

# A matrix interpretation for the descent algebra of type $D$

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## Abstract

We define a certain set of compositions  $\mathcal{C}(n)$ , and using these compositions we re-interpret multiplication in the descent algebra of  $D_n$  in terms of matrices. Using this interpretation we can also give a basis for the radical. We then sketch how we can derive such an interpretation.

Nous définissons un ensemble de compositions,  $\mathcal{C}(n)$ , et utilisons compositions pour donner une interprétation, en terme de matrices, de la multiplication dans ces l'algèbra de descentes de  $D_n$ . Nous donnons aussi une base du radical, puis nous ébauchons la preuve de notre interprétation.

## 1 Introduction

Let  $(W, S)$  be a Coxeter system, that is,  $W$  is a group with generating set  $S$ , subject only to relations of the form

$$(ss')^{m(s,s')} = 1$$

for all  $s, s' \in S$  ([5]). If  $J$  is any subset of  $S$ , then we let  $W_J$  denote the subgroup generated by  $J$ . Let  $X_J$  be the unique set of minimal length left coset representatives of  $W_J$  in  $W$ , and observe that  $X_J^{-1} = \{x^{-1} | x \in X_J\}$  is the unique set of minimal length right coset representatives of  $W_J$  in  $W$ .

In 1976 Solomon, [6], proved the following theorem.

**Theorem 1** *For every subset  $K$  of  $S$ , let*

$$\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.$$

Then for subsets  $J$  and  $K$  in  $S$

$$\mathcal{X}_J \mathcal{X}_K = \sum_{x \in X_J^{-1} \cap X_K} \mathcal{X}_{x^{-1} J x \cap K}.$$

From this it follows that the set of all  $\mathcal{X}_K$  is a basis for an algebra,  $\Sigma_W$ , over the field of rationals, with integer structure constants. It was this set of algebras, one for each Coxeter group  $W$ , that came to be known collectively as the descent algebras of Coxeter groups.

Here we shall formulate an interpretation of Theorem 1 for the descent algebra of the Coxeter groups of type  $D$  that will use matrices, or a “matrix interpretation” for short. Our motivation for developing such a matrix interpretation is because the formulation of analogous interpretations for the descent algebras of the Coxeter groups of types  $A$  and  $B$  has played a key role in the further understanding of these algebras ([3], [4], [1], [2]). Firstly, however, we shall define the  $n$ -th Coxeter group of type  $D$ ,  $D_n$ .

For this group we shall take the set upon which the set of generators  $S$  acts to be  $N = \{-n, \dots, -1, 1, \dots, n\}$   $n \geq 2$ , and  $S$  to be the set  $\{s_i\}_{i=1}^{n-1} \cup \{s'_1\}$ , where  $s_i$  is the product of transpositions  $(-i-1, -i)(i, i+1)$ , for  $i = 1, \dots, n-1$ , and  $s'_1$  is the product of transpositions  $(-2, 1)(-1, 2)$ . We shall now set up some machinery which will be used to formulate our matrix interpretation.

We define a composition,  $\kappa$ , of an integer,  $n$ , to be an ordered list  $[\kappa_1, \kappa_2, \dots]$  of positive integers whose sum is  $n$ , and shall write  $\kappa \vDash n$  to denote this. We shall call the integers  $\kappa_1, \kappa_2, \dots$  the components of  $\kappa$ .

We now define four sets of compositions which will be used to recode the basis elements of the descent algebra of  $D_n$ , denoted  $\Sigma_{D_n}$ .

- $\mathcal{C}_{<n} = \{\kappa \mid \kappa \vDash m, m \leq n-2\}$ ,
- $\mathcal{C}_1 = \{\kappa \mid \kappa \vDash n, \kappa_1 = 1\}$ ,
- $\mathcal{C}_n = \{\kappa \mid \kappa \vDash n, \kappa_1 \geq 2\}$ ,
- $\mathcal{C}'_n = \{\kappa \mid \kappa \vDash n, \kappa_1 \geq 2\}$ .

We observe that  $\mathcal{C}_n$  and  $\mathcal{C}'_n$  are two disjoint copies of the same set. We define the multiset  $\mathcal{C}(n)$  to be the union of these four sets.

