DECOMPOSABLE COMPOSITIONS, SYMMETRIC QUASISYMMETRIC FUNCTIONS AND EQUALITY OF RIBBON SCHUR FUNCTIONS

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Abstract. We define an equivalence relation on integer compositions and show that two ribbon Schur functions are identical if and only if their defining compositions are equivalent in this sense. This equivalence is completely determined by means of a factorization for compositions: equivalent compositions have factorizations that differ only by reversing some of the terms. As an application, we can derive identities on certain Littlewood-Richardson coefficients.

Finally, we consider the cone of symmetric functions having a nonnnegative representation in terms of the fundamental quasisymmetric basis. We show the Schur functions are among the extremes of this cone and conjecture its facets are in bijection with the equivalence classes of compositions.

1. Introduction

An important basis for the space of symmetric functions of degree n is the set of classical Schur functions s_λ, where λ runs over all *partitions* of n. The skew Schur functions $s_{\lambda/\mu}$ can be expressed in terms of these by means of the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ by

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu. \tag{1.1}$$

These coefficients also describe the structure constants in the algebra of symmetric functions. In particular they describe the multiplication rule for Schur functions,

$$s_\mu \; s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda. \tag{1.2}$$

From the perspective of the representation theory of the symmetric group, the coefficient $c^\lambda_{\mu\nu}$ gives the multiplicity of the irreducible representation corresponding to the partition λ in the induced tensor product of those corresponding to μ and ν. In algebraic geometry the $c^\lambda_{\mu\nu}$ arise as intersection numbers in the Schubert calculus on a Grassmanian. As a result of these and other instances in which they arise, the determination of these coefficients is a central problem.

We consider here the question of when two ribbon Schur functions might be equal. The more general question of equalities among all skew Schur functions

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has since been broached in [14], where the question of when a skew Schur function can equal a Schur function is answered in the case of power series and their associated polynomials. The general question of equalities among skew Schur functions remains open. Any such equalities imply equalities between certain pairs of Littlewood-Richardson coefficients.

In the case of *ribbon* Schur functions, that is skew Schur functions indexed by a shape known as a ribbon (or rim hook, or border strip; these are diagrams corresponding to connected skew shapes containing no 2×2 rectangle), we give necessary and sufficient conditions for equality. Ribbons are in natural correspondence with compositions, and equality arises from an equivalence relation on compositions, whose equivalence classes all have size equal to a power of two. This power corresponds to the number of nonsymmetric compositions in a certain factorization of any of the underlying compositions in a class, and equivalence comes by means of reversal of terms.

A motivation for studying ribbon Schur functions is that they arise in various contexts. They were studied already by MacMahon [11, §168-9], who showed their coefficients in terms of the monomial symmetric functions to count descents in permutations with repeated elements. The scalar product of any two gives the number of permutations such that it and its inverse have the associated pair of descent sets [12, Corollary 7.23.8]. They are also useful in computing the number of permutations with a given cycle structure and descent set [5]. Lascoux and Pragacz [9] give a determinant formula for computing Schur functions from associated ribbon Schur functions. In addition, they arise as sl_n-characters of the irreducible components of the Yangian representations in level 1 modules of $\widehat{sl}_n$ [8].

In the theory of noncommutative symmetric functions of Gel'fand *et al.* [4], the noncommutative analogues of the ribbon Schur functions form a homogeneous linear basis. It is therefore of some interest to know what relations are introduced when passing to the commutative case. In particular, which pairs become identical? In [12, Exercise 7.56 b)] the ribbon Schur function indexed by a composition and its reversal are seen to be identical. However, as we shall see, this is not the whole story.

This paper is organized as follows. In Section 2, we introduce an equivalence relation on compositions and derive some if its properties. The relation is defined in terms of coefficients of symmetric functions when expressed in terms of the fundamental basis of the algebra of quasisymmetric functions. We show that this relation can be viewed combinatorially in terms of coarsenings of the respective compositions. Theorem 2.6 then shows compositions to be equivalent if and only if their corresponding ribbon Schur functions are identical.

Section 3 introduces a binary operation on compositions. In the case of compositions denoting the descent sets of a pair of permutations, the operation results in the composition giving the descent set of their tensor product. In Sections 4 and 5 we prove our main result, Theorem 4.1, which states that equivalence of two compositions is precisely given by reversal of some or all of the terms in some factorization. Thus the congruence classes all have size given by a power of two; this power is the number of nonsymmetric terms in the finest factorization of any composition in this class.

Finally, in Section 6, we consider the cone of F-positive symmetric functions, showing the Schur functions to be among its extremes and conjecturing its facets to be in one-to-one correspondence with equivalence classes of compositions.

The remainder of this section contains the basic definitions we will be using. Where possible, we are using the notation of [10] or [12].

1.1. **Partitions and compositions.** A composition β of n, denoted $\beta \vDash n$, is a list of positive integers $\beta_1\beta_2\ldots\beta_k$ such that $\beta_1+\beta_2+\ldots+\beta_k=n$. We refer to each of the β_i as components, and say that β has *length* $l(\beta)=k$ and *size* $|\beta|=n$. If the components of β are weakly decreasing we call β a *partition*, denoted $\beta \vdash n$ and refer to each of the β_i as parts. For any composition β there will be two other closely related compositions that will be of interest to us. The first is the reversal of β, $\beta^* = \beta_k\ldots\beta_2\beta_1$, and the second is the partition determined by β, $\lambda(\beta)$, which is obtained by reordering the components of β in weakly decreasing order, e.g. $\lambda(3243)=4332$. Moreover we say two compositions β,γ determine the same partition if $\lambda(\beta)=\lambda(\gamma)$.

Any composition $\beta \vDash n$ also naturally corresponds to a subset $S(\beta)\subseteq[n-1]=\{1,2,\ldots,n-1\}$ where

$$S(\beta)=\{\beta_1,\beta_1+\beta_2,\beta_1+\beta_2+\beta_3,\ldots,\beta_1+\beta_2+\ldots+\beta_{k-1}\}.$$

Similarly any subset $S=\{i_1,i_2\ldots,i_{k-1}\}\subseteq[n-1]$ corresponds to a composition $\beta(S)\vDash n$ where

$$\beta(S)=i_1(i_2-i_1)(i_3-i_2)\ldots(n-i_{k-1}).$$

Finally, recall two partial orders that exist on compositions. We say that for compositions $\beta,\gamma\vDash n$, we write $\beta\prec\gamma$ when β is *lexicographically less* than γ, that is, $\beta=\beta_1\beta_2\cdots\neq\gamma_1\gamma_2\cdots=\gamma$, and the first i for which $\beta_i\neq\gamma_i$ satisfies $\beta_i<\gamma_i$. In particular, $11\cdots1\preceq\beta\preceq n$ for any $\beta\vDash n$. Secondly, given any two compositions β and γ we say β is a *coarsening* of γ, denoted $\beta\geq\gamma$, if we can obtain β by adding together adjacent components of γ, e.g., $3242\geq3212111$. Equivalently, we can say γ is a *refinement* of β.

1.2. **Quasisymmetric and symmetric functions.** We denote by $\mathcal{Q}$ the algebra of quasisymmetric functions over $\mathbb{Q}$, that is all bounded degree formal power series F in variables $x_1,x_2,\ldots$ such that for all k and $i_1<i_2<\ldots<i_k$ the coefficient of $x_{i_1}^{\beta_1}x_{i_2}^{\beta_2}\ldots x_{i_k}^{\beta_k}$ is equal to that of $x_1^{\beta_1}x_2^{\beta_2}\ldots x_k^{\beta_k}$. There are two natural bases for $\mathcal{Q}$ both indexed by compositions $\beta=\beta_1\beta_2\ldots\beta_k$, $\beta_i>0$: the monomial basis spanned by $M_0=1$ and all power series M_β where

$$M_\beta=\sum_{i_1<i_2<\ldots<i_k}x_{i_1}^{\beta_1}x_{i_2}^{\beta_2}\ldots x_{i_k}^{\beta_k}$$

and the fundamental basis spanned by $F_0=1$ and all power series F_β where

$$F_\beta=\sum_{\gamma\leq\beta}M_\gamma.$$

Note that $\mathcal{Q}$ is a graded algebra, with $\mathcal{Q}_n=\operatorname{span}_{\mathbb{Q}}\{M_\beta \mid \beta\models n\}$.

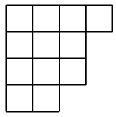
We define the algebra of symmetric functions Λ to be the subalgebra of $\mathcal{Q}$ spanned by the *monomial symmetric functions*

$$m_\lambda=\sum_{\beta:\lambda(\beta)=\lambda}M_\beta,\quad \lambda\vdash n,\ n>0 \tag{1.3}$$

and $m_0 = 1$. Again, Λ is graded, with $\Lambda_n = \Lambda \cap \mathcal{Q}_n$.

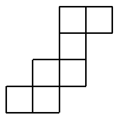
From quasisymmetric functions we can define Schur functions, which also form a basis for the symmetric functions, but first we need to recall some facts about tableaux.

For any partition $\lambda = \lambda_1 \dots \lambda_k \vdash n$ the related *Ferrers diagram* (by abuse of notation also referred to as λ) is an array of left justified boxes with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. For example, the Ferrers diagram 4332 is



A *(Young) tableau* of shape λ and size n is a filling of the boxes of λ with positive integers. If the rows weakly increase and the columns strictly increase we say it is a *semi-standard* tableau, and if in addition, the filling of the boxes involves the integers $1, 2, \dots, n$ appearing once and only once we say it is a *standard* tableau. Note that in this instance both the rows and columns strictly increase.

More generally, we can define skew diagrams and skew tableaux. Let λ, μ be partitions such that if there is a box in the (i, j)-th position in the Ferrers diagram μ then there is a box in the (i, j)-th position in the Ferrers diagram λ. The skew diagram λ/μ is the array of boxes $\{c \mid c \in \lambda,\ c \notin \mu\}$. For example, the skew diagram 4332/221 is



We can then define skew tableaux, semi-standard skew tableaux, and standard skew tableaux analogously.

Given a standard tableau or skew tableau T, we say it has a *descent* in position i if $i+1$ appears in a lower row than i. Denote the set of all descents of T by $D(T)$. We take [12, Theorem 7.19.7] as our definition of skew Schur functions.

Definition 1.1. *Let $\lambda, \mu \vdash n$ such that λ/μ is a skew diagram. Then the skew Schur function $s_{\lambda/\mu}$ is defined by*

$$s_{\lambda/\mu} = \sum_T F_{\beta(D(T))} \tag{1.4}$$

where the sum is over all standard tableaux T of shape λ/μ.

The *Schur functions* s_λ are those skew Schur functions with $\mu = 0$. For example, $s_{22} = F_{22} + F_{121}$.

A skew diagram is said to be *connected* if, regarded as a union of squares, it has a connected interior. If the skew diagram λ/μ is connected and contains no 2×2 array of boxes we call it a *ribbon*. Observe ribbons of size n are in one-to-one correspondence with compositions β of size n by setting β_i equal to the number of boxes in the i-th row from the bottom. For example, the skew diagram 4332/221 is a ribbon, corresponding to the composition 2212.

Henceforth, we will denote ribbons by compositions, and denote the skew Schur functions $s_{\lambda/\mu}$, for a ribbon λ/μ, as the *ribbon Schur function* r_β, where β is the corresponding composition. Thus $r_{2212} := s_{4332/221}$.

Further details on symmetric functions can be found in [12].

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2. Equality of Ribbon Schur Functions

Although, in general, it is difficult to determine when two skew Schur functions are equal, it transpires that when computing ribbon Schur functions equality is determined via a straightforward equivalence on compositions.

2.1. **Relations on ribbon Schur functions.** There is a useful representation of ribbon Schur functions in terms of the basis of complete homogeneous symmetric functions $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_k}$, known already to MacMahon [11, §168]. It can be derived from the Jacobi-Trudi identity [12, Theorem 7.16.1]

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j}), \tag{2.1}$$

where $h_0 = 1$ and $h_k = 0$ if $k < 0$.

Proposition 2.1. *For any* $\alpha \vDash n$*,*

$$r_\alpha = (-1)^{l(\alpha)} \sum_{\beta \geq \alpha} (-1)^{l(\beta)} h_{\lambda(\beta)}.$$

Proof. Applying (2.1) to the ribbon shape λ/μ corresponding to $\alpha = \alpha_1\cdots\alpha_k$, we get

$$r_\alpha = \det \begin{bmatrix} h_{\alpha_k} & h_{\alpha_{k-1}+\alpha_k} & h_{\alpha_{k-2}+\alpha_{k-1}+\alpha_k} & \cdots & h_{\alpha_1+\cdots+\alpha_k} \\ 1 & h_{\alpha_{k-1}} & h_{\alpha_{k-2}+\alpha_{k-1}} & \cdots & h_{\alpha_1+\cdots+\alpha_{k-1}} \\ & 1 & h_{\alpha_{k-2}} & \cdots & h_{\alpha_1+\cdots+\alpha_{k-2}} \\ & & \ddots & & \vdots \\ & & & 1 & h_{\alpha_1} \end{bmatrix}.$$

Expanding down the first column gives

$$r_\alpha = r_{\alpha_1\cdots\alpha_{k-1}}h_{\alpha_k} - r_{\alpha_1\cdots\alpha_{k-1}+\alpha_k},$$

which along with induction on $l(\alpha)$ gives the desired result. □

By inverting the relation in Proposition 2.1, we see immediately that the ribbon Schur functions r_α generate Λ.

