* \preceq \alpha \cdot 1$ . Thus every summand  $2^{k(\gamma \cdot 1)} M_{\gamma \cdot 1}$  of  $\theta_\alpha$  gives a summand  $2^{k(\gamma \cdot 2)} M_{\gamma \cdot 2}$  of  $\theta_{\alpha \cdot 1}$ .

This shows that  $\theta_{\alpha \cdot 1} = \varphi(\theta_\alpha)$ . The arguments for  $\theta_{\alpha^+}$  are similar, but simpler.  $\blacksquare$

The key lemma relating the peak enumerator on  $P$  and the modified descent Pieri operators on the réseau  $\delta P$  concerns the case when  $P$  is a chain.

**Lemma 7.4.** *Suppose  $\omega$  is a chain. Then  $K_{\delta\omega} = \theta_{\Lambda(\omega)}$ .*

*Proof.* We prove this by induction on the length of the chain  $\omega$ . The initial cases are easy calculations. Let  $b_1 \dots, b_k$  be the word of  $\omega$ , and set  $u$  to be the truncation of  $\omega$  at the penultimate cover, so that  $b_1 \dots, b_{k-1}$  is the word of  $u$ .

Consider first the case where  $b_{k-1} \leq b_k$ . Then every chain  $\gamma$  in  $\delta u$  gives two chains  $\gamma \cdot b_k$  and  $\gamma \cdot \bar{b}_k$  in  $\delta\omega$ . Since  $D(\gamma \cdot b_k) = D(\gamma)^+$  and  $D(\gamma \cdot \bar{b}_k) = D(\gamma) \cdot 1$ , we see that  $K_{\delta\omega} = \psi(K_{\delta u})$ . Similarly, if  $b_{k-2} > b_{k-1} > b_k$ , then considering the last three labels of a chain in  $\delta\omega$  show that  $K_{\delta\omega} = \psi(K_{\delta u})$ . In both cases,  $\Lambda(\omega) = \Lambda(u)^+$  (as the peak sets are the same), and the lemma follows by Lemma 7.3.

Now suppose  $b_{k-2} \leq b_{k-1} > b_k$ . Let  $v$  be the truncation of  $\omega$  at the  $(k-2)$ th position. Let  $\gamma$  be a chain of  $\delta v$  with descent composition  $\alpha$ . Then  $\gamma$  has 4 extensions to chains in  $\delta\omega$ , and 2 have descent composition  $\alpha^+ \cdot 1$  and 2 have descent composition  $\alpha \cdot 2$ . Thus if we define  $\phi: F_\alpha \mapsto 2F_{\alpha^+ \cdot 1} + 2F_{\alpha \cdot 2}$ , then  $\phi(K_{\delta v}) = K_{\delta\omega}$ . A straightforward calculation shows  $\phi(M_\beta) = 2M_{\beta^+ \cdot 1} + 2M_{\beta \cdot 2} + 4M_{\beta \cdot 1, 1}$ , which is  $\varphi(\psi(M_\beta))$ . Thus  $K_{\delta\omega} = \varphi(\psi(K_{\delta v})) = \varphi(K_{\delta u})$ . Since  $\omega$  has a peak at  $n-1$ , we have  $\Lambda(\omega) = \Lambda(u) \cdot 1$ , and so this case follows by Lemma 7.3.  $\blacksquare$