## 1.1 The matrix interpretation

Consider the template with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0v} \\ & y_{11} & y_{12} & \cdots & y_{1v} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{u1} & y_{u2} & \cdots & y_{uv} \\ z_{u0} & z_{u1} & z_{u2} & \cdots & z_{uv} \end{pmatrix}.$$

Note that the  $y$ -lines do not have entries in column 0. We say a template is a “filled template” if all entries are non-negative integers.

**Definition 1** Let  $\mathbf{t}$  be a filled template. We define the border-sum,  $\mathcal{B}(\mathbf{t})$ , of  $\mathbf{t}$  to be the sum

$$z_{00} + \sum_{i=1}^u z_{i0} + \sum_{j=1}^v z_{0j}$$

and the  $y$ -sum,  $\mathcal{Y}(\mathbf{t})$ , to be  $\sum_{i,j} y_{ij}$ . The reading word of  $\mathbf{t}$ ,  $r(\mathbf{t})$ , is given by

$$[z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted, unless  $z_{00} = 1$ , in which case  $r(\mathbf{t})$  is given by

$$[1, z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted.

Let  $\kappa = [\kappa_1, \dots, \kappa_v]$  and  $\nu = [\nu_1, \dots, \nu_u]$  be compositions in  $\mathcal{C}(n)$  such that  $\kappa \vDash l$  and  $\nu \vDash m$ . Let  $\kappa_0 = n - l$  and  $\nu_0 = n - m$ . We define  $Z(\kappa, \nu)$  to be the set of filled templates having  $v + 1$  columns, and  $2u + 1$  rows such that:

1. For  $0 \leq j \leq v$ , the entries in column  $j$  sum to  $\kappa_j$ ,
2. For  $1 \leq i \leq u$  the sum of the entries in rows  $2i - 1$  and  $2i$  is  $\nu_i$ ,
3. The entries in row 0 sum to  $\nu_0$ ,
4. If  $\mathcal{B}(\mathbf{t})=0$ ,  $\mathcal{Y}(\mathbf{t})$  is odd if
  - (a)  $\kappa \in \mathcal{C}_1 \cup \mathcal{C}_n$  and  $\nu \in \mathcal{C}'_n$ , or
  - (b)  $\kappa \in \mathcal{C}'_n$  and  $\nu \in \mathcal{C}_1 \cup \mathcal{C}_n$ .

Otherwise  $\mathcal{Y}(\mathbf{t})$  is even.

We will now encode subsets  $J \subseteq S$  by compositions  $\kappa \in \mathcal{C}(n)$  using a bijective correspondence that we shall give later, and denote the basis element  $\mathcal{X}_J$  by  $B_\kappa$ . To distinguish between those compositions belonging to  $\mathcal{C}_n$  and those belonging to  $\mathcal{C}'_n$ , if  $\kappa \in \mathcal{C}_n$  then one denotes by  $\kappa'$  its copy that belongs to  $\mathcal{C}'_n$ .

**Theorem 2** Let  $\kappa, \nu \in \mathcal{C}(n)$ . For any filled template,  $\mathbf{t}$ , let  $r(\mathbf{t}) = [r_1(\mathbf{t}), r_2(\mathbf{t}), \dots]$ . Then,

$$B_\kappa B_\nu = \sum_{\mathbf{t} \in Z(\kappa, \nu)} \tilde{B}_{r(\mathbf{t})}$$

where  $\tilde{B}_{r(\mathbf{t})}$  satisfies the following.

1. If  $\nu \in \mathcal{C}_1$ , then  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$ .
2. If  $\nu \in \mathcal{C}_n$ , then  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$ .
3. If  $\nu \in \mathcal{C}'_n$ , then  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})'}$ , or  $B_{r(\mathbf{t})}$  if  $r_1(\mathbf{t}) = 1$ .
4. If  $\nu \in \mathcal{C}_{<n}$ , then
  - (a) If  $\kappa \in \mathcal{C}_1 \cup \mathcal{C}_n$  and  $\mathcal{Y}(\mathbf{t})$  is odd, or  $\kappa \in \mathcal{C}'_n$  and  $\mathcal{Y}(\mathbf{t})$  is even, then  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})'}$ , if  $r_1(\mathbf{t}) \geq 2$ ,
  - (b) If  $\kappa \in \mathcal{C}_{<n}$  and  $z_{00} = 0$ , then  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})} + B_{r(\mathbf{t})'}$ , or  $2B_{r(\mathbf{t})}$  if  $r_1(\mathbf{t}) = 1$ .
  - (c) Otherwise  $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$ .