It is straightforward to establish all the algebraic relations that hold among ribbon Schur functions. Using Proposition 2.1, one can show that ribbon Schur functions satisfy the multiplicative relations

$$r_\alpha r_\beta = r_{\alpha\cdot\beta} + r_{\alpha\odot\beta}, \tag{2.2}$$

where for $\alpha = \alpha_1\cdots\alpha_k$ and $\beta = \beta_1\cdots\beta_l$, $\alpha\cdot\beta = \alpha_1\ldots\alpha_k\beta_1\ldots\beta_l$ is the usual operation of *concatenation* and $\alpha\odot\beta = \alpha_1\ldots\alpha_{k-1}(\alpha_k+\beta_1)\beta_2\ldots\beta_l$, is *near concatenation*, which differs from concatenation in that the last component of α is added to the first component of β. The relation (2.2) has been known since MacMahon [11, §169]. Proofs of (2.2) in the noncommutative setting can be found in [4, Prop.

3.13] and [1, Prop. 4.1]. We show next that these relations generate all the relations among ribbon Schur functions.

Proposition 2.2. *Let $z_\alpha, \alpha \models n, n \geq 1$ be commuting indeterminates. Then as algebras, Λ is isomorphic to the quotient*

$$\mathbb{Q}[z_\alpha]/\langle z_\alpha\ z_\beta - z_{\alpha\cdot\beta} - z_{\alpha\odot\beta}\rangle.$$

Proof. Consider the map $\varphi : \mathbb{Q}[z_\alpha] \to \Lambda$ defined by $z_\alpha \mapsto r_\alpha$. This map is surjective since the r_α generate Λ. Grading $\mathbb{Q}[z_\alpha]$ by setting the degree of z_α to be $n = |\alpha|$ makes φ homogeneous. To see that φ induces an isomorphism with the quotient, note that $\mathbb{Q}[z_\alpha]/\langle z_\alpha\ z_\beta - z_{\alpha\cdot\beta} - z_{\alpha\odot\beta}\rangle$ maps onto $\mathbb{Q}[z_\alpha]/\ker\varphi \simeq \Lambda$, since $\langle z_\alpha\ z_\beta - z_{\alpha\cdot\beta} - z_{\alpha\odot\beta}\rangle \subset \ker\varphi$.

It will suffice to show that the degree n component of $\mathbb{Q}[z_\alpha]/\langle z_\alpha\ z_\beta - z_{\alpha\cdot\beta} - z_{\alpha\odot\beta}\rangle$ is generated by the images of the $z_\lambda, \lambda \vdash n$, and so has dimension at most the number of partitions of n. We have the relations

$$z_{\alpha\cdot\beta} + z_{\alpha\odot\beta} = z_{\beta\cdot\alpha} + z_{\beta\odot\alpha} \tag{2.3}$$

by commutativity of the z_α. Let $\gamma = g_1 \dots g_k \models n$. We will show by induction on k that $z_\gamma = z_{\lambda(\gamma)}+$ a sum of z_δ's with δ having no more than $k-1$ parts.

Let g_i be a maximal component of γ. Then by (2.2) and (2.3) we have

$$\begin{aligned} z_\gamma &= z_{g_i \cdots g_k g_1 \cdots g_{i-1}} + \text{a sum of } z_\delta\text{'s with } k-1 \text{ or fewer parts} \\ &= z_{g_i} z_{g_{i+1} \cdots g_k g_1 \cdots g_{i-1}} + \text{a sum of } z_\delta\text{'s with } k-1 \text{ or fewer parts} \\ &= z_{g_i} z_{\lambda(g_{i+1} \cdots g_k g_1 \cdots g_{i-1})} + \text{a sum of } z_\delta\text{'s with } k-1 \text{ or fewer parts} \\ &= z_{\lambda(\gamma)} + \text{a sum of } z_\delta\text{'s with } k-1 \text{ or fewer parts,} \end{aligned}$$

where the third equality uses the induction hypothesis. A trivial induction on the length of γ now shows that any $z_\gamma, \gamma \models n$ can be reduced in the quotient to a linear combination of $z_\lambda, \lambda \vdash n$.

□

As a consequence of the proof we get that the ribbon Schur functions r_λ, $\lambda \vdash n$, span Λ_n and so form a basis. However, for general compositions α and β, it may be that $r_\alpha = r_\beta$. The rest of this section begins to deal with the question of when this can occur. In principle, the relation $r_\alpha = r_\beta$ is a consequence of the relations (2.2), and more specifically (2.3), so a purely algebraic development of the main results of this paper should be possible. This has not yet been done.

2.2. **Equivalence of compositions.**

We define an algebraic equivalence on compositions and reinterpret it in a combinatorial manner.

Definition 2.3. *Let β, γ be compositions. We say β and γ are* equivalent*, denoted $\beta \sim \gamma$, if for all $F = \sum c_\alpha F_\alpha \in \Lambda$, $c_\beta = c_\gamma$.*

That is, $\beta \sim \gamma$ if F_β has the same coefficient as F_γ in the expression of every symmetric function. Note that any basis for Λ can be used as a finite test set for this equivalence. We will be particularly interested in the monomial symmetric function basis (1.3) and the Schur function basis (1.4).

Example 2.1. For $\beta = 211$ and $\gamma = 121$ we find that $\beta \not\sim \gamma$ since $s_{22} = F_{22} + F_{121}$.

For any composition $\beta \models n$, we define $\mathcal{M}(\beta)$ to be the *multiset* of partitions determined by all coarsenings of β, that is,

$$\mathcal{M}(\beta) = \{\lambda(\alpha) \mid \alpha \geq \beta\}. \tag{2.4}$$

We denote by $\mathrm{mult}_{\mathcal{M}(\beta)}(\lambda)$ the multiplicity of λ in $\mathcal{M}(\beta)$.

Example 2.2. Note that while 2111 and 1211 have identical *sets* of partitions arising from their coarsenings, $\mathcal{M}(2111) \neq \mathcal{M}(1211)$ since $\mathrm{mult}_{\mathcal{M}(2111)}(311) = 1$ while $\mathrm{mult}_{\mathcal{M}(1211)}(311) = 2$.

With this in mind we reformulate our equivalence. Recall that for $F \in \mathcal{Q}$,

$$F = \sum c_\beta F_\beta = \sum d_\beta M_\beta,$$

where the c_α and d_α are related by

$$d_\beta = \sum_{\alpha \geq \beta} c_\alpha, \quad c_\beta = \sum_{\alpha \geq \beta} (-1)^{l(\alpha)-l(\beta)} d_\alpha, \tag{2.5}$$

and that $F \in \Lambda$ if and only if $d_\alpha = d_\beta$ whenever $\lambda(\alpha) = \lambda(\beta)$. The following is a direct consequence of (1.3) and (2.5).

Proposition 2.4. *If the monomial symmetric function $m_\lambda = \sum_{\beta \models n} c_\beta F_\beta$, then*

$$c_\beta = [m_\lambda]_{F_\beta} = (-1)^{l(\lambda)-l(\beta)} \mathrm{mult}_{\mathcal{M}(\beta)}(\lambda),$$

that is, up to sign, c_β is the multiplicity of λ in the multiset $\mathcal{M}(\beta)$.

As an immediate consequence we get

Corollary 2.5. *If β and γ are compositions, then $\beta \sim \gamma$ if and only if $\mathcal{M}(\beta) = \mathcal{M}(\gamma)$.*

Example 2.3. Returning to the example $\beta = 211$ and $\gamma = 121$, it is now straightforward to deduce $\beta \not\sim \gamma$ since

$$\mathcal{M}(\beta) = \{4, 31, 22, 211\} \quad \neq \quad \{4, 31, 31, 211\} = \mathcal{M}(\gamma).$$

We are now ready to state the main result of this section. Note first that Proposition 2.1 can be written

$$r_\alpha = (-1)^{l(\alpha)} \sum_{\lambda \in \mathcal{M}(\alpha)} (-1)^{l(\lambda)} h_\lambda. \tag{2.6}$$

Theorem 2.6. *For the ribbon Schur functions r_β and r_γ corresponding to compositions β and γ, we have $r_\beta = r_\gamma$ if and only if $\mathcal{M}(\beta) = \mathcal{M}(\gamma)$.*

Proof. If $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$, then by (2.6), $r_\beta = r_\gamma$. Conversely, since the h_k are algebraically independent, equality of r_β and r_γ implies, again by (2.6), that $\mathcal{M}(\alpha) = \mathcal{M}(\beta)$. □

An immediate corollary of this and (1.1) are the following Littlewood-Richardson coefficient identities.

Corollary 2.7. *Suppose ribbon skew shapes λ/μ and ρ/η correspond to compositions β and γ, where $\beta \sim \gamma$ are both compositions of n. Then, for all partitions ν of n,*

$$c^\lambda_{\mu,\nu} = c^\rho_{\eta,\nu}.$$

Example 2.4. Since $\mathcal{M}(211) = \{4, 31, 22, 211\} = \mathcal{M}(112)$ the above theorem assures us that $s_{222/11} = s_{4331/2221}$ and so, by Corollary 2.7, $c^{222}_{11,\nu} = c^{4331}_{2221,\nu}$ for all partitions ν of 4.

An immediate consequence of Theorem 2.6, Corollary 2.5 and [12, Corollary 7.23.4] is another description of the equivalence $\sim$. For $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n) \in S_n$, the *descent set* of σ is defined to be the set $d(\sigma) := \{i \mid \sigma(i) > \sigma(i+1)\} \subset [n-1]$.

Corollary 2.8. *For $\beta, \gamma \models n$, $\beta \sim \gamma$ if and only if for all $\alpha \models n$, the number of permutations $\sigma \in S_n$ satisfying $d(\sigma) = S(\alpha)$ and $d(\sigma^{-1}) = S(\beta)$ is equal to the number of permutations $\sigma \in S_n$ satisfying $d(\sigma) = S(\alpha)$ and $d(\sigma^{-1}) = S(\gamma)$.*

3. Compositions of Compositions

In this section we describe a method to combine compositions into larger ones that corresponds to determining the descent set of the tensor product of two permutations. This leads naturally to a necessary and sufficient condition for two compositions to be equivalent.

3.1. Composition, tensor product, and plethysm.

Let $\mathcal{C}_n$ denote the set of all compositions of n and let

$$\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n.$$

Given $\alpha = \alpha_1 \dots \alpha_k \models m$ and $\beta = \beta_1 \dots \beta_l \models n$, recall the binary operations of *concatenation*

$$\begin{array}{rcl} \cdot : \mathcal{C}_m \times \mathcal{C}_n & \to & \mathcal{C}_{m+n} \\ (\alpha, \beta) & \mapsto & \alpha \cdot \beta = \alpha_1 \dots \alpha_k \beta_1 \dots \beta_l \end{array}$$

and *near concatenation*

$$\begin{array}{rcl} \odot : \mathcal{C}_m \times \mathcal{C}_n & \to & \mathcal{C}_{m+n} \\ (\alpha, \beta) & \mapsto & \alpha \odot \beta = \alpha_1 \dots \alpha_{k-1} (\alpha_k + \beta_1) \beta_2 \dots \beta_l. \end{array}$$

For convenience we write

$$\alpha^{\odot n} = \underbrace{\alpha \odot \alpha \odot \dots \odot \alpha}_{n}.$$

These two operations can be combined to produce a third, which will be our focus:

$$\begin{array}{rcl} \circ : \mathcal{C}_m \times \mathcal{C}_n & \to & \mathcal{C}_{mn} \\ (\alpha, \beta) & \mapsto & \alpha \circ \beta = \beta^{\odot \alpha_1} \cdot \beta^{\odot \alpha_2} \cdots \beta^{\odot \alpha_k}. \end{array}$$

Example 3.1. If $\alpha = 12, \beta = 12$ then $\alpha \cdot \beta = 1212$, $\alpha \odot \beta = 132$ and $\alpha \circ \beta = 12132$.

It is straightforward to observe that $\mathcal{C}$ is closed under $\circ$ and that for $\alpha \models m$ we have $1 \circ \alpha = \alpha \circ 1 = \alpha$. Note that the operation $\circ$ is not commutative since $12 \circ 3 = 36$ whereas $3 \circ 12 = 1332$.

We now see that composing compositions corresponds to determining descent sets in the tensor product of permutations.

