A matrix interpretation for the descent algebra of type D

Stephanie J. van Willigenburg
Department of Mathematics and Statistics,
York University, 4700 Keele St,
Toronto, ON, M3J 1P3, CANADA.
sponsored by the Leverhulme Trust.

July 18, 2001

Abstract

We define a certain set of compositions $\mathcal{C}(n)$, and using these compositions we re-interpret multiplication in the descent algebra of D_n in terms of matrices. Using this interpretation we can also give a basis for the radical. We then sketch how we can derive such an interpretation.

Nous definissons un ensemble de compositions, $\mathcal{C}(n)$, et utilisons compositions pour donner une interprétation, en terme de matrices, de la multiplication dans ces l'algèbra de descentes de D_n. Nous donnons aussi une base du radical, puis nous ébauchons la preuve de notre interprétation.

1 Introduction

Let (W, S) be a Coxeter system, that is, W is a group with generating set S, subject only to relations of the form

$$(ss')^{m(s,s')} = 1$$

for all $s, s' \in S$ ([5]). If J is any subset of S, then we let W_J denote the subgroup generated by J. Let X_J be the unique set of minimal length left coset representatives of W_J in W, and observe that $X_J^{-1} = \{x^{-1} | x \in X_J\}$ is the unique set of minimal length right coset representatives of W_J in W.

In 1976 Solomon, [6], proved the following theorem.

Theorem 1 *For every subset K of S, let*

$$\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.$$

Then for subsets J and K in S

$$\mathcal{X}_J\mathcal{X}_K = \sum_{x\in X_J^{-1}\cap X_K} \mathcal{X}_{x^{-1}Jx\cap K}.$$

From this it follows that the set of all $\mathcal{X}_K$ is a basis for an algebra, Σ_W, over the field of rationals, with integer structure constants. It was this set of algebras, one for each Coxeter group W, that came to be known collectively as the descent algebras of Coxeter groups.

Here we shall formulate an interpretation of Theorem 1 for the descent algebra of the Coxeter groups of type D that will use matrices, or a "matrix interpretation" for short. Our motivation for developing such a matrix interpretation is because the formulation of analogous interpretations for the descent algebras of the Coxeter groups of types A and B has played a key role in the further understanding of these algebras ([3], [4], [1], [2]). Firstly, however, we shall define the n-th Coxeter group of type D, D_n.

For this group we shall take the set upon which the set of generators S acts to be $N = \{-n, \ldots, -1, 1, \ldots, n\}$ $n \geq 2$, and S to be the set $\{s_i\}_{i=1}^{n-1} \cup \{s_1'\}$, where s_i is the product of transpositions $(-i-1, -i)(i, i+1)$, for $i = 1, \ldots, n-1$, and s_1' is the product of transpositions $(-2, 1)(-1, 2)$. We shall now set up some machinery which will be used to formulate our matrix interpretation.

We define a composition, κ, of an integer, n, to be an ordered list $[\kappa_1, \kappa_2, \ldots]$ of positive integers whose sum is n, and shall write $\kappa \vDash n$ to denote this. We shall call the integers $\kappa_1, \kappa_2, \ldots$ the components of κ.

We now define four sets of compositions which will be used to recode the basis elements of the descent algebra of D_n, denoted Σ_{D_n}.

- $\mathcal{C}_{<n} = \{\kappa | \kappa \vDash m, m \leq n-2\}$,
- $\mathcal{C}_1 = \{\kappa | \kappa \vDash n, \kappa_1 = 1\}$,
- $\mathcal{C}_n = \{\kappa | \kappa \vDash n, \kappa_1 \geq 2\}$,
- $\mathcal{C}_n' = \{\kappa | \kappa \vDash n, \kappa_1 \geq 2\}$.

We observe that $\mathcal{C}_n$ and $\mathcal{C}_n'$ are two disjoint copies of the same set. We define the multiset $\mathcal{C}(n)$ to be the union of these four sets.