Recall that the radical of an algebra is the unique maximal nilpotent two sided ideal in that algebra. With the following definition we can recover another result of Solomon [6], and determine the radical of  $\Sigma_{D_n}$  as a corollary to our matrix interpretation.

**Definition 2** If  $\kappa, \nu \in \mathcal{C}(n)$  then we say that  $\kappa \approx \nu$  if the components of  $\kappa$  can be re-ordered to give to components of  $\nu$ , but  $\kappa$  and  $\nu$  do not satisfy either

1.  $\kappa \in \mathcal{C}_n, \nu \in \mathcal{C}'_n$ , all components are even, or
2.  $\kappa \in \mathcal{C}'_n, \nu \in \mathcal{C}_n$ , all components are even.

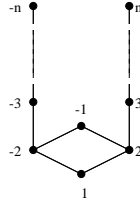
**Corollary 1** Let  $\kappa, \nu \in \mathcal{C}(n)$ , then the radical of  $\Sigma_{D_n}$  is spanned by all  $B_\kappa - B_\nu$  such that  $\kappa \approx \nu$ .

## 2 Skeleton proof

This section falls naturally into two subsections. In the first we show how we recode the basis elements of Theorem 1 into those seen in Theorem 2. The latter gives a flavour of how the remainder of the, rather technical, proof proceeds.

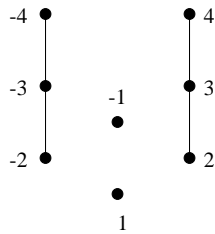
## 2.1 Encoding the basis elements

Let  $J$  be any subset of  $S$ . We define  $\mathcal{J} = (N, \mathcal{E})$  to be the graph with vertex set  $N$ , and edge set  $\mathcal{E} = \{(i, j) | (i, j)(-i, -j) \in J\}$ . Observe from this definition that  $\mathcal{J}$  is a subgraph of



isomorphic to itself under the bijection that maps vertex  $i$  to vertex  $-i$ . From here on we shall take the convention that roman capitals  $J, K, \dots$  will denote subsets of  $S$ , and their calligraphic counterparts  $\mathcal{J}, \mathcal{K}, \dots$  will denote the associated graphs.

**Example 1** Let  $n = 4$ , and  $J = \{(-3, -2)(2, 3), (-4, -3)(3, 4)\}$ . Then the graph,  $\mathcal{J}$ , of  $J$  is



Each connected component of  $\mathcal{J}$  has an associated set of vertices. To label these sets, we order them by their least elements, and then label them in a natural way

$$\dots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \dots$$

using the following algorithm:

1. If  $-1$  and  $1$  are contained in the same component, then we label that component  $\mathcal{J}_0$ .
2. If  $-1$  is not contained in the same component as  $1$ , but is in the same component as  $2$ , we set  $\mathcal{J}_0 = \emptyset$ , and label the others such that  $-1 \in \mathcal{J}_1$ .
3. If  $1$  is not contained in the same component as  $-1$ , but is in the same component as  $2$ , we set  $\mathcal{J}_0 = \emptyset$ , and label the others such that  $1 \in \mathcal{J}_1$ .

4. Otherwise we set  $\mathcal{J}_0 = \emptyset$ , and label the others such that  $-1 \in \mathcal{J}_{-1}$ ,  $1 \in \mathcal{J}_1$ .

From this algorithm we observe that if  $-1$  and  $1$  do not belong to the same component, then we introduce an empty set. In addition, by our labelling, it follows that  $|\mathcal{J}_i| = |\mathcal{J}_{-i}|$ .

Let  $\mathcal{G}(n)$  denote the set of all graphs defined from the subsets of  $S$  in the manner defined above. It is convenient to divide  $\mathcal{G}(n)$  into four disjoint classes called, for reasons which will become clear later,  $\mathcal{G}_{<n}$ ,  $\mathcal{G}'_n$ ,  $\mathcal{G}_n$ , and  $\mathcal{G}_1$ . Every graph of  $\mathcal{G}(n)$  belongs to exactly one of the classes depending on how its components have been labelled:

1.  $\mathcal{J} \in \mathcal{G}_{<n}$  if and only if  $\mathcal{J}_0 \neq \emptyset$ .
2.  $\mathcal{J} \in \mathcal{G}'_n$  if and only if  $-1, 2 \in \mathcal{J}_1$ .
3.  $\mathcal{J} \in \mathcal{G}_n$  if and only if  $1, 2 \in \mathcal{J}_1$ .
4.  $\mathcal{J} \in \mathcal{G}_1$  otherwise.