*Proof of Theorem 7.2.* Given an interval  $[x, y]$  in a poset or réseau, let  $ch[x, y]$  be the set of saturated chains in  $[x, y]$ . Let  $K$  be the quasi-symmetric function given by the descent Pieri operators on the réseau  $\delta P$ . Let  $x \leq y$  in  $P$ . Given a chain  $\omega \in ch \delta[x, y]$ , we obtain a chain  $|\omega| \in ch[x, y]$  by replacing each cover in  $\delta[x, y]$  with a negative integer label by the corresponding cover in  $[x, y]$  whose label is positive. Then, by Equation 6.1, we have

$$\begin{aligned} K_{\delta[x, y]} &= \sum_{\omega \in ch \delta[x, y]} F_{D(\omega)} \\ &= \sum_{\beta \in ch[x, y]} \sum_{\omega: |\omega| = \beta} F_{D(\omega)} \\ &= \sum_{\beta \in ch[x, y]} K_{\delta\beta} = \sum_{\beta \in ch[x, y]} \theta_{\Lambda(\beta)} = \tilde{K}_{[x, y]}. \quad \blacksquare \end{aligned}$$

**Example 7.5.** *Enriched  $P$ -partitions.* Stembridge enriches the theory of  $P$ -partitions [28] giving a new class of quasi-symmetric generating functions. Let  $(P, \gamma)$  be a labelled poset and let  $\mathbb{P} = \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3, \dots\}$  be two copies of the positive integers ordered as follows:  $\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \dots$ . An *enriched  $(P, \gamma)$ -partition* is an order-preserving map  $f: P \rightarrow \mathbb{P}$  such that for  $x < y$  in  $P$

- if  $f(x) = f(y) = \bar{k}$  is a barred integer, then  $\gamma(x) < \gamma(y)$ ,
- if  $f(x) = f(y) = k$  is a usual integer, then  $\gamma(x) > \gamma(y)$ .



Let  $\mathcal{E}(P, \gamma)$  be the set of all enriched  $(P, \gamma)$ -partitions and define the weight enumerator

$$\Delta(P, \gamma) = \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{x \in P} z_{f(x)}$$

where  $z_{\bar{k}} = z_k$  for all integers  $k$ . The analogue of Equation 6.3 for enriched  $P$ -partitions is

$$\Delta(P, \gamma) = \sum_{w \in \mathcal{L}(P, \gamma)} \Delta(w, \gamma).$$

We thus need to characterise the quasi-symmetric function corresponding to a linear extension  $(w, \gamma)$  of  $(P, \gamma)$ . A *peak* of the linear extension  $(w, \gamma)$  is an index  $i$  with  $1 < i < |w|$  where  $\gamma(w_{i-1}) < \gamma(w_i) > \gamma(w_{i+1})$ . Stembridge shows that

$$\Delta(w, \gamma) = \theta_{\Lambda(w, \gamma)}$$

where  $\theta_\alpha$  is defined in Equation 5.3 and  $\Lambda(w, \gamma)$  is the composition associated to the set of peaks of the linear extension  $(w, \gamma)$ . We can then generalize the construction we have for  $P$ -partitions. This time we proceed as in Definition 7.1 and consider the descent Pieri operators on the doubled réseau  $\delta\mathcal{IP}$ . By Lemma 7.4, every maximal chain of  $\mathcal{IP}$  contributes exactly  $\Delta(w, \gamma)$  to  $K_{\delta\mathcal{IP}}$ , where  $w$  is the linear extension of  $\mathcal{L}(P, \gamma)$  corresponding to that chain. This shows that

$$K_{\delta\mathcal{IP}} = \Delta(P, \gamma).$$

**Example 7.6.** *Isotropic Pieri formula.* The Pieri-type formulas for the flag manifolds  $SO(2n+1, \mathbb{C})$  and  $Sp(2n, \mathbb{C})$  of [5] each give symmetric Eulerian Pieri operators. These are defined on enrichments of the same subposet of the Bruhat order on the group  $\mathcal{B}_n$  of signed permutations. For an integer  $i$ , let  $\bar{i}$  denote  $-i$ .