Definition 3.1. *Let $\sigma = \sigma(1)\sigma(2) \dots \sigma(m) \in S_m$ and $\tau = \tau(1)\tau(2) \dots \tau(n) \in S_n$. Then their tensor product is the permutation*

$$\begin{array}{rcl} \sigma \otimes \tau & = & [(\sigma(1)-1)n + \tau(1)][(\sigma(1)-1)n + \tau(2)] \dots [(\sigma(1)-1)n + \tau(n)] \\ & & [(\sigma(2)-1)n + \tau(1)] \dots [(\sigma(m)-1)n + \tau(n)] \quad \in S_{mn}. \end{array}$$

Remark 3.2. An alternative realization is as follows. Given $\sigma \in S_m, \tau \in S_n$ and the $m \times n$ matrix

$$M_{mn} = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (m-1)n+1 & (m-1)n+2 & \cdots & mn \end{pmatrix}$$

then ${}^{\sigma}M_{mn}$ is the matrix in which the i-th row of ${}^{\sigma}M_{mn}$ is the $\sigma(i)$-th row of M_{mn}. Similarly, M_{mn}^{τ} is the matrix in which the j-th column of M_{mn}^{τ} is the $\tau(j)$-th column of M_{mn}. With this in mind, $\sigma \otimes \tau \in S_{mn}$ is the permutation obtained by reading the entries of ${}^{\sigma}M_{mn}^{\tau}$ by row.

Example 3.3. If $\sigma = 213, \tau = 132 \in S_3$ then $\sigma \otimes \tau = 213 \otimes 132 = 465132798$, and

$$M_{mn} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, {}^{\sigma} M_{mn} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}, {}^{\sigma} M_{mn}^{\tau} = \begin{pmatrix} 4 & 6 & 5 \\ 1 & 3 & 2 \\ 7 & 9 & 8 \end{pmatrix}.$$

The following shows that the operation $\circ$ on compositions yields the descent set of the tensor product of two permutations from their respective descent sets.

Proposition 3.2. *Let $\sigma \in S_m$ and $\tau \in S_n$. If $d(\sigma) = S(\beta)$ and $d(\tau) = S(\gamma)$ then $d(\sigma \otimes \tau) = S(\beta \circ \gamma)$.*

Proof. Let $d(\sigma) = S(\beta) = \{i_1, i_2, \ldots, i_k\}$ and $d(\tau) = S(\gamma) = \{j_1, j_2, \ldots, j_l\}$. Then $d(\sigma \otimes \tau) = \{j_1, j_2, \ldots j_l, n + j_1, n + j_2, \ldots, n + j_l, \ldots, (m-1)n + j_1, (m-1)n + j_2, \ldots, (m-1)n + j_l\} \cup \{ni_1, ni_2, \ldots, ni_k\} = S(\beta \circ \gamma)$. □

From Proposition 3.2 and the associativity of $\otimes$, we can conclude that $\circ$ is associative. Consequently we obtain

Proposition 3.3. *$(\mathcal{C}, \circ)$ is a monoid.*

Finally we relate the operation $\circ$ on compositions to the operation of plethysm on symmetric functions. For the power sum symmetric function $p_m \in \Lambda_m$ and $g \in \Lambda_n$, define the *plethysm*

$$p_m \circ g = p_m[g] = g(x_1^m, x_2^m, \ldots).$$

Extend this to define $f \circ g \in \Lambda_{mn}$ for any $f \in \Lambda_m$ and $g \in \Lambda_n$ by requiring that the map taking f to $f \circ g$ be an algebra map. (See [10, p.135], [12, p.447] for details.)

When $\alpha \vDash m$ and $\beta \vDash n$ the ribbon functions $r_\alpha \circ r_\beta$ and $r_{\alpha \circ \beta}$ both have degree mn. While they are not in general equal, they are equal on the average, as seen by the following identity.

Proposition 3.4. *For any $\beta \vDash n$,*

$$\sum_{\alpha \vDash m} r_\alpha \circ r_\beta = \sum_{\alpha \vDash m} r_{\alpha \circ \beta}.$$

Proof. We first note that

$$r_1^m = \sum_{\alpha \vDash m} r_\alpha,$$

which can be seen, for example, by repeated application of (2.2). Since the plethysm $f \circ g$ gives an algebra map in f, it follows that

$$\sum_{\alpha \vDash m} r_\alpha \circ r_\beta = r_1^m \circ r_\beta = r_\beta^m.$$

On the other hand, we show $r_\beta^m = \sum_{\alpha\models m} r_{\alpha\circ\beta}$ by induction on m. The case $m = 1$ is clear. For $m > 1$, we get by induction and (2.2) that

$$\begin{aligned} r_\beta^m &= r_\beta \cdot \sum_{\alpha\models m-1} r_{\alpha\circ\beta} \\ &= \sum_{\alpha\models m-1} \left[r_{\beta\cdot(\alpha\circ\beta)} + r_{\beta\odot(\alpha\circ\beta)}\right] \\ &= \sum_{\alpha\models m} r_{\alpha\circ\beta}. \end{aligned}$$

□

3.2. **Unique factorization and other properties.** If a composition α is written in the form $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ then we call this a *decomposition* or *factorization* of α. A factorization $\alpha = \beta \circ \gamma$ is called *trivial* if any of the following conditions are satisfied:

(1) one of β, γ is the composition 1,
(2) the compositions β and γ both have length 1,
(3) the compositions β and γ both have all components equal to 1.

Definition 3.5. *A factorization $\alpha = \alpha_1 \circ \cdots \circ \alpha_k$ is called* irreducible *if no $\alpha_i \circ \alpha_{i+1}$ is a trivial factorization, and each α_i admits only trivial factorizations. In this case, each α_i is called an* irreducible factor.

Theorem 3.6. *The irreducible factorization of any composition is unique.*

Proof. We proceed by induction on the number of irreducible factors in a decomposition.

First observe that if the only irreducible factor of a composition is itself then its irreducible factorization is unique.

Now let α be some composition with two irreducible factorizations

$$\mu_1 \circ \ldots \circ \mu_{k-1} \circ \mu_k = \alpha = \nu_1 \circ \ldots \circ \nu_{l-1} \circ \nu_l,$$

and for convenience set $\beta = \mu_1 \circ \ldots \circ \mu_{k-1}$, $\gamma = \mu_k$, $\delta = \nu_1 \circ \ldots \circ \nu_{l-1}$ and $\epsilon = \nu_l$ so

$$\beta \circ \gamma = \alpha = \delta \circ \epsilon.$$

Our first task is to establish $|\gamma| = |\epsilon|$ from which the induction will easily follow. First assume $|\gamma| = n$ and $|\epsilon| = s$ such that $s \neq n$ and without loss of generality let $s < n$.

If $\epsilon = s$ then it follows $\gamma \neq n$ as if $\gamma = n$ then by our induction assumption and the fact that $s \neq n$ we have that the lowest common multiple of s and n would also be an irreducible factor, which is a contradiction. Hence $\gamma \neq n$ and so $l(\gamma) > 1$. Furthermore since $l(\gamma) > 1$ then $\gamma = \gamma_1 \ldots \gamma_k$ must consist of components of α (the righthandmost and $k-1$ lefthandmost components, for example), which implies $s|\gamma_1, \ldots, s|\gamma_k$ and hence γ is not an irreducible factor.

Thus $\epsilon \neq s$ so $l(\epsilon) > 1$ and since $s < n$ we also have that $l(\gamma) > 1$. In addition, since $l(\gamma) > 1, l(\epsilon) > 1$ we have as above that γ and ϵ must consist of components of α. Hence if $s|n$ then it follows that γ has ϵ as an irreducible factor and hence γ is not an irreducible factor.

Consequently we have that if $s \neq n$ then $l(\gamma) > 1, l(\epsilon) > 1$ and $s \nmid n$. Moreover, the components of γ consist of the components of ϵ repeated (and perhaps the sum of the first and last components of ϵ) plus one copy of ϵ truncated at one end of γ. However, since γ and ϵ consist of components of α it follows that if

$s \neq n$, $l(\gamma) > 1, l(\epsilon) > 1$ and $s \nmid n$, then ϵ cannot be an irreducible factor. Thus $|\gamma| = n = s = |\epsilon|$.

Now that we have established $|\gamma| = |\epsilon|$ we will show that in fact $\gamma = \epsilon$. If $\gamma = n$ then clearly $\epsilon = n$ and we are done. If not, then since the last component of $\beta, \delta \geq 1$ it follows the righthand components of α whose sum is less than n must be those of γ and ϵ and since $|\gamma| = |\epsilon|$ it follows that $\gamma = \epsilon$.

Since we now have $\beta \circ \gamma = \alpha = \delta \circ \gamma$, it is straightforward to see $\beta = \delta$. By the associativity of $\circ$ the result now follows by induction. □

We can also deduce expressions for the content and length of a composition in terms of its decomposition. We omit the proofs, which each follow by a straightforward induction.

Proposition 3.7. *For compositions* $\beta_1, \beta_2, \ldots, \beta_k$

$$|\beta_1 \circ \beta_2 \ldots \circ \beta_k| = \prod_{i=1}^{k} |\beta_i|.$$

Proposition 3.8. *For compositions* $\beta_1, \beta_2, \ldots, \beta_k$

$$l(\beta_1 \circ \beta_2 \ldots \circ \beta_k) = l(\beta_1) + \sum_{i=2}^{k} \left(\prod_{j=1}^{i-1} |\beta_j| \right) (l(\beta_i) - 1).$$

Finally, it will be useful to observe that reversal of compositions commutes with the composition. The proof is clear.

Proposition 3.9. *Let* β, γ *be compositions then*

$$(\beta \circ \gamma)^* = \beta^* \circ \gamma^*.$$

Remark 3.4. For $\sigma \in S_n$, define $\sigma^* \in S_n$ by $\sigma^*(i) := (n+1) - \sigma(n+1-i)$. It is easy to see that for $\sigma \in S_m$, $\tau \in S_n$ $(\sigma \otimes \tau)^* = \sigma^* \otimes \tau^*$ and $\beta(d(\sigma^*)) = (\beta(d(\sigma))^*$. One wonders whether this, in conjunction with Proposition 3.2, can provide a more direct approach to that of the next two sections.

4. Equivalence of Compositions under $\circ$

We show in this section that the equivalence relation of Definition 2.3 is related to the composition of compositions via reversal of terms. In particular, we prove

Theorem 4.1. *Two compositions* β *and* γ *satisfy* $\beta \sim \gamma$ *if and only if for some* k,

$$\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k \quad \text{and} \quad \gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k,$$

where, for each i*, either* $\gamma_i = \beta_i$ *or* $\gamma_i = \beta_i^*$*. Thus the equivalence class of a composition* β *will contain* 2^r *elements, where* r *is the number of nonsymmetric (under reversal) irreducible factors in the irreducible factorization of* β*.*

Before we embark on the proof, which will consist of the remainder of this section and the next, we note a corollary that follows immediately from Corollary 2.5, Theorem 2.6, Corollary 2.8 and Theorem 4.1.

Corollary 4.2. *The following are equivalent for a pair of compositions* β, γ*:*

1. $r_\beta = r_\gamma$,
2. *in all symmetric functions* $F = \sum c_\alpha F_\alpha$*, the coefficient of* F_β *is equal to the coefficient of* F_γ,
3. $\mathcal{M}(\beta) = \mathcal{M}(\gamma)$,

(4) *the number of permutations* $\sigma \in S_n$ *satisfying* $d(\sigma) = S(\alpha)$ *and* $d(\sigma^{-1}) = S(\beta)$ *is equal to the number of permutations* $\sigma \in S_n$ *satisfying* $d(\sigma) = S(\alpha)$ *and* $d(\sigma^{-1}) = S(\gamma)$ *for all* α*,*

(5) *for some* k*,*

$$\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k \quad \text{and} \quad \gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k,$$

and, for each i*, either* $\gamma_i = \beta_i$ *or* $\gamma_i = \beta_i^*$*.*

Example 4.1. Since 12132 has irreducible factorization $12 \circ 12$, Corollary 4.2 assures us that

$$r_{12132} = r_{13212} = r_{21231} = r_{23121}$$

and, moreover, these are the only ribbon Schur functions equal to r_{12132}. In addition, from

$$s_{54221/311} = r_{12132} = r_{13212} = s_{54431/332},$$

we can conclude from (1.1) the identity of Littlewood-Richardson coefficients

$$c^{54221}_{311,\nu} = c^{54431}_{332,\nu}$$

for all partitions ν of 9.

4.1. **Reversal implies equivalence.** We recall that for compositions β and γ, $\beta \sim \gamma$ if and only if $\mathcal{M}(\beta) = \mathcal{M}(\gamma)$ by Corollary 2.5. From this and Proposition 3.9 it is easy to conclude

Proposition 4.3. *For compositions* β *and* $\gamma_1, \dots, \gamma_k$*,*

$$\beta^* \sim \beta$$

and

$$\gamma_1 \circ \gamma_2 \cdots \circ \gamma_k \sim \gamma_1^* \circ \gamma_2^* \cdots \circ \gamma_k^*.$$

We show now that reversal of any of the terms in a decomposition of β yields a composition equivalent to β.