If  $\mathcal{J} \in \mathcal{G}(n) \setminus \mathcal{G}_1$ , then we define the *ordered presentation* of  $\mathcal{J}$  to be the ordered list

$$(\dots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \dots).$$

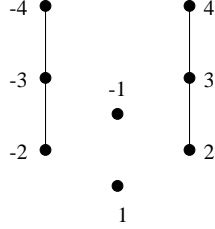
However, if  $\mathcal{J} \in \mathcal{G}_1$ , we say the ordered presentations of  $\mathcal{J}$  are the ordered lists

$$(\dots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \dots)$$

and

$$(\dots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_1, \mathcal{J}_0, \mathcal{J}_{-1}, \mathcal{J}_2, \mathcal{J}_3 \dots).$$

**Example 2** Recall  $\mathcal{J}$ ,



The vertex sets of  $\mathcal{J}$ , when ordered by their least elements, are

$$\{-4, -3, -2\}, \{-1\}, \{1\}, \{2, 3, 4\}.$$

By our labelling algorithm these vertex sets are labelled

$$\mathcal{J}_{-2} = \{-4, -3, -2\}, \mathcal{J}_{-1} = \{-1\}, \mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2, 3, 4\}$$

and  $\mathcal{J}_0 = \emptyset$ . Note that  $\mathcal{J} \in \mathcal{G}_1$ . The ordered presentations of  $\mathcal{J}$  are

$$(\{-4, -3, -2\}, \{-1\}, \emptyset, \{1\}, \{2, 3, 4\})$$

and

$$(\{-4, -3, -2\}, \{1\}, \emptyset, \{-1\}, \{2, 3, 4\}).$$

We are now going to explain the terminology for the four subclasses of  $\mathcal{G}(n)$ . There is a one-to-one correspondence between the compositions  $\mathcal{C}(n)$  defined earlier, and  $\mathcal{G}(n)$  in which  $\mathcal{G}_{<n}$  corresponds to  $\mathcal{C}_{<n}$ ;  $\mathcal{G}'_n$  to  $\mathcal{C}'_n$ ;  $\mathcal{G}_n$  to  $\mathcal{C}_n$ ; and  $\mathcal{G}_1$  to  $\mathcal{C}_1$ . This correspondence is defined as follows.

The composition,  $\kappa \vDash m \leq n$ , with components  $\kappa_1, \dots, \kappa_k$ , corresponds to the graph  $\mathcal{J}$ , with vertex sets  $\mathcal{J}_{-k}, \dots, \mathcal{J}_0, \dots, \mathcal{J}_k$ , if

1.  $|\mathcal{J}_{-i}| = |\mathcal{J}_i| = \kappa_i$  for  $i > 0$ .
2.  $\frac{1}{2}|\mathcal{J}_0| = \kappa_0 = n - m$ .
3. (a) If  $-1, 1 \in \mathcal{J}_0$  then  $\kappa \in \mathcal{C}_{<n}$ ,  
 (b) If  $-1, 2 \in \mathcal{J}_1$  then  $\kappa \in \mathcal{C}'_n$ ,  
 (c) If  $1, 2 \in \mathcal{J}_1$  then  $\kappa \in \mathcal{C}_n$ ,  
 (d) Otherwise  $\kappa \in \mathcal{C}_1$ .

**Example 3** We take again our graph from Example 1.

For  $\mathcal{J}$  we recall its vertex sets are

$$\mathcal{J}_{-2} = \{-4, -3, -2\}, \mathcal{J}_{-1} = \{-1\}, \mathcal{J}_0 = \{\}, \mathcal{J}_1 = \{1\}, \mathcal{J}_2 = \{2, 3, 4\}.$$

Hence  $\mathcal{J}$  corresponds to the composition  $\kappa$  such that

$$\begin{aligned} |\mathcal{J}_{-1}| &= |\mathcal{J}_1| = \kappa_1 = 1 \\ |\mathcal{J}_{-2}| &= |\mathcal{J}_2| = \kappa_2 = 3 \\ \frac{1}{2}|\mathcal{J}_0| &= \kappa_0 = n - m = 0. \end{aligned}$$

Since  $-1 \in \mathcal{J}_{-1}$ ,  $1 \in \mathcal{J}_1$  then  $\kappa \in \mathcal{C}_1$ . Hence, we have that  $\kappa = [1, 3]$ .