1.1 The matrix interpretation

Consider the template with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0v} \\ & y_{11} & y_{12} & \cdots & y_{1v} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{u1} & y_{u2} & \cdots & y_{uv} \\ z_{u0} & z_{u1} & z_{u2} & \cdots & z_{uv} \end{pmatrix}.$$

Note that the y-lines do not have entries in column 0. We say a template is a "filled template" if all entries are non-negative integers.

Definition 1 *Let* $\boldsymbol{t}$ *be a filled template. We define the* border-sum*,* $\mathcal{B}(\boldsymbol{t})$*, of* $\boldsymbol{t}$ *to be the sum*

$$z_{00} + \sum_{i=1}^{u} z_{i0} + \sum_{j=1}^{v} z_{0j}$$

and the y-sum*,* $\mathcal{Y}(\boldsymbol{t})$*, to be* $\sum_{i,j} y_{ij}$*. The reading word of* $\boldsymbol{t}$*,* $r(\boldsymbol{t})$*, is given by*

$$[z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted, unless $z_{00} = 1$*, in which case* $r(\boldsymbol{t})$ *is given by*

$$[1, z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted.

Let $\kappa = [\kappa_1, \dots, \kappa_v]$ and $\nu = [\nu_1, \dots, \nu_u]$ be compositions in $\mathcal{C}(n)$ such that $\kappa \vDash l$ and $\nu \vDash m$. Let $\kappa_0 = n - l$ and $\nu_0 = n - m$. We define $Z(\kappa, \nu)$ to be the set of filled templates having $v + 1$ columns, and $2u + 1$ rows such that:

1. For $0 \leq j \leq v$, the entries in column j sum to κ_j,
2. For $1 \leq i \leq u$ the sum of the entries in rows $2i - 1$ and $2i$ is ν_i,
3. The entries in row 0 sum to ν_0,
4. If $\mathcal{B}(\boldsymbol{t})$=0, $\mathcal{Y}(\boldsymbol{t})$ is odd if
   (a) $\kappa \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\nu \in \mathcal{C}'_n$, or
   (b) $\kappa \in \mathcal{C}'_n$ and $\nu \in \mathcal{C}_1 \cup \mathcal{C}_n$.

   Otherwise $\mathcal{Y}(\boldsymbol{t})$ is even.

We will now encode subsets $J \subseteq S$ by compositions $\kappa \in \mathcal{C}(n)$ using a bijective correspondence that we shall give later, and denote the basis element $\mathcal{X}_J$ by B_κ. To distinguish between those compositions belonging to $\mathcal{C}_n$ and those belonging to $\mathcal{C}'_n$, if $\kappa \in \mathcal{C}_n$ then one denotes by κ' its copy that belongs to $\mathcal{C}'_n$.

Theorem 2 *Let $\kappa, \nu \in \mathcal{C}(n)$. For any filled template, $\boldsymbol{t}$, let $r(\boldsymbol{t}) = [r_1(\boldsymbol{t}), r_2(\boldsymbol{t}), \ldots]$. Then,*

$$B_\kappa B_\nu = \sum_{\boldsymbol{t} \in Z(\kappa,\nu)} \tilde{B}_{r(\boldsymbol{t})}$$

where $\tilde{B}_{r(\boldsymbol{t})}$ satisfies the following.