Theorem 4.4. *For any compositions* β*,* γ *and* α*,*

(1) $\beta^* \circ \gamma \sim \beta \circ \gamma$*,*
(2) $\beta \circ \gamma^* \sim \beta \circ \gamma$ *and*
(3) $\beta \circ \alpha^* \circ \gamma \sim \beta \circ \alpha \circ \gamma$*.*

Proof. By definition,

$$\beta \circ \gamma = \gamma^{\odot \beta_1} \cdot \gamma^{\odot \beta_2} \cdots \gamma^{\odot \beta_k}$$

and

$$\beta^* \circ \gamma = \gamma^{\odot \beta_k} \cdots \gamma^{\odot \beta_2} \cdot \gamma^{\odot \beta_1}.$$

To prove (1), note that any coarsening δ of $\beta \circ \gamma$ that does not involve adding terms in different components $\gamma^{\odot \beta_i}$ clearly corresponds to a coarsening of $\beta^* \circ \gamma$ that has the same sorting $\lambda(\delta)$. On the other hand, a coarsening that involves, say, combining terms in $\gamma^{\odot \beta_i}$ with terms of $\gamma^{\odot \beta_{i+1}}$ can be viewed as a coarsening of the first sort of

$$(\beta_1, \dots, \beta_{i-1}, \beta_i + \beta_{i+1}, \beta_{i+2}, \dots, \beta_k) \circ \gamma,$$

which can be seen to correspond to one arising as a coarsening of $\beta^* \circ \gamma$.

Assertions (2) and (3) follow from (1) and Proposition 4.3 via

$$\beta \circ \gamma^* \sim \beta^* \circ \gamma \sim \beta \circ \gamma$$

and

$$\beta \circ \alpha^* \circ \gamma \sim \beta^* \circ \alpha \circ \gamma^* \sim \beta \circ \alpha^* \circ \gamma^* \sim \beta \circ \alpha \circ \gamma,$$

respectively. □

One direction in the assertion of Theorem 4.1 now follows from Theorem 4.4. The remainder of this section and the next is devoted to the proof of the other direction.

4.2. **Equivalence implies reversal.** In this subsection, we prove the converse to the result established in the previous subsection: namely, that if $\beta \sim \gamma$, then there is a factorization $\beta = \beta_1 \circ \cdots \circ \beta_k$ such that $\gamma = \gamma_1 \circ \cdots \circ \gamma_k$, where $\gamma_i = \beta_i$ or β_i^*. We achieve this via two theorems. The first of these is

Theorem 4.5. *Let $\beta \sim \gamma$, and $\beta = \delta \circ \epsilon$. Then γ can be decomposed as $\zeta \circ \eta$ with $\zeta \sim \delta$ and $\eta \sim \epsilon$.*

Example 4.2. Let $\beta = 13212$ and $\gamma = 12132$. It is straightforward to check that these two compositions are equivalent. Note we have that $\beta = 21 \circ 12$. Theorem 4.5 says that there should be a decomposition $\gamma = \zeta \circ \eta$ with $\zeta \sim 21$ and $\eta \sim 12$. We observe that $\gamma = 12 \circ 12$ satisfies these conditions.

In order to prove Theorem 4.5 we require two lemmas:

Lemma 4.6. *Let $\beta = \delta \circ \epsilon$ where $\beta \vDash n$. Let ϵ have size m and p components. Let $\lambda = \lambda_1 \dots \lambda_k$ be a partition of n which occurs in $\mathcal{M}(\beta)$. Let $\bar{\lambda}_i$ be the remainder when λ_i is divided by m, and suppose that the sum of the $\bar{\lambda}_i$ is m. Then the number of non-zero $\bar{\lambda}_i$ is at most p.*

Proof. Reordering the parts of λ if necessary, let $\lambda_1 \dots \lambda_k$ be a composition of n which is a coarsening of β. Now consider the composition of m given by $\bar{\lambda}_1 \dots \bar{\lambda}_k$ (where we omit any zero components). This composition is a coarsening of ϵ, and thus has at most p components. □

Lemma 4.7. *Let $\beta = \delta \circ \epsilon$ where $\beta \vDash n$ and $\epsilon \vDash m$, and let λ be a partition of m, with k parts. Then*

$$\mathrm{mult}_{\mathcal{M}(\beta)}(\lambda, n-m) = (k - 1 + \mathrm{mult}_{\mathcal{M}(\beta)}(m, n-m))\mathrm{mult}_{\mathcal{M}(\epsilon)}(\lambda).$$

Remark 4.3. Note that in the statement of the previous lemma and subsequently, when the context is unambiguous, we will refer to the multiplicity of a composition in the multiset of coarsenings of a composition when we intend the multiplicity of the partition determined by that composition.

Proof. Given a way to realize λ from ϵ, there are $k - 1 + \mathrm{mult}_{\mathcal{M}(\beta)}(m, n-m)$ corresponding ways to realize $(\lambda, n-m)$ from β: one must pick where to put in the $n-m$ component. □

Proof of Theorem 4.5. Let the size of ϵ be m. Write $q = n/m$. Let the number of components of ϵ be p.

Define $\zeta \vDash q$ by setting $S(\zeta) = \{i \mid mi \in S(\gamma)\}$. Now $\mathrm{mult}_{\mathcal{M}(\zeta)}(\lambda) = \mathrm{mult}_{\mathcal{M}(\gamma)} m\lambda$ where we write $m\lambda$ for the partition obtained by multiplying all the parts of λ by m. Similarly, $\mathrm{mult}_{\mathcal{M}(\delta)}(\lambda) = \mathrm{mult}_{\mathcal{M}(\beta)} m\lambda$. Thus, the equivalence of δ and ζ follows from that of β and γ.

Define $\eta_i \vDash m$, $i = 0, \dots, q-1$, by setting

$$S(\eta_i) = \{x \mid 0 < x < m,\ x + im \in S(\gamma)\}.$$

We wish to show that all the η_i are equal and equivalent to ϵ.

For any $0 \le i \le q-1$, the number of components of η_i is at most p: otherwise, consider the composition of γ consisting of

- im plus the first component of η_i,
- the remaining components of η_i except the last,
- the last component of η_i plus $(q-1-i)m$.

The partition corresponding to this composition appears in $\mathcal{M}(\gamma)$ but by Lemma 4.6, it cannot appear in $\mathcal{M}(\beta)$, which is a contradiction.

The cardinalities of $S(\beta)$ and $S(\gamma)$ must be the same, and we have already seen that $|S(\beta) \cap m\mathbb{Z}| = |S(\gamma) \cap m\mathbb{Z}|$. We know that $|S(\beta) \cap (\mathbb{Z} \setminus m\mathbb{Z})| = q(p-1)$, so the same must hold for γ. Now, since each of the η_i has at most p components, each of the η_i must have exactly p components.

We now need the following lemma:

Lemma 4.8. *Let β, γ, and the η_i be as already defined. Let $0 \le i < j \le q-1$. Let $S(\eta_i) = \{a_1 < \cdots < a_{p-1}\}$ and $S(\eta_j) = \{b_1 < \cdots < b_{p-1}\}$. Then $a_t \ge b_t$ for all t.*

Proof. If this were not so, let ν be the partition consisting of the following:

- im plus the first component of η_i,
- the second through t-th components of η_i,
- $(j-i)m + b_t - a_t$,
- the $t+1$-th through $p-1$-th components of η_j,
- the last component of η_j plus $(q-j-1)m$.

Now ν appears in $\mathcal{M}(\gamma)$ but by Lemma 4.6 does not appear in $\mathcal{M}(\beta)$, a contradiction. □

Let μ be the partition of m determined by ϵ. Let $x = \mathrm{mult}_{\mathcal{M}(\beta)}(m, n-m) \in \{0,1,2\}$. The multiplicity of $(\mu, n-m)$ in $\mathcal{M}(\beta)$ is $p-1+x$.

Now consider the possible occurrences of $(\mu, n-m)$ in $\mathcal{M}(\gamma)$. If the t-th element of $S(\eta_0)$ coincides with the t-th element of $S(\eta_{q-1})$, then we have one possible occurrence of $(\mu, n-m)$ with $n-m$ as the $t+1$-th component. Also, since by the equivalence of β and γ, x of $\{m, n-m\}$ are in $S(\gamma)$, there are x possible occurrences of compositions realizing $(\mu, n-m)$ such that the $n-m$ part is either the first or the last component. However, there must be $p-1+x$ realizations of $(\mu, n-m)$, so all these possibilities must actually realize the partition.

In particular, this shows that $S(\eta_0)$ and $S(\eta_{q-1})$ must coincide. Now, by Lemma 4.8, all the $S(\eta_i)$ must coincide, and we can now denote all the η_i by η. The equality of the η_i (in particular, the equality of η_0 and η_{q-1}) means that we can apply the same argument as in Lemma 4.7 to show that for λ a partition of m with k parts,

$$\mathrm{mult}_{\mathcal{M}(\gamma)}(\lambda, n-m) = (k-1+x)\mathrm{mult}_{\mathcal{M}(\eta)}(\lambda).$$

The equivalence of β and γ also implies the multiplicities of λ in $\mathcal{M}(\eta)$ and $\mathcal{M}(\epsilon)$ are equal for any λ that is a partition of m, and hence that ϵ and η are equivalent, as desired. This completes the proof of the theorem. □

The second theorem requires the concept of reconstructibility of a composition.

Definition 4.9. *A composition β is said to be* reconstructible *if knowing $\mathcal{M}(\beta)$ allows us to determine β up to reversal.*

Example 4.4. The composition 112 is reconstructible because if β is a composition satisfying $\mathrm{mult}_{\mathcal{M}(\beta)}\lambda(211) = 1$ and $\mathrm{mult}_{\mathcal{M}(\beta)}\lambda(22) = 1$, then $\beta = 112$ or $\beta = 211$.

Theorem 4.10. *If $\beta \vDash n$ is not reconstructible, then β decomposes as $\delta \circ \epsilon$, where neither δ nor ϵ have size 1.*

Proof of Theorem 4.10. We establish this result by defining a function h on β and then proving that if β is not reconstructible then h is periodic with period $|\epsilon| > 1$. This, in turn, yields our result. Since the proof of the periodicity of h is somewhat technical we will state the pertinent lemmas but postpone their proofs until Section 5. Before we define h we need a few other definitions.

Definition 4.11. *With respect to a composition $\gamma \vDash n$, for any $0 < i < n$, we say that i is of* type *0, 1, or 2, depending on whether there are 0, 1, or 2 occurrences of the partition $(i, n-i)$ in $\mathcal{M}(\gamma)$ or, equivalently, if 0, 1, or 2 of $i, n-i$ are in $S(\gamma)$. For $i = n/2$, if $n/2$ is an integer, we say that its type is twice the number of occurrences of $(n/2, n/2)$ in $\mathcal{M}(\gamma)$.*

Example 4.5. In the composition 11231, 1, 4 and 7 are type 2, 2 and 6 are type 1, 3 and 5 are type 0.

Fix a composition β of n. Let A_i be the set of those elements of $[n-1]$ that are of type i with respect to β. If $A_1 = \emptyset$, then clearly β is reconstructible. Note that $A_1 = \emptyset$ exactly when β is symmetric under reversal. Now suppose $A_1 \neq \emptyset$. Let k be the least element of A_1. Reversing β if necessary, we may assume that $k \in S(\beta)$, and $n - k \notin S(\beta)$.

Definition 4.12. *For $j \in A_1$, we say that j is* determined *if we can tell whether or not $j \in S(\beta)$ from $\mathcal{M}(\beta)$ and the knowledge that $k \in S(\beta)$.*

Example 4.6. In the composition 12132, $A_1 = [8]$. The determined elements are $\{1, 2, 4, 5, 7, 8\}$. 3 and 6 are undetermined, because $12132 \sim 13212$, and $3 \in S(12132)$, while $3 \notin S(13212)$, and the reverse is true of 6.

Definition 4.13. *For $x, y \in [n-1]$, we say that they* agree *if they are of the same type and either both or neither are in $S(\beta)$.*

Remark 4.7. Note that this second condition follows from the first for x, y of even type.

We extend the notion of type to all $\mathbb{Z}$ by saying that multiples of n are type 0, and otherwise, x has the same type as $x \bmod n$.

If every element of A_1 is determined, then β is reconstructible. Suppose β is not reconstructible, so there are undetermined elements of A_1. Let us define T_0 to be the set of all integers that are undetermined, where we extend the notion of determinedness to all integers by saying that, in general, x is determined if and only if $x \bmod n$ is determined. Let t_0 be the greatest common divisor of T_0. We are going to define inductively a collection T_i of sets of integers. We will write $T_{\leq j}$ for the union of $T_0, \ldots, T_j$. Let t_j be the greatest common divisor of $T_{\leq j}$.

Definition 4.14. *For $i > 0$, let T_i be the set of x not divisible by t_{i-1}, such that there is some $t \in T_{i-1}$ with x and $x + t$ of even type and disagreeing.*

Clearly, only finitely many of the T_i are non-empty. Let s be the greatest common divisor of all the T_i. By convention, set $t_{-1} = n$.