## 2.2 Further details

With our encoding now in place we are now in a position to attack the reformulation of Theorem 1. Let us define  $\mathcal{W}_{\mathcal{J}_i}$  to be the group of all permutations on  $N$  that fix all vertices outside  $\mathcal{J}_i$ . Define

$$\mathcal{W}_{\mathcal{J}} = (\dots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \dots) \cap D_n$$

We write  $|_D$  to denote the restriction to  $D_n$ , that is

$$\mathcal{W}_{\mathcal{J}} = (\dots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \dots)|_D$$

and observe that  $W_{\mathcal{J}} = \mathcal{W}_{\mathcal{J}}$ .

Let  $S_k$  denote the symmetric group on  $k$  elements. Let  $\kappa \in \mathcal{C}(n)$ , and let  $\kappa \vDash m \leq n$  have components  $\kappa_1, \dots, \kappa_k$ . Then we define

$$\mathbf{D}_{\kappa} = D_{n-m} \times S_{\kappa_1} \times \dots \times S_{\kappa_k}$$

and observe that

$$\begin{aligned} \mathbf{D}_{\kappa} &= D_{n-m} \times S_{\kappa_1} \times \dots \times S_{\kappa_k} \\ &\cong (\mathcal{W}_{\mathcal{J}_{-k}} \times \dots \times \mathcal{W}_{\mathcal{J}_k})|_D \\ &= \mathcal{W}_{\mathcal{J}} \end{aligned}$$

where  $\kappa$  corresponds to  $\mathcal{J}$  via the correspondence between  $\mathcal{C}(n)$  and  $\mathcal{G}(n)$  given earlier.

Let  $J$  and  $K$  to be any subsets of  $S$ , and  $x \in W$ . Let  $x\mathcal{J}$  denote the image of the graph  $\mathcal{J}$  under  $x$ ; that is  $(x(i), x(j))$  is an edge in  $x\mathcal{J}$  if and only if  $(i, j)$  is an edge in  $\mathcal{J}$ . Let  $\mathcal{J} \cap \mathcal{K}$  be the graph with vertex set  $N$  whose edges are those present in both  $\mathcal{J}$  and  $\mathcal{K}$ , and let  $x^{-1}\mathcal{J} \cap \mathcal{K}$  be understood as  $(x^{-1}\mathcal{J}) \cap \mathcal{K}$ . We can prove that  $x^{-1}\mathcal{J}$  is a subgraph of  $\mathcal{S}$ , and with these constructs we can prove the following lemma.

**Lemma 1** *Let  $J$  and  $K$  be subsets of  $S$ , and let  $x \in X_J^{-1} \cap X_K$ . Let the ordered presentation of  $\mathcal{J}$  be*

$$(\mathcal{J}_{-v}, \dots, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \dots, \mathcal{J}_v),$$

and  $\mathcal{K}$  be

$$(\mathcal{K}_{-u}, \dots, \mathcal{K}_{-1}, \mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_u).$$

Then an ordered presentation of  $x^{-1}\mathcal{J} \cap \mathcal{K}$  is

$$\begin{aligned} & (x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_{-u}, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_{-u}, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_{-u}, \\ & x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_{-u+1}, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_{-u+1}, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_{-u+1}, \\ & \dots, x^{-1}\mathcal{J}_0 \cap \mathcal{K}_0, \dots \\ & x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_u, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_u, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_u) \end{aligned}$$

or

$$\begin{aligned} & (x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_{-u}, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_{-u}, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_{-u}, \\ & x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_{-u+1}, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_{-u+1}, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_{-u+1}, \\ & \dots, \{-1\}, \{\}, \{1\}, \dots \\ & x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_u, x^{-1}\mathcal{J}_{-v+1} \cap \mathcal{K}_u, \dots, x^{-1}\mathcal{J}_v \cap \mathcal{K}_u) \end{aligned}$$

if  $x^{-1}\mathcal{J}_0 \cap \mathcal{K}_0 = \{-1, 1\}$ , with empty sets removed. However if  $x^{-1}\mathcal{J}_0 \cap \mathcal{K}_0$  is empty then we include it in the list.