1. *If $\nu \in \mathcal{C}_1$, then $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})}$.*
2. *If $\nu \in \mathcal{C}_n$, then $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})}$.*
3. *If $\nu \in \mathcal{C}'_n$, then $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})'}$, or $B_{r(\boldsymbol{t})}$ if $r_1(\boldsymbol{t}) = 1$.*
4. *If $\nu \in \mathcal{C}_{<n}$, then*
   (a) *If $\kappa \in \mathcal{C}_1 \cup \mathcal{C}_n$ and $\mathcal{Y}(\boldsymbol{t})$ is odd, or $\kappa \in \mathcal{C}'_n$ and $\mathcal{Y}(\boldsymbol{t})$ is even, then $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})'}$, if $r_1(\boldsymbol{t}) \geq 2$,*
   (b) *If $\kappa \in \mathcal{C}_{<n}$ and $z_{00} = 0$, then $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})} + B_{r(\boldsymbol{t})'}$, or $2B_{r(\boldsymbol{t})}$ if $r_1(\boldsymbol{t}) = 1$.*
   (c) *Otherwise $\tilde{B}_{r(\boldsymbol{t})} = B_{r(\boldsymbol{t})}$.*

Recall that the radical of an algebra is the unique maximal nilpotent two sided ideal in that algebra. With the following definition we can recover another result of Solomon [6], and determine the radical of Σ_{D_n} as a corollary to our matrix interpretation.

Definition 2 *If $\kappa, \nu \in \mathcal{C}(n)$ then we say that $\kappa \approx \nu$ if the components of κ can be re-ordered to give to components of ν, but κ and ν do not satisfy either*

1. *$\kappa \in \mathcal{C}_n$, $\nu \in \mathcal{C}'_n$, all components are even, or*
2. *$\kappa \in \mathcal{C}'_n$, $\nu \in \mathcal{C}_n$, all components are even.*

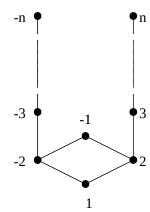
Corollary 1 *Let $\kappa, \nu \in \mathcal{C}(n)$, then the radical of Σ_{D_n} is spanned by all $B_\kappa - B_\nu$ such that $\kappa \approx \nu$.*

2 Skeleton proof

This section falls naturally into two subsections. In the first we show how we recode the basis elements of Theorem 1 into those seen in Theorem 2. The latter gives a flavour of how the remainder of the, rather technical, proof proceeds.

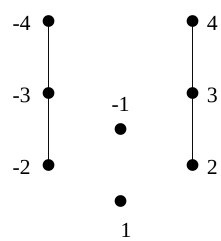
2.1 Encoding the basis elements

Let J be any subset of S. We define $\mathcal{J} = (N, \mathcal{E})$ to be the graph with vertex set N, and edge set $\mathcal{E} = \{(i,j) | (i,j)(-i,-j) \in J\}$. Observe from this definition that $\mathcal{J}$ is a subgraph of



isomorphic to itself under the bijection that maps vertex i to vertex $-i$. From here on we shall take the convention that roman capitals $J, K, \ldots$ will denote subsets of S, and their calligraphic counterparts $\mathcal{J}, \mathcal{K}, \ldots$ will denote the associated graphs.

Example 1 Let $n = 4$, and $J = \{(-3,-2)(2,3), (-4,-3)(3,4)\}$. Then the *graph*, $\mathcal{J}$, of J is



Each connected component of $\mathcal{J}$ has an associated set of vertices. To label these sets, we order them by their least elements, and then label them in a natural way

$$\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots$$

using the following algorithm:

1. If -1 and 1 are contained in the same component, then we label that component $\mathcal{J}_0$.

2. If -1 is not contained in the same component as 1, but is in the same component as 2, we set $\mathcal{J}_0 = \emptyset$, and label the others such that $-1 \in \mathcal{J}_1$.

3. If 1 is not contained in the same component as -1, but is in the same component as 2, we set $\mathcal{J}_0 = \emptyset$, and label the others such that $1 \in \mathcal{J}_1$.

4. Otherwise we set $\mathcal{J}_0 = \emptyset$, and label the others such that $-1 \in \mathcal{J}_{-1}$, $1 \in \mathcal{J}_1$.

From this algorithm we observe that if -1 and 1 do not belong to the same component, then we introduce an empty set. In addition, by our labelling, it follows that $|\mathcal{J}_i| = |\mathcal{J}_{-i}|$.