We regard  $\mathcal{B}_n$  as a subgroup of the group of permutations on  $\{\bar{n}, \dots, \bar{2}, \bar{1}, 1, \dots, n\}$  as usual. Let  $\ell$  be the length function on the Coxeter group  $\mathcal{B}_n$ . The *0-Bruhat order*  $<_0$  on  $\mathcal{B}_n$  is the labelled poset with covers  $u <_0 w$  if  $\ell(u) + 1 = \ell(w)$  and  $u^{-1}w$  is a reflection with either the form  $(\bar{i}, i)$  or the form  $(\bar{i}, j)(\bar{j}, i)$  for some  $0 < i, j$ . When  $u <_0 w$ , either  $wu^{-1} = (\bar{\beta}, \beta)$  for some  $0 < \beta$  or else  $wu^{-1} = (\bar{\beta}, \bar{\alpha})(\alpha, \beta)$  for some  $0 < \alpha < \beta \leq n$ . We label a such a cover with the (positive) integer  $\beta$ .

Consider the Eulerian descent Pieri operators on the doubled réseaux  $\delta\mathcal{B}_n^0$ . These operators are symmetric, as  $\frac{1}{2}\bar{h}_k$  models the action of the Schur P-polynomial  $p_k$  on the basis of Schubert classes (indexed by  $\mathcal{B}_n$ ) in the cohomology of the flag manifold  $SO(2n+1, \mathbb{C})/B$  [5]. (This is because there are twice as many increasing chains in a doubled interval  $\delta[x, y]$  as peakless chains in the interval  $[x, y]$ , and the coefficient of  $y$  in  $x.p_k$  is this number of peakless chains.)

We modify this descent action of  $NC$  on  $\delta\mathcal{B}_n^0$  by declaring  $h_k$  to act as  $\frac{1}{2}\bar{h}_k$  (which is still integral). These new Pieri operators are still symmetric (as they model the action of  $p_k$ ) and they are still Eulerian, as  $2h_k$  satisfies the Euler relations 5.2. In exact analogy to how the Skew Schubert functions are shown in [6] to be the generating functions for the coefficients  $c_{u, (\lambda, k)}^w$  and given by descent Pieri operators, we have the

following formula

$$K_{[u,w]} = \sum_{\lambda} b_{u,\lambda}^w Q_{\lambda},$$

the sum over all strict partitions  $\lambda$  of  $\ell(w) - \ell(u)$ . Here  $b_{u,\lambda}^w$  is the coefficient of the Schubert class  $\mathfrak{B}_w$  in the product  $\mathfrak{B}_u \cdot P_{\lambda}$ , and  $P_{\lambda}, Q_{\lambda}$  are Schur P- and Q-polynomials, which form dual bases for the symmetric algebra  $\Pi_{\mathbb{Q}} \cap \Lambda$ . The polynomials  $Q_{\lambda}$  appear as  $\sum_{\lambda} P_{\lambda} \otimes Q_{\lambda}$  is the Cauchy element of  $\Pi_{\mathbb{Q}} \otimes \Lambda$ .

For the symplectic flag manifold, we modify the réseau  $\delta\mathcal{B}_n^0$  by erasing the negative edge  $u \xrightarrow[-\beta]{\beta} w$  in a cover when  $wu^{-1} = (\bar{\beta}, \beta)$ . Write  $\mathcal{LB}_n^0$  for the resulting réseau.

It is a slight modification of the 0-Bruhat réseau of [5], and may be used in its place for the combinatorics therein. Let  $\bar{h}_k$  be the descent Pieri operators on  $\mathcal{LB}_n^0$ . This family of Pieri operators is symmetric and Eulerian, as  $\bar{h}_k$  models the action of the Schur Q-polynomial  $q_k$  on the Schubert basis of the cohomology of the flag manifold  $Sp(2n, \mathbb{C})$ , and the Schur Q-polynomials  $q_k$  satisfy the Euler relations. As before, we have the following formula

$$K_{[u,w]} = \sum_{\lambda} c_{u,\lambda}^w P_{\lambda},$$

the sum over all strict partitions  $\lambda$  of  $\ell(w) - \ell(u)$ . Here  $c_{u,\lambda}^w$  is the coefficient of the Schubert class  $\mathfrak{C}_w$  in the product  $\mathfrak{C}_u \cdot Q_{\lambda}$ .