We are now ready to define the function h and state the results needed in order to analyze its periodicity. Let g and h be the functions defined on $\mathbb{Z}$ with respect to β by

$$h(x) = \begin{cases} 0 & \text{if } x \text{ is type } 0 \\ 1 & \text{if } x \text{ is type 1 and } x \bmod n \in S(\beta) \\ -1 & \text{if } x \text{ is type 1 and } x \bmod n \notin S(\beta) \\ 2 & \text{if } x \text{ is type } 2 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is of even type} \\ 1 & \text{if } x \text{ is type 1 and } x \bmod n \in S(\beta) \\ -1 & \text{if } x \text{ is type 1 and } x \bmod n \notin S(\beta). \end{cases}$$

Consider the following three statements concerning the functions g and h and the sets T_i.

$\mathbf{P}_i$: The function g is t_i-periodic except at multiples of t_i.
$\mathbf{Q}_i$: The function h is t_{i-1}-periodic except at multiples of t_i.
$\mathbf{R}_i$: For $x \in T_{i+1}$ and z of type 1, $t_i \nmid z$, z and $x + z$ agree.

These statements are all defined for $i \geq 0$. Note that $\mathbf{Q}_0$ is immediate, by our conventional definition of t_{-1}. The remaining statements will follow by simultaneous induction.

Example 4.8. Consider the composition $\beta = 132121332 = 213 \circ 12$. We can write out the values of g and h on $[18]$ as strings of 18 characters, writing $+$ for 1, and $-$ for -1.

$$\begin{aligned} g &= + - 0 + - + + - 0 + - - + - 0 + -0 \\ h &= + - 0 + - + + - 2 + - - + - 0 + -0 \end{aligned}$$

Since $\beta \sim 312 \circ 12$, 6 and 12 are type 1 undetermined. In fact, $T_0 \cap [18] = \{6, 12\}$; $t_0 = 6$. We next observe that 3 and 9 belong to T_1 because 3 and 3+6 (resp. 9 and 9+6) are of even type but disagree, and $6 \in T_0$. In fact, $T_1 \cap [18] = \{3, 9\}$. Hence $t_1 = 3$. All the T_i for $i > 1$ are empty. Thus $t_i = 3$ for $i > 1$.

We now take a look at the meanings of $\mathbf{P}_i$ and $\mathbf{Q}_i$ for this choice of β. $\mathbf{P}_0$ says that g is 6-periodic except at multiples of 6. $\mathbf{Q}_0$ says h is 18-periodic except at multiples of 6. $\mathbf{P}_1$ says that g is 3-periodic except at multiples of 3. $\mathbf{Q}_1$ says that h is 6-periodic except at multiples of 3. $\mathbf{P}_2$ says nothing more than $\mathbf{P}_1$. $\mathbf{Q}_2$ says that h is 3-periodic except at multiples of 3.

For clarity of exposition, we will divide the proof of the simultaneous induction into several parts:

- Proof of $\mathbf{P}_0$ (Lemma 5.7).
- Proof that $\mathbf{P}_j$ and $\mathbf{Q}_j$ for $j \leq i$ imply $\mathbf{R}_i$ (Lemma 5.10).
- Proof that $\mathbf{R}_i$ and $\mathbf{P}_i$ imply $\mathbf{P}_{i+1}$ (Lemma 5.17).
- Proof that $\mathbf{P}_{i+1}$ and $\mathbf{Q}_i$ imply $\mathbf{Q}_{i+1}$ (Lemma 5.18).

These four lemmas establish the simultaneous induction.

Observe that for i sufficiently large, $t_i = t_{i-1} = s$. Thus $\mathbf{Q}_i$ implies that h is s-periodic except at multiples of s. We now apply the following lemma:

Lemma 4.15. *The composition $\beta \vDash n$ has a decomposition $\beta = \delta \circ \epsilon$ with $|\epsilon| = p$ if and only if p divides n and the function h determined by β is p-periodic except at multiples of p.*

Proof. Suppose β has such a decomposition. It is clear that $p|n$. Write h_β for the function determined by β, and h_ϵ for the function determined by ϵ. For $x \in [n-1]$, $p \nmid x$, $h_\beta(x) = h_\epsilon(x \bmod p)$, which proves the desired periodicity.

Conversely, suppose that h_β has the desired periodicity. Define $\epsilon \vDash p$ by setting $h_\epsilon|_{[0,p-1]} = h_\beta|_{[0,p-1]}$. Define $\delta \vDash n/p$ by setting $h_\delta(x) = h_\beta(px)$. It is then clear that $\beta = \delta \circ \epsilon$. □

Example 4.9. Continuing Example 4.8, and applying Lemma 4.15 to the assertion of $\mathbf{Q}_2$, that h is 3-periodic except at multiples of 3, we conclude that $132121332 = \delta \circ \epsilon$ where $|\epsilon| = 3$, which is indeed true, since $132121332 = 213 \circ 12$.

Returning to the proof of Theorem 4.10, we see that an application of Lemma 4.15 implies that $\beta = \delta \circ \epsilon$, where $|\epsilon| = s$. We have $s < n$ since β is not reconstructible. Since also $s > 1$ (see Lemma 5.20), this factorization is non-trivial. This proves Theorem 4.10. □

We are now in a position to prove our main result.

Proof of Theorem 4.1. If β and γ satisfy $\beta \sim \gamma$ then by Theorem 4.10, we can factor $\beta = \beta_1 \circ \cdots \circ \beta_k$ where all the β_i are reconstructible. Applying Theorem 4.5 repeatedly, we find that $\gamma = \gamma_1 \circ \cdots \circ \gamma_k$, where $\gamma_i \sim \beta_i$. However, since the β_i are reconstructible, $\gamma_i \sim \beta_i$ implies that $\gamma_i = \beta_i$ or β_i^*.

Conversely, if $\beta = \beta_1 \circ \cdots \circ \beta_k$ and $\gamma = \gamma_1 \circ \cdots \circ \gamma_k$ such that either $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$ then by Theorem 4.4 it follows that $\beta \sim \gamma$.

Finally, observe that by Theorem 3.6 the equivalence class of β contains 2^r elements where r is the number of non-symmetric compositions under reversal in the irreducible factorization of β. □

5. Technical Lemmas

In this section we prove the technical lemmas which we deferred from the previous section. We begin with a basic lemma which will be useful throughout this section.

Lemma 5.1. *Let $\beta \vDash n$, and let $\alpha = m \circ \beta$ for some $m > 1$. Then:*

(1) *$\mathcal{M}(\alpha)$ can be determined from $\mathcal{M}(\beta)$.*
(2) *If n does not divide x, then x has the same type with respect to α as $x \bmod n$ does with respect to β.*
(3) *The functions g and h determined by α and β coincide.*
(4) *$t_i(\alpha) = t_i(\beta)$.*
(5) *Each of $\mathbf{P}_i$, $\mathbf{Q}_i$ and $\mathbf{R}_i$ holds for α if and only if it holds for β.*

Proof. Suppose we know $\mathcal{M}(\beta)$. We wish to determine $\mathcal{M}(\alpha)$. This is equivalent to determining the equivalence class of α with respect to equivalence for compositions. By Theorem 4.5, the equivalence class of α consists exactly of those compositions which can be written as $m \circ \gamma$ with $\gamma \sim \beta$. Thus, knowing $\mathcal{M}(\beta)$ suffices to determine $\mathcal{M}(\alpha)$.

Observe that (2), (3), and (5) are immediate from the definitions. For (4), we have to verify that x is determined for α if and only if $x \bmod n$ is determined for β. Suppose $x \bmod n$ is determined for β. That says exactly that all compositions in the equivalence class of β agree at $x \bmod n$. By Theorem 4.5, the equivalence class of α consists of the single-part partition m composed with elements of the equivalence class of β, and therefore x is determined for α. The converse follows the same way. □

The purpose of this lemma is that at any step in the simultaneous induction that proves $\mathbf{P}_i$, $\mathbf{Q}_i$ and $\mathbf{R}_i$, we can replace β by $m \circ \beta$ if we so desire.

5.1. **Proof of $\mathbf{P}_0$.** In this subsection we prove $\mathbf{P}_0$ (Lemma 5.7). We also prove Lemma 5.8, which will be necessary for our proof of Lemma 5.20.

Let the elements of $T_0 \cap [n-1]$ be $m_1 < \cdots < m_l$. Let $r_i = \gcd(m_1, \ldots, m_i)$. Note that m_1 and $n - m_1$ are both in T_0, so r_l divides n, and therefore r_l coincides with t_0, the greatest common divisor of T_0. We begin with some lemmas.

Lemma 5.2. *Suppose that x, y, and $x+y$ all lie in A_1. Then from $\mathcal{M}(\beta)$ we can tell if x, y, and $n-(x+y)$ all agree, or if they don't all agree.*

Proof. If x, y, and $n-(x+y)$ agree, then $(x, y, n-(x+y))$ does not appear in $\mathcal{M}(\beta)$. Otherwise, it does appear. □

Lemma 5.3. *Suppose x, y, and $x+y$ lie in $[n-1]$ and exactly two of them lie in A_1. Then we can determine from $\mathcal{M}(\beta)$ whether or not they agree.*

Proof. The proof is similar to that of Lemma 5.2, though there are more cases to check. It is sufficient to check the cases: x type 0 (and the others type 1); x type 2; $x+y$ type 0; $x+y$ type 2. In each case, one sees that the multiplicity of $(x, y, n-(x+y))$ in $\mathcal{M}(\beta)$ depends on whether the two type 1 points agree or disagree. □

Definition 5.4. *We say that a function f defined on a set of integers including $[p-1]$ is* antisymmetric *on $[p-1]$ if $f(x) = -f(p-x)$ for $0 < x < p$.*

Lemma 5.5. *Let f be a function on $[d-1]$ which takes values 0, 1, -1, $*$, and suppose that there is some $c < d$, such that for all x for which both sides are well-defined*

$$f(x) = -f(d-x) \tag{5.1}$$

$$f(x) = -f(c-x) \tag{5.2}$$

$$f(x) = f(c+x) \tag{5.3}$$

except that if either side equals $*$, the equation is not required to hold. Further, we require that the points where f takes the value $*$ are either exactly the multiples of c less than d, or else no points at all. Let r be the greatest common divisor of c and d. Then on multiples of r, f takes on only the values 0 and (possibly) $*$. On non-multiples of r, f is r-periodic, and f is antisymmetric on $[r-1]$.

Proof. The proof is by induction. We first consider the base case, which is when $r = c$. Periodicity is (5.3). Antisymmetry is (5.2). Notice (5.3) also implies that f is constant on multiples of c; by (5.1) this constant value is either $*$ or 0.

Now we prove the induction step. Let (5.1′), (5.2′), (5.3′) denote (5.1), (5.2), and (5.3), with d replaced by c and c replaced by $d \bmod c$. It is easy to see that (5.1′), (5.2′), and (5.3′) follow from (5.1), (5.2), (5.3). Also, f restricted to $[c-1]$ never takes on the value $*$. The desired results now follow by induction. □

Lemma 5.6. *For $1 \le i \le l$,*

(i) g is antisymmetric on $[r_i - 1]$,

(ii) g is r_i-periodic on $[m_i - 1]$ except at multiples of r_i.

Proof. The proof is by induction on i. We begin by proving the base case, which is when $i = 1$.

Suppose $0 < x < m_1$. By assumption, x and $m_1 - x$ are determined if they are type 1. Suppose one of them is of even type, and the other is type 1. Then by Lemma 5.3, we can determine m_1, contradiction. Suppose that x and $m_1 - x$ are both type 1. If they agree, Lemma 5.2 allows us to determine m_1, contradiction. Hence they must disagree. This establishes (i) in the base case. In the base case, (ii) is vacuous.

Now we prove the induction step. For $i \geq 2$ define a function g_i on $[m_i - 1]$, as follows:

$$g_i(x) = \begin{cases} * & \text{if } r_{i-1}|x \\ g(x) & \text{otherwise.} \end{cases}$$

We wish to apply Lemma 5.5 to g_i, with $d = m_i$, $c = r_{i-1}$. If $0 < x < m_i$, and neither x nor $m_i - x$ is a multiple of r_{i-1} (so in particular, neither is type 1 undetermined), then, as in the proof of the base case, $g_i(x) = -g_i(m_i - x)$. This is condition (5.1).

Suppose both x and $m_{i-1} + x < n$ are type 1 and determined. If they disagree (which means that x and $n - (m_{i-1} + x)$ agree), then we can determine m_{i-1}, contradiction. Similarly, if one is of even type and the other is type 1 determined, we can determine m_{i-1}, again a contradiction. It follows that g_i is m_{i-1}-periodic on $[m_i - 1]$ except at multiples of r_{i-1}. However, by induction, g_i is r_{i-1}-periodic except at multiples of r_{i-1} on $[m_{i-1} - 1]$, so g_i is r_{i-1}-periodic except at multiples of r_{i-1} on $[m_i - 1]$. This establishes condition (5.3). Condition (5.2) follows by the induction hypothesis.