Let $\mathcal{G}(n)$ denote the set of all graphs defined from the subsets of S in the manner defined above. It is convenient to divide $\mathcal{G}(n)$ into four disjoint classes called, for reasons which will become clear later, $\mathcal{G}_{<n}, \mathcal{G}'_n, \mathcal{G}_n$, and $\mathcal{G}_1$. Every graph of $\mathcal{G}(n)$ belongs to exactly one of the classes depending on how its components have been labelled:

1. $\mathcal{J} \in \mathcal{G}_{<n}$ if and only if $\mathcal{J}_0 \neq \emptyset$.
2. $\mathcal{J} \in \mathcal{G}'_n$ if and only if $-1, 2 \in \mathcal{J}_1$.
3. $\mathcal{J} \in \mathcal{G}_n$ if and only if $1, 2 \in \mathcal{J}_1$.
4. $\mathcal{J} \in \mathcal{G}_1$ otherwise.

If $\mathcal{J} \in \mathcal{G}(n) \backslash \mathcal{G}_1$, then we define the *ordered presentation* of $\mathcal{J}$ to be the ordered list

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots).$$

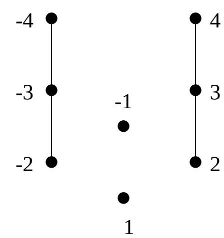
However, if $\mathcal{J} \in \mathcal{G}_1$, we say the ordered presentations of $\mathcal{J}$ are the ordered lists

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \ldots)$$

and

$$(\ldots, \mathcal{J}_{-3}, \mathcal{J}_{-2}, \mathcal{J}_1, \mathcal{J}_0, \mathcal{J}_{-1}, \mathcal{J}_2, \mathcal{J}_3 \ldots).$$

Example 2 Recall $\mathcal{J}$,



The vertex sets of $\mathcal{J}$, when ordered by their least elements, are

$$\{-4,-3,-2\},\{-1\},\{1\},\{2,3,4\}.$$

By our labelling algorithm these vertex sets are labelled

$$\mathcal{J}_{-2}=\{-4,-3,-2\},\mathcal{J}_{-1}=\{-1\},\mathcal{J}_1=\{1\},\mathcal{J}_2=\{2,3,4\}$$

and $\mathcal{J}_0=\emptyset$. Note that $\mathcal{J}\in\mathcal{G}_1$. The ordered presentations of $\mathcal{J}$ are

$$(\{-4,-3,-2\},\{-1\},\emptyset,\{1\},\{2,3,4\})$$

and

$$(\{-4,-3,-2\},\{1\},\emptyset,\{-1\},\{2,3,4\}).$$

We are now going to explain the terminology for the four subclasses of $\mathcal{G}(n)$. There is a one-to-one correspondence between the compositions $\mathcal{C}(n)$ defined earlier, and $\mathcal{G}(n)$ in which $\mathcal{G}_{<n}$ corresponds to $\mathcal{C}_{<n}$; $\mathcal{G}'_n$ to $\mathcal{C}'_n$; $\mathcal{G}_n$ to $\mathcal{C}_n$; and $\mathcal{G}_1$ to $\mathcal{C}_1$. This correspondence is defined as follows.

The composition, $\kappa\vDash m\leq n$, with components $\kappa_1,\ldots,\kappa_k$, corresponds to the graph $\mathcal{J}$, with vertex sets $\mathcal{J}_{-k},\ldots,\mathcal{J}_0,\ldots\mathcal{J}_k$, if

1. $|\mathcal{J}_{-i}|=|\mathcal{J}_i|=\kappa_i$ for $i>0$.
2. $\frac{1}{2}|\mathcal{J}_0|=\kappa_0=n-m$.
3. (a) If $-1,1\in\mathcal{J}_0$ then $\kappa\in\mathcal{C}_{<n}$,
   (b) If $-1,2\in\mathcal{J}_1$ then $\kappa\in\mathcal{C}'_n$,
   (c) If $1,2\in\mathcal{J}_1$ then $\kappa\in\mathcal{C}_n$,
   (d) Otherwise $\kappa\in\mathcal{C}_1$.