Since every chain in an interval of  $\mathcal{B}_n^0$  has the same number of covers of the form  $(\bar{\beta}, \beta)$ —these count the number  $s(wu^{-1})$  of sign changes between  $u$  and  $w$ , we have

$$K_{\delta[x,y]} = 2^{-s(wu^{-1})} K_{\mathcal{L}[x,y]}.$$

Geometry shows these coefficients  $b_{u,\lambda}^w$  and  $c_{u,\lambda}^w$  are non-negative. It is an important open problem to give a combinatorial or algebraic proof of this fact.

**Example 7.7.** *Stanley symmetric functions of types B, C, and D.* In [10], Billey and Haiman describe the Stanley symmetric function of type  $B$  and  $D$  in terms of peaks of reduced words of elements in the corresponding Coxeter groups.

For  $\mathcal{B}_n$ , the simple transpositions are  $s_0, s_1, \dots, s_{n-1}$ , where  $s_0 = (\bar{1}, 1)$  and if  $i > 0$ , then  $s_i = (\overline{i+1}, \bar{i})(i, i+1)$ . The weak order on  $\mathcal{B}_n$  is the labelled poset whose covers are  $w \prec ws_i$  with label  $i$  if  $\ell(w) + 1 = \ell(ws_i)$ . A reduced word  $\mathbf{a}$  for  $w$  is sequence of labels of a chain in  $\mathcal{B}_n$  from the identity  $e$  to  $w$ . Billey and Haiman define the Stanley symmetric function of type  $B$  to be

$$F_w^B := \sum_{\mathbf{a} \in R(w)} \theta_{\Lambda(\mathbf{a})},$$

where  $R(w)$  is the set of reduced words for  $w$  and  $\Lambda(\mathbf{a})$  is the peak composition of the reduced word  $\mathbf{a}$ . By Theorem 7.2,  $F_w^B$  is the function  $K_{\delta[e,w]}$  obtained from the Eulerian descent operators on the doubled réseau  $\delta\mathcal{B}_n$ . Billey and Haiman establish the formula

$$F_w^B = \sum_{\lambda} f_{\lambda}^w Q_{\lambda},$$

the sum over all strict partitions  $\lambda$  of  $\ell(w)$ , and where  $f_\lambda^w$  counts the reduced words that satisfy a condition imposed by the partition  $\lambda$  (coming from the shifted Edelman-Greene correspondence [16]). Thus the Eulerian descent Pieri operators on  $\delta\mathcal{B}_n$  are also symmetric.

While Billey and Haiman do not define Stanley functions of type  $C$ , one reasonably sets  $F_w^C := 2^{-s(w)}F_w^B$ , where  $s(w)$  is the number of sign changes in the permutation  $w$ . This is just the number of  $s_0$ 's appearing in any reduced word of  $w$ . Let the réseau  $\mathcal{LB}_n^0$  be the modification of the doubled réseau  $\delta\mathcal{B}_n^0$  where we erase the edge with negative label  $\bar{1}$  for covers  $w \lessdot ws_0$ . Then every chain in an interval  $\mathcal{L}[e, w]$  of  $\mathcal{LB}_n^0$  gives rise to  $2^{s(w)}$  chains in  $\delta[x, y]$ , each with the same descents as the original chain. Then Equation 6.1 and Theorem 7.2 show that

$$K_{\mathcal{L}[x,y]} = 2^{-s(w)}K_{\delta[x,y]} = 2^{-s(w)}F_w^B = F_w^C.$$

This shows these descent Pieri operators are Eulerian and symmetric.