Thus, we can apply Lemma 5.5. This proves the induction step, and hence the lemma. □

Lemma 5.7. $\mathbf{P}_0$ *holds, that is to say, g is t_0-periodic except possibly at multiples of t_0.*

Proof. Since $t_0 = r_l$, we have already shown (Lemma 5.6) that g is t_0-periodic on $[m_l - 1]$ except at multiples of t_0. Since g is antisymmetric on $[n - 1]$ (by the definition of g) it follows that g is t_0-periodic on $[n - 1]$ except at multiples of t_0, from which the desired result follows. □

We now prove Lemma 5.8 which will be used in the proof of Lemma 5.20.

Lemma 5.8. *The greatest common divisor t_0 of T_0 does not divide k.*

Proof. Suppose otherwise. Let i be the least index such that $r_i|k$. Note that $i > 1$, since $k < m_1$. By the result of applying Lemma 5.5 to g_i, we know that g_i is zero on multiples of r_i which are not multiples of r_{i-1}. However, this means that $g_i(k) = 0$, which contradicts the fact that k is type 1. □

5.2. **Proof of $\mathbf{R}_i$.** We begin by deducing $\mathbf{R}_0$ from $\mathbf{P}_0$ (Lemma 5.9). We then prove the general statement that $\mathbf{P}_j$ and $\mathbf{Q}_j$ for $j \leq i$ imply $\mathbf{R}_i$ (Lemma 5.10), which reduces to the argument for Lemma 5.9.

Lemma 5.9. $\mathbf{P}_0$ *implies* $\mathbf{R}_0$.

Proof. We must show that if z is type 1 and not a multiple of t_0 (which means in particular that it is determined), and $y_1 \in T_1$, then z and $z + y_1$ agree.

Since $y_1 \in T_1$, there is some $y_0 \in T_0$ such that y_1 and $y_1 + y_0$ are of even type and disagree. Clearly, we may assume that z, y_0, and y_1 are all positive.

If $z + y_1 + y_0 > n$, we may replace β by $m \circ \beta$ for some sufficiently large m, by Lemma 5.1. We also wish to assume that $z < y_0$. If this is not true, we can make it true by another replacement as above, followed by adding n to y_0.

By $\mathbf{P}_0$, we know that z and $z + y_0$ agree. Also, observe that since z is type 1, so is $n - z$, and thus, by $\mathbf{P}_0$, so is any $w \equiv -z \bmod t_0$. Since y_1 is of even type, this means that $t_0 \nmid z + y_1$, so $z + y_1$ is of even type or determined, and $\mathbf{P}_0$ tells us that $g(z + y_1 + y_0) = g(z + y_1)$.

By considering the multiplicity of $(y_0, y_1, z, n - (y_1 + y_0 + z))$ in $\mathcal{M}(\beta)$, we see that one of two things happens:

- $z + y_1$ and $z + y_1 + y_0$ are type 1 and both agree with z
- $z + y_1$ agrees with y_1, while $z + y_1 + y_0$ agrees with $y_0 + y_1$.

We now exclude the second possibility. Suppose we are in that case. Let $w = y_0 - z$. This w is not a multiple of t_0, so w is of even type or is determined. As already remarked, since $w \equiv -z \bmod t_0$, w must be type 1 determined. Now apply the previous part of the proof to (y'_0, y'_1, z') with $y'_0 = y_0$, $y'_1 = z + y_1$, $z' = w$. Then we see that either $w + z + y_1$ must either be the same type as y_1, or as w. However, $w + z + y_1 = y_0 + y_1$, and we know that it is of even type but disagrees with y_1, which is a contradiction. □

Lemma 5.10. $\mathbf{P}_j$ *and* $\mathbf{Q}_j$ *for* $j \leq i$ *imply* $\mathbf{R}_i$.

Proof. We wish to show that for z of type 1, $t_i \nmid z$ (so in particular z is determined), and $x \in T_{i+1}$, that z and $z + x$ agree. Write y_{i+1} for x. Now there is some $y_i \in T_i$ such that y_{i+1} and $y_i + y_{i+1}$ are of even type and disagree. Similarly, choose y_j for all $0 \leq j \leq i - 1$ so that y_{j+1} and $y_{j+1} + y_j$ are even type and disagree.

For I a subset of $[0, i + 1]$, write y_I for the sum of the y_j with $j \in I$. We now determine the types of y_I and $y_I + z$.

Lemma 5.11. *Let* $I \subset [0, i + 1]$. *Let* j *be the maximal element of* I. *Then:*

1. *If* I *does not contain* $j - 1$, *then* y_I *agrees with* y_j.
2. *If* I *does contain* $j - 1$, *then* y_I *is of even type disagreeing with* y_j.
3. *Either* $z + y_I$ *is of even type or it is determined.*
4. *If* I *does not contain* $i + 1$, $z + y_I$ *agrees with* z.
5. *If* I *contains* $i + 1$ *but not* i, $z + y_I$ *agrees with* $z + y_{i+1}$.
6. *If* I *contains* $i + 1$ *and* i, $z + y_I$ *agrees with* $z + y_{i+1} + y_i$.
7. *Either* $z + y_{i+1}$ *and* $z + y_{i+1} + y_i$ *agree, or they are both of even type.*

Proof. Statement (1) follows from $\mathbf{Q}_{j-1}$, since y_j is not a multiple of t_{j-1}. Statement (2) follows because y_j and $y_j + y_{j-1}$ are of even type and disagree, and then applying $\mathbf{Q}_{j-1}$ as before.

Since g is t_i-periodic except at multiples of t_i, and its period is anti-symmetric, it follows that any $w \equiv -z \bmod t_i$ must be of odd type. Thus $y_{i+1} \not\equiv -z \bmod t_i$. All the other y_l are multiples of t_i. Thus $t_i \nmid z + y_I$, so $z + y_I$ is either determined or of even type. This establishes (3).

Statement (4) follows from $\mathbf{P}_i$, since z is not a multiple of t_i. Since $t_i \nmid z + y_{i+1}$, (5) follows from $\mathbf{Q}_i$. Statement (6) follows from $\mathbf{Q}_i$ together with the fact that, since t_i does not divide $z + y_{i+1}$, it doesn't divide $z + y_{i+1} + y_i$. Statement (7) follows from $\mathbf{P}_i$. □

We now return to the proof of $\mathbf{R}_i$. We want to assume that $p = n - (z + \sum_{j=0}^{i+1} y_j) > 0$, and that p does not coincide with any y_j or z. In order to guarantee this, by Lemma 5.1, we may replace β by $m \circ \beta$, and add multiples of n as desired to the y_i and z.

Since $y_0 \in T_0$, it is undetermined. This means precisely that there is some composition γ which is equivalent to β (but not equal to β), such that $k \in S(\gamma)$, but y_0 is in exactly one of $S(\beta)$, $S(\gamma)$. Note that since γ is equivalent to β, every $0 < x < n$ has the same type in β and γ.

Write ν for the partition of n whose parts are $(y_0, y_1, \ldots, y_{i+1}, z, p)$. One consequence of the equivalence of β and γ that we shall focus on is the fact that $\mathrm{mult}_{\mathcal{M}(\beta)}(\nu) = \mathrm{mult}_{\mathcal{M}(\gamma)}(\nu)$.

Let Ω be the set of all the compositions of n determining the partition ν. It will be convenient for us to keep track of such a composition as two lists: the left list, which consists of the components in order which precede p, and the right list, which consists of the components following p in reverse order. For any composition in Ω, each component other than p occurs in exactly one list, and any pair of lists with this property determines a composition.

We put an order $\prec$ on the components y_j, z by ordering the y_j by their indices, and setting $y_j \prec z$ for $j \neq i + 1$. (Thus, the order is nearly a total order but not quite: y_{i+1} and z are incomparable.)

Definition 5.12. *A composition in Ω is called* ordered *if both its right and left lists are in (a linear extension of) $\prec$ order. The other compositions in Ω are called* disordered.

Lemma 5.13. *The number of disordered compositions which can be obtained as coarsenings of β is the same as the number that can be obtained as coarsenings of γ.*

Proof. To prove this lemma, we will define an involution i on disordered compositions such that κ is a coarsening of β if and only if $i(\kappa)$ is a coarsening of γ.

Fix a disordered composition κ. Let $M(\kappa)$ be the maximal subset of $y_0, \ldots, y_{i+1}, z$ which is a $\prec$ order ideal such that $M(\kappa)$ consists of the union of initial subsequences of the left and right lists of γ, and these subsequences are in $\prec$ order. Write $M_L(\kappa)$ and $M_R(\kappa)$ for these two initial subsequences. Then $i(\kappa)$ is obtained by swapping $M_L(\kappa)$ and $M_R(\kappa)$. Observe that $i(\kappa)$ is disordered if and only if κ is disordered.

Example 5.1. We give an example of the definition of i.

$$\text{If } \kappa = \left| \begin{array}{c|c} y_0 & y_1 \\ y_2 & y_4 \\ y_5 & y_3 \\ z & \end{array} \right| \text{ then } M(\kappa) = \{y_0, y_1, y_2\} \text{ and } i(\kappa) = \left| \begin{array}{c|c} y_1 & y_0 \\ y_5 & y_2 \\ z & y_4 \\ & y_3 \end{array} \right|.$$

We shall now define a bijection, also denoted i, taking $S(\kappa)$ to $S(i(\kappa))$, such that for $x \in S(\kappa)$, $x \in S(\beta)$ if and only if $i(x) \in S(\gamma)$. The existence of such a bijection between $S(\kappa)$ and $S(i(\kappa))$ implies that κ is a coarsening of β if and only if $i(\kappa)$ is a coarsening of γ, proving the lemma.

To define the bijection between $S(\kappa)$ and $S(i(\kappa))$, we need another definition:

Definition 5.14. *We say $x \in S(\kappa)$ is an* outside break *if it is either the sum of an initial subsequence of $M_L(\kappa)$ or n minus the sum of an initial subsequence of $M_R(\kappa)$. Otherwise, we say that $x \in S(\kappa)$ is an* inside break.

Example 5.2. In our continuing example, the outside breaks of κ are y_0, y_0+y_2, and $n-y_1$, while the outside breaks of $i(\kappa)$ are $n-y_0$, $n-(y_0+y_2)$, and y_1. The inside breaks in κ are $y_0+y_2+y_5$, $y_0+y_2+y_5+z$, $n-(y_1+y_4)$, $n-(y_1+y_4+y_3)$, while the corresponding inside breaks in $i(\kappa)$ are y_1+y_5, y_1+y_5+z, $n-(y_0+y_2+y_4)$, $n-(y_0+y_2+y_4+y_3)$.

If x is an outside break of κ, set $i(x)=n-x$. Clearly, $i(x)$ is an outside break of $i(\kappa)$. Now observe that all the outside breaks except y_0 or $n-y_0$ are of even type in β by Lemma 5.11. Thus for these outside breaks (excluding y_0 and $n-y_0$), $x \in S(\beta)$ if and only if x is type 2 for β if and only if x is type 2 for γ if and only if $i(x)$ is type 2 for γ if and only if $i(x) \in S(\gamma)$. On the other hand, $y_0 \in S(\beta)$ if and only if $n-y_0 \in S(\gamma)$. Thus, for x an outside break of κ, $x \in S(\beta)$ if and only if $i(x) \in S(\gamma)$.

Now we consider the inside breaks. Let y^L denote the sum of the y_j appearing in $M_L(\kappa)$, and similarly for y^R. If x is an inside break for κ, set $i(x)=x-y^L+y^R$. This is clearly an inside break for $i(\kappa)$.

Since κ is disordered, define l by $M(\kappa)=\{y_0,y_1,\ldots,y_l\}$. By definition, all the y_j that occur in y^L and y^R have $j \leq l$. To show that $x \in S(\beta)$ if and only if $i(x) \in S(\gamma)$ there are a four cases to consider: when x is of the form y_I, $z+y_I$, $n-y_I$, or $n-y_I-z$. In the first case, observe that I contains at least one element greater than $l+1$, and so, by Lemma 5.11(1) or (2), x and $i(x)$ agree and are of even type. It follows that $x \in S(\beta)$ if and only if $i(x) \in S(\beta)$ if and only if $i(x) \in S(\gamma)$, as desired.

In the second case, since $l \leq i-1$ it is again clear by Lemma 5.11(4), (5), or (6), that x and $i(x)$ agree, so $x \in S(\beta)$ if and only if $i(x) \in S(\beta)$. By Lemma 5.11(3), $i(x)$ is either determined or of even type, so $i(x) \in S(\beta)$ if and only if $i(x) \in S(\gamma)$, which establishes the desired result.