Due to our various correspondences, this lemma gives us a handle, in terms of graphs, of the various summands,  $\mathcal{X}_{x^{-1}\mathcal{J} \cap \mathcal{K}}$ , in Theorem 1. If we can display this information in terms of filled templates with appropriate reading words, then we shall have all the necessary ingredients for the reformulation of Theorem 1 into Theorem 2. This is what we shall head towards next.

From Lemma 2 [6], which states the first equality; and Lemma 1, it follows that if  $x \in X_J^{-1} \cap X_K$  then

$$\begin{aligned} x^{-1}W_J x \cap W_K &= W_{x^{-1}\mathcal{J} \cap \mathcal{K}} \\ &= \mathcal{W}_{x^{-1}\mathcal{J} \cap \mathcal{K}} \\ &= [\mathcal{W}_{(x^{-1}\mathcal{J}_{-v} \cap \mathcal{K}_{-u})} \times \dots \times \mathcal{W}_{(x^{-1}\mathcal{J}_v \cap \mathcal{K}_u)}] \Big|_D \\ &= [(x^{-1}\mathcal{W}_{\mathcal{J}_{-v}} x \cap \mathcal{W}_{\mathcal{K}_{-u}}) \times \dots \times (x^{-1}\mathcal{W}_{\mathcal{J}_v} x \cap \mathcal{W}_{\mathcal{K}_u})] \Big|_D \\ &= [(x^{-1}S_{\kappa_v}^- x \cap S_{\nu_u}^-) \times \dots \times (x^{-1}S_{\kappa_v}^+ x \cap S_{\nu_u}^+)] \Big|_D \end{aligned}$$

where  $\kappa = [\kappa_1, \dots, \kappa_v]$  and  $\nu = [\nu_1, \dots, \nu_u]$  are suitable compositions in  $\mathcal{C}(n)$ , determined by the correspondence between  $\mathcal{G}(n)$  and  $\mathcal{C}(n)$ . The superscript  $-$  or  $+$  on a symmetric group  $S_{\kappa_i}$  simply denotes that for the given  $i > 0$ ,  $S_{\kappa_i}$  acts on  $\mathcal{J}_{-i}$  or  $\mathcal{J}_i$  respectively. Observe that, say,  $x^{-1}S_{\kappa_i}^- x \cap S_{\nu_j}^-$  is the group of all permutations on

$$x^{-1}\mathcal{J}_{-i} \cap \mathcal{K}_{-j}.$$

Let  $z_{ij} = |x^{-1}\mathcal{J}_i \cap \mathcal{K}_j|$ , then we have a mapping

$$\zeta : x \mapsto (z_{ij})$$

from  $X_J^{-1} \cap X_K$  into a subset of all  $(2u + 1) \times (2v + 1)$  matrices

$$\begin{pmatrix} z_{(-u)(-v)} & \cdots & z_{(-u)0} & \cdots & z_{(-u)v} \\ \vdots & & \vdots & & \vdots \\ z_{0(-v)} & \cdots & z_{00} & \cdots & z_{0v} \\ \vdots & & \vdots & & \vdots \\ z_{u(-v)} & \cdots & z_{u0} & \cdots & z_{uv} \end{pmatrix}$$

with non-negative integer entries that satisfy

$$\begin{aligned} \sum_{i=-u}^u z_{ij} &= \kappa_{|j|}, j \neq 0, & \sum_{j=-v}^v z_{ij} &= \nu_{|i|}, i \neq 0 \\ \sum_{i=-u}^u z_{i0} &= 2\kappa_0, & \sum_{j=-v}^v z_{0j} &= 2\nu_0, \end{aligned}$$

$$z_{ij} = z_{(-i)(-j)},$$

and

1. If  $\mathcal{J}, \mathcal{K} \in \mathcal{G}_1 \cup \mathcal{G}_n$ ; or  $\mathcal{J}, \mathcal{K} \in \mathcal{G}'_n$  then  $\sum_{i,j>0} z_{i(-j)}$  must be even.
2. If  $\mathcal{J} \in \mathcal{G}_1 \cup \mathcal{G}_n$ , and  $\mathcal{K} \in \mathcal{G}'_n$ ; or  $\mathcal{J} \in \mathcal{G}'_n$  and  $\mathcal{K} \in \mathcal{G}_1 \cup \mathcal{G}_n$  then  $\sum_{i,j>0} z_{i(-j)}$  must be odd.