The Coxeter group  $\mathcal{D}_n$  has simple reflections  $s_1, s_{\hat{1}}, s_2, \dots, s_{n-1}$ . The weak order on  $\mathcal{D}_n$  is the labelled poset with cover  $w \lessdot ws_i$  labelled by  $i$  if  $\ell(w) + 1 = \ell(ws_i)$ . Here, we set  $\hat{1} < 1$ . A reduced word  $\mathbf{a}$  for  $w$  as before is a chain in  $\mathcal{D}_n$  from  $e$  to  $w$ . Since there are no occurrences of  $1\hat{1}1$  or  $\hat{1}\hat{1}\hat{1}$  in a reduced word, changing all occurrences of  $\hat{1}$  to  $1$  does not change the peaks in a reduced word, and so the type  $D$  Stanley symmetric function of Billey and Haiman satisfy

$$F_w^D = \sum_{\mathbf{a} \in R(w)} 2^{-o(\mathbf{a})} \theta_{\Lambda(\mathbf{a})},$$

where  $o(\mathbf{a})$  counts the number of occurrences of  $1$  and  $\hat{1}$  in the reduced word  $\mathbf{a}$ . Let  $\delta\mathcal{D}_n$  and  $\mathcal{LD}_n$  be the doubled réseau and its modification, erasing all edges with (negative) labels  $-1$  and  $-\hat{1}$ .

**Theorem 7.8.**  $F_w^D = K_{\mathcal{L}[e,w]}$ .

*Proof.* By Theorem 7.2, we have

$$F_w^D = \sum_{\mathbf{a} \in R(w)} 2^{-o(\mathbf{a})} K_{\delta\mathbf{a}}.$$

The theorem follows from Equation 6.1 and the following  $1$  to  $2^{o(\mathbf{a})}$  map from chains in  $\mathcal{L}\mathbf{a}$  to chains in  $\delta\mathbf{a}$ , which preserves descents. When there are no subwords  $1\hat{1}$  or  $\hat{1}1$  in a chain in  $\mathcal{L}\mathbf{a}$ , simply make all possible substitutions of negative and positive labels for each occurrence of  $1$  and  $\hat{1}$ . If however, there is a subword  $1\hat{1}$ , then there is another chain differing from the first only in that subword (having  $\hat{1}1$  instead), and the map uses the substitutions in both chains

$$\begin{aligned} \hat{1}1 &\longrightarrow \hat{1}1, \bar{\hat{1}}1, \bar{1}\hat{1}, \bar{1}\bar{\hat{1}} \\ 1\hat{1} &\longrightarrow 1\hat{1}, 1\bar{\hat{1}}, \hat{1}\bar{1}, \hat{1}\bar{\bar{1}}. \quad \blacksquare \end{aligned}$$

Lastly, we remark that these descent operators on  $\mathcal{LD}_n$  are Eulerian and symmetric, as Billey and Haiman give a formula

$$F_w^D = \sum_{\lambda} e_w^{\lambda} Q_{\lambda},$$

where  $e_w^{\lambda}$  is a rational number that counts certain weighted reduced words.

## APPENDIX A. HOPF ALGEBRAS

A Hopf algebra is an algebra whose linear dual is also an algebra, with some compatibility conditions. They are important in representation theory (and in this paper) because they act on tensor products of their representations. The usefulness of Hopf algebras in combinatorics is apparent from the ubiquity of their applications. In this section, we summarise the basic notion of Hopf algebras.

A  $\mathbb{Z}$ -module  $\mathcal{H}$  is a *coalgebra* if there are two maps  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  (coproduct) and  $\epsilon : \mathcal{H} \rightarrow \mathbb{Z}$  (counit or augmentation) such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H} \otimes \mathcal{H} \\ \Delta \downarrow & & \downarrow 1 \otimes \Delta \\ \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta \otimes 1} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{H} & \xleftarrow{\epsilon \otimes 1} & \mathcal{H} \otimes \mathcal{H} \\ 1 \otimes \epsilon \uparrow & \swarrow 1 & \uparrow \Delta \\ \mathcal{H} \otimes \mathcal{H} & \xleftarrow{\Delta} & \mathcal{H} \end{array}$$

where  $1$  is the identity map on  $\mathcal{H}$ .