The third and fourth cases are similar to the first and second cases. This completes the proof that i is a bijection from $S(\beta)$ to $S(\gamma)$, which completes the proof of the lemma. □

Now we consider the ordered compositions. Suppose κ is an ordered composition which is a coarsening of β. Thus y_0 is the beginning of one list. Which list is determined by which of y_0 and $n-y_0$ is a break in β. Since y_1 and y_1+y_0 disagree, which list y_1 occurs in is forced. Similarly for y_2, etc. Hence all the y_j are forced up to y_i. There are now six possible ways to complete the construction. For each of these six possibilities we show the positions of y_i, y_{i+1}, and z in the two lists.

(a)		(b)		(c)		(d)		(e)		(f)	
y_i		y_i	z	y_i		y_i	z	y_i	y_{i+1}	y_i	y_{i+1}
z			y_{i+1}	y_{i+1}		y_{i+1}			z	z	
y_{i+1}				z							

The argument now proceeds as in Lemma 5.9. Essentially what has happened is that by reducing to ordered compositions, we do not need to consider the y_j with $j<i$. We are now only interested in the middle part of the composition, which involves parts y_i, y_{i+1}, z, and p. Also y_i now behaves like y_0 in Lemma 5.9: we count

up the number of compositions which occur with y_i on the extreme left (among the four parts we are interested in) and those where it occurs on the extreme right. One of these numbers represents the contribution of ordered partitions to $\mathrm{mult}_{\mathcal{M}(\beta)}(\nu)$, the other the contribution to $\mathrm{mult}_{\mathcal{M}(\gamma)}(\nu)$. These numbers must therefore be the same. As in the proof of Lemma 5.9, we consider cases based on the types (and for type 1, whether or not each is a break) of y_{i+1}, z, $y_{i+1}+z$, and $y_{i+1}+y_i+z$. Lemma 5.11(7) eliminates a number of possibilities and with the remainder, as in Lemma 5.9, one of the following two things must happen:

- $z+y_{i+1}$ and $z+y_{i+1}+y_i$ are type 1 and both agree with z
- $z+y_{i+1}$ agrees with y_{i+1}, while $z+y_{i+1}+y_i$ agrees with $y_{i+1}+y_i$.

We now exclude the second possibility.

Since z is not a multiple of t_i, $\mathbf{P}_i$ tells us that y_i-z agrees with $n-z$, which is type 1 determined. Also by $\mathbf{P}_i$, $z \not\equiv -y_{i+1} \pmod{t_i}$, so t_i does not divide $z+y_{i+1}$. Since $y_i \in T_i$, and $z+y_{i+1}$ and $z+y_{i+1}+y_i$ disagree, $z+y_{i+1} \in T_{i+1}$. Set $z'=y_i-z$, $y'_{i+1}=z+y_{i+1}$, $y'_j=y_j$ for $j \le i$. Applying the whole proof of the lemma so far, we find that $z'+y'_{i+1}$ must agree either with z' or y'_{i+1}, which is to say that y_i+y_{i+1} agrees with either y_i-z or $z+y_{i+1}$, both of which are impossible, and we are done. ☐

5.3. **Proofs of $\mathbf{P}_i$ and $\mathbf{Q}_i$.** We begin with a preliminary lemma which will be useful for the proofs of Lemmas 5.17 and 5.18. While working towards proving these two lemmas, we will often need to consider $\mathbb{Z}/t_j\mathbb{Z}$ (for some j). We will write $\mathbb{Z}_{t_j}$ for $\mathbb{Z}/t_j\mathbb{Z}$, and $\bar{z}$ for the image of z in $\mathbb{Z}_{t_j}$.

Lemma 5.15. *Let f be a function defined on $\mathbb{Z}$. Let $S \subset \mathbb{Z}_{t_j}$ be such that $f(z) = f(z+t_{j-1})$ for $\bar{z} \in S$. Suppose further that for any $x \in T_j$, $f(z)=f(z+x)$ provided $\bar{z} \in S$. Then $f(z)=f(z+t_j)$ for all $\bar{z} \in S$.*

Proof. Write t_j as the sum of a series of elements of $T_{\le j}$. Let the partial sums of this series be $x_1, x_2, \ldots, x_m = t_j$. Then observe that if $\bar{z} \in S$, then the same is true for $\overline{z+x_l}$ for all l. It follows from the assumptions of the lemma that $f(z+x_l)=f(z+x_{l+1})$, and the result is proven. ☐

Lemma 5.16. *For $p>0$, if $x \in T_p$, then there is an element $x' \in T_p$ such that $x \equiv -x'$ modulo t_{p-1}.*

Proof. Since $x \in T_p$, there is some $y \in T_{p-1}$ such that x and $y+x$ are of even type and disagree. It follows that $n-y-x$ and $n-x$ are of even type and disagree, and hence that $n-y-x \in T_p$. Set $x'=n-y-x$. ☐

Lemma 5.17. *$\mathbf{R}_i$ and $\mathbf{P}_i$ imply $\mathbf{P}_{i+1}$.*

Proof. Let $j=i+1$. We wish to show that g is t_j-periodic except at multiples of t_j. Let $S=\mathbb{Z}_{t_j} \setminus \{\bar{0}\}$. $\mathbf{P}_i$ tells us that g is t_{j-1}-periodic except at multiples of t_{j-1}. Suppose $\bar{z} \in S$, and $x \in T_j$. $\mathbf{R}_i$ tells us that if z is type 1, then $g(z+x)=g(z)$. Likewise, if $z+x$ is type 1, then, choosing x' as provided by Lemma 5.16, $z+x+x'$ is type 1, and now by the t_{j-1} periodicity of g, $g(z+x)=g(z)$. If neither z nor $z+x$ is type 1, then $g(z+x)=0=g(z)$.

Thus, it follows that for any z such that $\bar{z} \in S$, and x in T_j, that $g(z+x)=g(z)$. Therefore, we can apply Lemma 5.15, and desired result follows. ☐

Lemma 5.18. *$\mathbf{P}_{i+1}$ and $\mathbf{Q}_i$ imply $\mathbf{Q}_{i+1}$.*

Proof. Let $j = i$. Let $S = \mathbb{Z}_{t_j} \setminus t_{j+1}\mathbb{Z}_{t_j}$. We wish to show that $h(z) = h(z + t_j)$ for $\bar{z} \in S$. $\mathbf{Q}_i$ tells us that $h(z + t_{j-1}) = h(z)$ for $\bar{z} \in S$. Now suppose that we have some z such that $\bar{z} \in S$, and $x \in T_j$. By $\mathbf{P}_{i+1}$, if $h(z) = \pm 1$ then $h(z + x) = h(z)$. Also by $\mathbf{P}_{i+1}$, if $h(z)$ is even, then so is $h(z + x)$. Now, if $h(z) \neq h(z + x)$, then $z \in T_{j+1}$, contradicting our assumption. Thus $h(z + x) = h(z)$ and we can apply Lemma 5.15 to obtain the desired result. □

5.4. **Proof that $s > 1$.** Finally, we show that s, the greatest common divisor of the T_i, is greater than 1.

Lemma 5.19. *Let G be an arbitrary finite abelian group, which we write additively. Let Y be a set of generators for G, closed under negation. Fix some $a \in G$. For any b in G, it is possible to write b as the sum of a series of elements from Y, so that no proper partial sum of the series equals a (i.e., excluding the empty partial sum and the complete partial sum).*

Proof. The proof is by induction on $|G|$. If G is cyclic, pick $x \in Y$ a generator for G. If b occurs before a in the sequence $x, 2x, \ldots$, then we are done. Otherwise, use $-x$.

If G is not cyclic, find a cyclic subgroup H which is a direct summand, and has a generator $x \in Y$. Let $\bar{a}$, $\bar{b}$ denote the images of a and b in G/H. Apply the induction hypothesis to G/H. Lifting to G, we obtain a series whose sum differs from b by an element of H, which we can dispose of as in the cyclic case above. The only problem occurs if $\bar{b} = \bar{a}, b \neq a$, and the series for G/H happens to sum to a. In this case, instead of putting the series obtained for H after the series for G/H, begin with the first term from the series for H, followed by the series for G/H, followed by the rest of the series for H. □

Example 5.3. Let $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Let $Y = \{(1, 0), (0, 1), (0, -1)\}$ Let $a = (1, 0)$, $b = (1, 1)$. If we choose H to be the copy of $\mathbb{Z}/2\mathbb{Z}$, then the G/H series is $(0, 1)$, the H series is $(1, 0)$, and we can take $((0, 1), (1, 0))$ as our desired series.

If we take H to be the copy of $\mathbb{Z}/3\mathbb{Z}$, then the G/H series is $(1, 0)$, and the series for H is $(0, 1)$. In this case we cannot just concatenate the two series, because we are in the undesirable situation described above where $\bar{b} = \bar{a}$ and the G/H series sums to a. Thus we take the first term of the H series (which in this case happens to be all of the H series), followed by the G/H series, followed by the rest of the H series (which in this case happens to be empty) and we obtain $((0, 1), (1, 0))$ as our desired series.

Lemma 5.20. *The greatest common divisor s of all the T_i is greater than 1.*

Proof. Suppose otherwise. Let i be as small as possible, so that t_i divides k. By Lemma 5.8, $i > 0$. We will now demonstrate that all multiples of t_i which are not multiples of t_{i-1} must be type 1. However, since elements of T_i are of even type, this would force T_i to be empty, and $t_i = t_{i-1}$, a contradiction.

By $\mathbf{R}_{i-1}$, adding an element of T_i to an element of type 1 not divisible by t_{i-1} yields another element of type 1. Let x be an arbitrary element of T_i which is not a multiple of t_{i-1}. We wish to write $x - k$ as the sum of a series of elements from $T_{\leq i-1}$ such that, if the partial sums are $z_1, \ldots, z_m = x - k$, then for no l is $k + z_l$ divisible by t_{i-1}. If we can do this, we can conclude that x is type 1.

We know that the elements of T_i generate $t_i\mathbb{Z}/t_{i-1}\mathbb{Z}$, but in fact more is true. By Lemma 5.16, we know that T_i contains a set of generators and their negatives for $t_i\mathbb{Z}/t_{i-1}\mathbb{Z}$. We can therefore apply Lemma 5.19, and we are done. □

6. The Cone of F-positive Symmetric Functions

We now consider the set $\mathcal{K}$ of all $F \in \Lambda$ having a nonnegative representation in terms of the basis of fundamental quasisymmetric functions, that is,

$$\mathcal{K} = \{\sum_{\alpha} c_\alpha F_\alpha \in \Lambda \mid c_\alpha \geq 0 \text{ for all } \alpha\}. \tag{6.1}$$

Since $\mathcal{K}$ is the intersection of Λ with the nonnegative orthant of $\mathcal{Q}$ (with respect to the basis $\{F_\beta\}$), $\mathcal{K}_n := \mathcal{K} \cap \Lambda_n$ is a polyhedral cone for each $n \geq 0$. It contains the Schur functions s_λ, $\lambda \vdash n$, so it has full dimension in Λ_n.

6.1. The generators of $\mathcal{K}_n$.

We consider first the minimal generators of the cone $\mathcal{K}_n$, i.e., its 1-dimensional faces or extreme rays. These include all the Schur functions and, in general, can be characterized by a condition of being balanced.

We begin by considering the notion of the spread of a quasisymmetric function. For $\beta \preceq \gamma$, we denote by

$$[\beta, \gamma]_\preceq = \{\alpha \mid \beta \preceq \alpha \preceq \gamma\} \tag{6.2}$$

the lexocographic interval between β and γ. For a quasisymmetric function $F = \sum c_\alpha F_\alpha \in \mathcal{Q}$, we define the *spread* of F to be the smallest lexocographic interval $[\beta, \gamma]_\preceq$ so that $c_\alpha = 0$ whenever $\alpha \notin [\beta, \gamma]_\preceq$.

For a partition $\lambda \vdash n$, we let λ' denote the conjugate partition and define the composition

$$\widetilde{\lambda} := \beta\left([n-1] \setminus S\left(\lambda'\right)\right). \tag{6.3}$$

Thus if $\lambda = 33$, then $\lambda' = 222$, so $[5] \setminus S(\lambda') = \{1, 3, 5\} \subset [5]$ and $\widetilde{\lambda} = 1221$. Note that λ corresponds to the descent set of the tableaux T_r obtained by filling the Ferrers shape λ by rows, $\widetilde{\lambda}$ corresponds similarly to descent set of the filling T_c by columns and $\widetilde{\lambda} \preceq \lambda$, with equality if and only if λ is n or 1^n. In the example above, we have

$$T_r = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix} \text{ and } T_c = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{matrix}.$$

Also note that $\widetilde{\lambda} = \widetilde{\nu}$ if and only if $\lambda = \nu$.