By doing a simple case study on the possible combinations of  $\mathcal{J}, \mathcal{K}$ , depending on whether they belong to  $\mathcal{G}_1, \mathcal{G}_n, \mathcal{G}'_n$  or  $\mathcal{G}_{<n}$  we can prove that  $\zeta$  is surjective, and injective apart from when  $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$  and  $x^{-1}\mathcal{J}_0 \cap \mathcal{K}_0 = \{ \}$ . In this case  $x, (-1, 1)x \in X_J^{-1} \cap X_K$  have the same image under  $\zeta$ . So now we have a handle in terms of matrices for the summands of Theorem 1.

We now require, for each  $x \in X_J^{-1} \cap X_K$ , that the matrix  $(z_{ij}) = \zeta(x)$  should yield a composition  $\eta \in \mathcal{C}(n)$  such that  $\mathbf{D}_\eta$  is isomorphic to  $W_{x^{-1}\mathcal{J}_x \cap \mathcal{K}}$ . This will give us our desired reading word. To do this we need to know the components of  $\eta$  and whether  $\eta$  belongs to  $\mathcal{C}_1, \mathcal{C}_n, \mathcal{C}'_n$  or  $\mathcal{C}_{<n}$ . After an analysis of the matrix entries of  $\zeta(x)$  we obtain that the components of  $\eta$  are, in order, all the non-zero entries of  $\zeta(x)$  read by row after  $z_{00}$ , or 1 followed by all the non-zero entries of  $\zeta(x)$  read by row after  $z_{00}$  if  $z_{00} = 1$ .

As to whether  $\eta$  belongs to  $\mathcal{C}_1, \mathcal{C}_n, \mathcal{C}'_n$  or  $\mathcal{C}_{<n}$ , since the graph of  $\eta$  is a subgraph of  $\mathcal{K}$  we can deduce the following.

1. If  $\mathcal{K} \in \mathcal{G}_1$  then  $\eta \in \mathcal{C}_1$ ,
2. If  $\mathcal{K} \in \mathcal{G}_n$  then  $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$ ,
3. If  $\mathcal{K} \in \mathcal{G}'_n$  then  $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$ ,
4. If  $\mathcal{K} \in \mathcal{G}_{<n}$  and
  - (a)  $\mathcal{J} \in \mathcal{G}_1 \cup \mathcal{G}_n$  then if  $\sum_{i,j>0} z_{i(-j)}$  is even then  $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$ , otherwise  $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$ .

- (b)  $\mathcal{J} \in \mathcal{G}'_n$  then if  $\sum_{i,j>0} z_{i(-j)}$  is even then  $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$ , otherwise  $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$ .
- (c)  $\mathcal{J} \in \mathcal{G}_{<n}$  then if  $z_{00} > 2$  then  $\eta \in \mathcal{C}_{<n}$ , if  $z_{00} = 2$  then  $\eta \in \mathcal{C}_1$ , if  $z_{00} = 0$  then if  $1 \in \bigcup_{i>0} x^{-1}\mathcal{J}_i \cap \mathcal{K}_0$  then  $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$  otherwise  $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$ .

We now observe that each matrix in the image of  $\zeta$  corresponds to one  $x \in X_J^{-1} \cap X_K$ , that is one summand in Theorem 1, unless  $z_{00} = 0$  and  $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$ , in which case the matrix involved corresponds to both  $x, x' \in X_J^{-1} \cap X_K$ , where  $x' = (-1, 1)x$ .

If we now recode the basis elements  $\mathcal{X}_J$ , given in Solomon's Theorem, to  $B_\kappa$ , where  $\kappa$  and  $\mathcal{J}$  correspond via the correspondence between  $\mathcal{C}(n)$  and  $\mathcal{G}(n)$ ; biject the matrices into filled templates; and summarize what we have deduced as clearly as possible, we arrive at the matrix interpretation of Solomon's Theorem for the descent algebra of  $D_n$  given in the Introduction.

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