**Remark A.1.** The first of these diagrams is the coassociativity property, which is the statement that the dual of  $\Delta$  defines an associative product on the linear dual of  $\mathcal{H}$ , and the second asserts this linear dual has a unit, induced by the dual of  $\epsilon$ .

If  $\mathcal{H}$  is also an algebra, then it is a *bialgebra* if  $\Delta, \epsilon$  are algebra morphisms. While some authors call this structure a Hopf algebra, we define a *Hopf algebra* to be a bialgebra with a map  $s : \mathcal{H} \rightarrow \mathcal{H}$  (coinverse or antipode) such that the following diagram commutes.

$$\begin{array}{ccccc} & & \mathcal{H} \otimes \mathcal{H} & \xrightarrow{s \otimes 1} & \mathcal{H} \otimes \mathcal{H} & & \\ & \nearrow \Delta & & & & \searrow \mu & \\ \mathcal{H} & & \xrightarrow{\epsilon} & \mathbb{Z} & \xrightarrow{u} & \mathcal{H} & \\ & \searrow \Delta & & & & \nearrow \mu & \\ & & \mathcal{H} \otimes \mathcal{H} & \xrightarrow{1 \otimes s} & \mathcal{H} \otimes \mathcal{H} & & \end{array}$$

Here  $\mu : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$  is the map induced by the multiplication of  $\mathcal{H}$  and  $u : \mathbb{Z} \rightarrow \mathcal{H}$  is the map induced by mapping  $1$  to the unit of  $\mathcal{H}$ . The above diagram implies that  $s$  is an algebra antihomomorphism, i.e.  $s(hh') = s(h')s(h)$  for all  $h, h' \in \mathcal{H}$ .

The existence of an antipode  $s$  may seem to be a strong restriction on a bialgebra, however as we will see, it is no restriction for graded bialgebras. A *graded bialgebra* is a graded algebra  $\mathcal{H} = \bigotimes_n \mathcal{H}_n$  where  $\Delta$  is graded and  $\mathcal{H}_0 = \mathbb{Z}$ . Given  $x \in \mathcal{H}_n$ , the

$n$ th graded component, we have

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^n y_i \otimes z_{n-i},$$

where  $y_i$  and  $z_i$  have degree  $i$ . The first term is always present due to the counit diagram. With this in mind, Ehrenborg proved the following.

**Proposition A.2** (Lemma 2.1 [11]). *Given a graded bialgebra  $\mathcal{H}$  there is a unique Hopf algebra with antipode  $s$  defined recursively by  $s(1) = 1$ , and for  $x \in \mathcal{H}_n$ ,  $n \geq 1$ ,*

$$s(x) = - \sum_{i=1}^n s(y_i) \cdot z_i.$$

Lastly, we remark on the useful Sweedler notation, which is an elegant solution to the following quandary. Given  $h \in \mathcal{H}$ , how do you efficiently represent  $\Delta h$  as an element of  $\mathcal{H} \otimes \mathcal{H}$ ? Carefully indexing this element would confuse even the writer. Sweedler notation sidesteps this by omitting the indices of summation entirely,

$$\Delta h = \sum h_1 \otimes h_2.$$

It is this notation that is normally used when dealing with Hopf algebras.