Proposition 6.1. *The spread of the Schur function s_λ is the interval $[\widetilde{\lambda}, \lambda]_\preceq$.*

Proof. Recall that $s_\lambda = \sum c_\alpha F_\alpha$ where c_α is the number of standard Young tableaux T of shape λ with $\alpha = \beta(D(T))$. Let T_r, respectively, T_c be the standard Young tableaux obtained by filling the Ferrers diagram with shape λ by rows, respectively, by columns. As noted, T_r and T_c correspond this way to λ and $\widetilde{\lambda}$. Now for any other tableaux T, let i_r be the first index for which $i_r + 1$ is not in the same row as in T_r and let i_c be the first index for which $i_c + 1$ is not in the same column as in T_c. Then i_r is a descent in T but not in T_r and i_c is a descent in T_c but not in T, so $\widetilde{\lambda} \prec \beta(D(T)) \prec \lambda$. □

Lemma 6.2. *Suppose $\lambda, \nu \vdash n$, $\lambda \neq \nu$, and the spread of s_ν is a subset of the spread of s_λ. Then if $s_\lambda = \sum c_\beta F_\beta$ it follows that $c_\nu = 0$.*

Proof. By assumption, we have $\widetilde{\lambda} \prec \widetilde{\nu} \preceq \nu \prec \lambda$. The first inequality implies $\nu' \prec \lambda'$, so there is a minimum index $j > 1$ so that

$$\nu_1 + \cdots + \nu_j > \lambda_1 + \cdots + \lambda_j. \tag{6.4}$$

Now if $c_\nu \neq 0$, then there must be a filling T of the shape λ with $\beta(D(T)) = \nu$. Then indices $1, 2, \ldots, \nu_1$ need to be in the first row of T, indices $\nu_1 + 1, \ldots, \nu_1 + \nu_2$ need to be in the first two rows, etc. However (6.4) indicates this filling will fail at row j. □

We can now prove the main result of this section.

Theorem 6.3. *The Schur functions s_λ are extreme in the cone $\mathcal{K}$.*

Proof. Suppose $s_\lambda = F_1 + F_2$ with $F_1, F_2 \in \mathcal{K}$. Then $F_i = \sum_\mu a^i_\mu s_\mu$ with

$$a^1_\lambda + a^2_\lambda = 1 \quad \text{and} \quad a^1_\mu + a^2_\mu = 0, \quad \mu \neq \lambda. \tag{6.5}$$

Suppose $F_i = \sum c^i_\beta F_\beta$.

If there is a $\mu \neq \lambda$ with $a^i_\mu \neq 0$, then either $\mu \succ \lambda$, $\widetilde{\mu} \prec \widetilde{\lambda}$ or the spread of s_μ is a subset of the spread of s_λ. If there is such a μ with $\mu \succ \lambda$, choose one which is lexicographically largest. If not, but there is one with $\widetilde{\mu} \prec \widetilde{\lambda}$, then choose such a μ such that $\widetilde{\mu}$ is lexicographically smallest. Otherwise, choose a lexicographically largest μ with the spread of s_μ a subset of the spread of s_λ. By Proposition 6.1 and Lemma 6.2, one of the F_i must have $c^i_\mu < 0$ or $c^i_{\widetilde{\mu}} < 0$ for the chosen μ.

Thus $a^i_\mu = 0$ for $\mu \neq \lambda$ and so both F_1 and F_2 are multiples of s_λ, showing s_λ to be extreme. □

Note that there are extremes other than the Schur functions. The first one appears when $n = 4$:

$$s_{31} + s_{211} - s_{22} = F_{31} + F_{13} + F_{211} + F_{112}$$

is extreme in $\mathcal{K}_4$. In $\mathcal{K}_5$, there are two such extremes, $s_{311} + s_{2111} - s_{221}$ and $s_{41} + s_{311} - s_{32}$. In $\mathcal{K}_6$, there are 23. At present there is no general description of which combinations of Schur functions are extreme.

Consequently, we consider next the problem of determining when a quasisymmetric function $F = \sum h_S F_S$ is an extreme element of the cone $\mathcal{K}$ of F-positive symmetric functions. (Here we begin indexing by subsets of $[n]$ in place of compositions of $n+1$, where $F_S = F_{\beta(S)}$.) We relate this to a property of the multicollection $\{S^{\,h_S}\}$, which leads to the notion of *fully balanced multicollections* of subsets of a finite set. Fully balanced multicollections with nonnegative multiplicities will yield F-positive symmetric functions, in general, while minimal such collections give rise to extremes.

We say a subset $S \subset [n]$ has *profile* $a_1, \ldots, a_k$ if S consists of maximal consecutive strings of length $a_1, \ldots, a_k$ in some order. In this case, $|S| = a_1 + \cdots + a_k$. For example, $\{2, 3, 5, 7, 8, 9\} \subset [11]$ has profile 321000. For $\lambda = \lambda_1\lambda_2 \ldots \lambda_k \vdash n + 1$, define

$$\mathcal{F}_\lambda = \{S \subset [n] \mid S \text{ has profile } \lambda_1 - 1, \ldots, \lambda_k - 1\}. \tag{6.6}$$

Thus if $S = \{2, 3, 5, 7, 8, 9\} \subset [11]$, then $S \in \mathcal{F}_{432111}$. Further $\mathcal{F}_{11\ldots1} = \{\emptyset\}$, $S \in \mathcal{F}_{21\ldots1}$ if and only if $|S| = 1$, and $S \in \mathcal{F}_{221\ldots1}$ if and only if $S = \{i, j\}$, where $i < j - 1$, while $S = \{i, i + 1\} \in \mathcal{F}_{31\ldots1}$.

We denote a *multicollection* of subsets of $[n]$ by $\{S^{\ k_S} \mid S \subset [n]\} = \{S^{\ k_S}\}$, where k_S denotes the multiplicity of the subset S. For our purposes, a multicollection $\{S^{\ k_S}\}$ can have any rational multiplicities k_S.

Definition 6.4. *Let $\lambda \vdash n+1$. A multicollection $\{S^{\ k_S}\}$ of subsets of $[n]$ is* λ-balanced *if there is a constant κ_λ such that for all $T \in \mathcal{F}_\lambda$,*

$$\sum_{S \supseteq T} k_S = \kappa_\lambda. \tag{6.7}$$

The multicollection $\{S^{\ k_S}\}$ is fully balanced *if it λ-balanced for all $\lambda \vdash n+1$.*

Multicollections that are $21\ldots1$-balanced have been called *balanced* in the literature of cooperative game theory [7], although there the term is applied to the underlying collection whenever positive multiplicities k_S exist.

Theorem 6.5. *A homogeneous quasisymmetric function $F = \sum_S h_S F_S \in \mathcal{Q}_{n+1}$ is symmetric if and only if the multicollection $\{S^{\ h_S}\}$ of subsets of $[n]$ is fully balanced.*

Proof. Note that, for $\mu \vdash n+1$, $R \in \mathcal{F}_\mu$ if and only if $\lambda\left(\beta([n] \setminus R)\right) = \mu$. Further, note that if $T \in \mathcal{F}_\lambda$ and $R \subset T$, $R \neq T$, then $R \in \mathcal{F}_\mu$ for some $\mu \prec \lambda$. By inclusion-exclusion, we get

$$\sum_{S \supseteq T} h_S = \sum_{R \subseteq T} (-1)^{|R|} \sum_{S \subseteq [n] \setminus R} h_S = \sum_{R \subseteq T} (-1)^{|R|} f_{[n]\setminus R}, \tag{6.8}$$

where f_S and h_S are related as d_β and c_β in (2.5). Now, F is symmetric if and only if $f_{[n]\setminus R}$ only depends on μ for $R \in \mathcal{F}_\mu$. Thus if F is symmetric, then (6.8) shows the sum $\sum_{S \supseteq T} h_S$ to depend only on λ (and $\mu \prec \lambda$) when $T \in \mathcal{F}_\lambda$.

Now suppose the multicollection $\{S^{\ h_S}\}$ is λ-balanced for all $\lambda \vdash n+1$. We argue by induction on the lexicographic order on partitions. We assume $f_{[n]\setminus R}$ only depends on μ for all $R \in \mathcal{F}_\mu$, $\mu \prec \lambda$. (The base case for $\lambda = 11\ldots1$ is trivial.) For $T \in \mathcal{F}_\lambda$, the assertion now follows from (6.8), since the number of $R \subset T$ with $R \in \mathcal{F}_\mu$, for $\mu \prec \lambda$, depends only on λ. □

Thus, elements of $\mathcal{K}_{n+1}$ correspond to fully balanced collections with nonnegative multiplicities. Those with minimal support $\{S \mid h_S \neq 0\}$ correspond to the extremes of the cone. One can view integral extremes of $\mathcal{K}_{n+1}$ as combinatorial designs of an extremely balanced sort: each element of $[n]$ is in the same number of sets (counting multiplicity), as are each nonadjacent pair, each adjacent pair, etc. One is led to wonder whether the designs coming this way from Schur functions have special properties among these. The first of these for which the multiplicities are not all one is

$$\begin{aligned} s_{321} &= F_{\{1,3\}} + F_{\{1,4\}} + F_{\{2,3\}} + 2F_{\{2,4\}} + F_{\{2,5\}} + F_{\{3,4\}} + F_{\{3,5\}} \\ &+ F_{\{1,2,4\}} + F_{\{1,2,5\}} + F_{\{1,3,4\}} + 2F_{\{1,3,5\}} + F_{\{1,4,5\}} + F_{\{2,3,5\}} + F_{\{2,4,5\}}. \end{aligned}$$

Here $\kappa_{21111} = 8$, $\kappa_{3111} = 2$, $\kappa_{2211} = 4$, $\kappa_{321} = 1$ and $\kappa_{222} = 2$.

6.2. **The facets of $\mathcal{K}_n$.** To describe the facets of $\mathcal{K}_n$, we rewrite (6.1) as follows. Since the Schur functions s_λ, $\lambda \vdash n$, are a basis for Λ_n, writing $s_\lambda = \sum_\beta [s_\lambda]_{F_\beta} F_\beta$, we see that

$$\mathcal{K}_n = \{\sum_{\lambda \vdash n} c_\lambda s_\lambda \mid \sum_\lambda c_\lambda [s_\lambda]_{F_\beta} \geq 0 \text{ for all } \beta \vDash n\}. \tag{6.9}$$

Equation (6.9) gives 2^{n-1} inequalities for $\mathcal{K}_n$, one for each $\beta \models n$. However, when $\beta \sim \gamma$, these inequalities are identical (see Definition 2.3). In fact, we conjecture that these are the only redundant inequalities, so the facets of $\mathcal{K}_n$ would be in bijection with the equivalence classes of compositions under $\sim$.

The inequality for $\mathcal{K}_n$ given by $c_\alpha \geq 0$ in (6.1) is redundant if and only if there exist $a_\beta \geq 0$ such that

$$c_\alpha = \sum_{\beta \not\sim \alpha} a_\beta c_\beta \tag{6.10}$$

holds for all $F = \sum c_\gamma F_\gamma \in \Lambda$.

For each composition $\beta \models n$ we define the vector $v_\beta = (v_{\beta,\lambda}; \lambda \vdash n)$ by $v_{\beta,\lambda} = \mathrm{mult}_{\mathcal{M}(\beta)}(\lambda)$. By definition, $\beta \sim \gamma$ if and only if $v_\beta = v_\gamma$.

Proposition 6.6. *The inequality $c_\alpha \geq 0$ is redundant for some $\alpha \models n$ if and only if v_α is not extreme in the convex hull of all v_β, $\beta \models n$.*

We end with the following conjecture.

Conjecture 6.1. Any one, and so all, of the equivalent statements holds:

(1) The facets of $\mathcal{K}_n$ are in bijection with the equivalence classes of compositions $\beta \models n$,
(2) The inequalities $c_\alpha \geq 0$, $\alpha \models n$, are all irredundant,
(3) Each v_α is extreme in the convex hull of all v_β, $\beta \models n$.

One can imagine an approach to Conjecture 6.1 that uses Theorem 4.1 along with a separation argument for v_β that targets the decomposition structure of the composition β.

6.3. **Ribbon-positivity.** One can call a symmetric function *F-positive* if it belongs to the cone $\mathcal{K}$. Being F-positive is a weakening of the condition of being Schur-positive. Both F-positivity and Schur-positivity are closed under taking products. The quasisymmetric functions F_S have been identified with characters of 0-Hecke algebras [13, §4.1], lending a representation-theoretic interpretation to being F-positive.

A strengthening of Schur-positivity would be what we might call *ribbon-positivity*, that is, belonging to the cone in Λ spanned by the ribbon Schur functions $r_\alpha, \alpha \models n$. By (2.2), this cone is also closed under taking products. As far as we know, this ribbon cone has not been studied.

Stronger still would be membership in the cone spanned by the $r_\lambda, \lambda \vdash n$. That this simplicial cone is strictly smaller than the ribbon cone can be seen from the fact that $r_{132} = r_{321} + r_{33} - r_{42}$, which follows from (2.3). This relation suggests that the ribbon Schur functions might all be extreme in the ribbon cone.

We know of one example of ribbon-positive symmetric functions. In [6], Hersh shows that the chain-enumeration quasisymmetric function of k-shuffle posets are ribbon-positive by showing they are sums of products of ribbon Schur functions.

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