#### REFERENCES

- [1] M. BAYER AND L. BILLERA, *Generalized Dehn-Sommerville relations for polytopes, spheres, and Eulerian partially ordered sets*, Invent. Math., 79 (1985), pp. 143–157.
- [2] N. BERGERON, S. MYKYTIUK, F. SOTTILE, AND S. VAN WILLIGENBURG, *Shifted quasi-symmetric functions and the Hopf algebra of peak functions*. math.CO/9904105, 1999.
- [3] N. BERGERON AND F. SOTTILE, *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, Duke Math. J., 95 (1998), pp. 373–423.
- [4] ———, *Hopf algebras of edge-labelled posets*, J. Algebra, 216 (1999), pp. 641–651.
- [5] ———, *A Pieri-type formula for isotropic flag manifolds*. math.CO/9810025, 1999.
- [6] ———, *Skew Schubert functions and the Pieri formula for flag manifolds*. math.AG/97090034, 1999.
- [7] A. BERTRAM, *Quantum Schubert calculus*, Adv. Math., 128 (1997), pp. 289–305.
- [8] A. BERTRAM, I. CIOCAN-FONTANINE, AND W. FULTON, *Quantum multiplication of Schur polynomials*, J. Algebra, 219 (1999), pp. 728–746.
- [9] L. BILLERA AND N. LIU, *Noncommutative enumeration in graded posets*. J. Alg. Combin., to appear, 1999.
- [10] S. BILLEY AND M. HAIMAN, *Schubert polynomials for the classical groups*, J. AMS, 8 (1995), pp. 443–482.
- [11] R. EHRENBORG, *On posets and Hopf algebras*, Adv. Math., 119 (1996), pp. 1–25.
- [12] S. FOMIN AND C. GREENE, *Noncommutative Schur functions and their applications*. Discrete Math., to appear.
- [13] I. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V. RETAKH, AND J.-Y. THIBON, *Non-commutative symmetric functions*, Adv. in Math., 112 (1995), pp. 218–348.
- [14] I. GESSEL, *Multipartite  $P$ -partitions and products of skew Schur functions*, in Combinatorics and Algebra (Boulder, Colo., 1983), C. Greene, ed., vol. 34 of Contemp. Math., AMS, 1984, pp. 289–317.
- [15] F. GOODMAN AND H. WENZEL, *Littlewood-Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math., 82 (1990), pp. 244–265.

- [16] M. HAIMAN, *Dual equivalence with applications, including a conjecture of Proctor*, Discrete Math., 99 (1992), pp. 79–113.
- [17] S. JONI AND G.-C. ROTA, *Coalgebras and bialgebras in combinatorics*, Stud. Appl. Math., 61 (1979), pp. 93–139.
- [18] A. LASCoux AND M.-P. SCHÜTZENBERGER, *Polynômes de Schubert*, C. R. Acad. Sci. Paris, 294 (1982), pp. 447–450.
- [19] I. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, 1995. second edition.
- [20] C. MALVENUTO, *Produits et coproduits des fonctions quasi-symétriques et de l’algèbre des descents*, no. 16, Laboratoire de combinatoire et d’informatique mathématique (LACIM), Univ. du Québec à Montréal, Montréal, 1994.
- [21] M. S. Ravi, J. Rosenthal, and X.C. Wang, *Dynamic pole assignment and Schubert calculus*, SIAM J. Control and Optim. 34 (1996), 813–832.
- [22] W. R. SCHMITT, *Antipodes and incidence coalgebras*, J. Combin. Theory Ser. A, 46 (1987), pp. 264–290.
- [23] F. SOTTILE, *Pieri’s formula for flag manifolds and Schubert polynomials*, Annales de l’Institut Fourier, 46 (1996), pp. 89–110.
- [24] R. STANLEY, *Ordered structures and partitions*, Mem. Amer. Math. Soc., 119 (1972).
- [25] R. STANLEY, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin., 5 (1984), pp. 359–372.
- [26] R. STANLEY, *Enumerative Combinatorics, Vol. 1*, Wadsworth and Brooks/Cole, 1986.
- [27] R. STANLEY, *Enumerative Combinatorics Vol. 2*, no. 62 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. Appendix 1 by Sergey Fomin.
- [28] J. STEMBRIDGE, *Enriched P-partitions*, Trans. AMS, 349 (1997), pp. 763–788.

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