Example 3 We take again our graph from Example 1.

For $\mathcal{J}$ we recall its vertex sets are

$$\mathcal{J}_{-2}=\{-4,-3,-2\},\mathcal{J}_{-1}=\{-1\},\mathcal{J}_0=\{\},\mathcal{J}_1=\{1\},\mathcal{J}_2=\{2,3,4\}.$$

Hence $\mathcal{J}$ corresponds to the composition κ such that

$$\begin{array}{c} |\mathcal{J}_{-1}| = |\mathcal{J}_1| = \kappa_1 = 1 \\ |\mathcal{J}_{-2}| = |\mathcal{J}_2| = \kappa_2 = 3 \\ \frac{1}{2}|\mathcal{J}_0| = \kappa_0 = n - m = 0. \end{array}$$

Since $-1 \in \mathcal{J}_{-1}$, $1 \in \mathcal{J}_1$ then $\kappa \in \mathcal{C}_1$. Hence, we have that $\kappa = [1, 3]$.

2.2 Further details

With our encoding now in place we are now in a position to attack the reformulation of Theorem 1. Let us define $\mathcal{W}_{\mathcal{J}_i}$ to be the group of all permutations on N that fix all vertices outside $\mathcal{J}_i$. Define

$$\mathcal{W}_{\mathcal{J}} = (\ldots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \ldots) \cap D_n$$

We write $|_D$ to denote the restriction to D_n, that is

$$\mathcal{W}_{\mathcal{J}} = (\ldots \times \mathcal{W}_{\mathcal{J}_{-2}} \times \mathcal{W}_{\mathcal{J}_{-1}} \times \mathcal{W}_{\mathcal{J}_0} \times \mathcal{W}_{\mathcal{J}_1} \times \mathcal{W}_{\mathcal{J}_2} \times \ldots)|_D$$

and observe that $W_J = \mathcal{W}_{\mathcal{J}}$.

Let S_k denote the symmetric group on k elements. Let $\kappa \in \mathcal{C}(n)$, and let $\kappa \models m \leq n$ have components $\kappa_1, \ldots, \kappa_k$. Then we define

$$\boldsymbol{D}_{\boldsymbol{\kappa}} = D_{n-m} \times S_{\kappa_1} \times \ldots \times S_{\kappa_k}$$

and observe that

$$\begin{aligned} \boldsymbol{D}_{\boldsymbol{\kappa}} &= D_{n-m} \times S_{\kappa_1} \times \ldots \times S_{\kappa_k} \\ &\cong (\mathcal{W}_{\mathcal{J}_{-k}} \times \ldots \times \mathcal{W}_{\mathcal{J}_k})|_D \\ &= \mathcal{W}_{\mathcal{J}} \end{aligned}$$

where κ corresponds to $\mathcal{J}$ via the correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$ given earlier.

Let J and K to be any subsets of S, and $x \in W$. Let $x\mathcal{J}$ denote the image of the graph $\mathcal{J}$ under x; that is $(x(i), x(j))$ is an edge in $x\mathcal{J}$ if and only if (i, j) is an edge in $\mathcal{J}$. Let $\mathcal{J} \cap \mathcal{K}$ be the graph with vertex set N whose edges are those present in both $\mathcal{J}$ and $\mathcal{K}$, and let $x^{-1}\mathcal{J} \cap \mathcal{K}$ be understood as $(x^{-1}\mathcal{J}) \cap \mathcal{K}$. We can prove that $x^{-1}\mathcal{J}$ is a subgraph of $\mathcal{S}$, and with these constructs we can prove the following lemma.

Lemma 1 *Let J and K be subsets of S, and let $x \in X_J^{-1} \cap X_K$. Let the ordered presentation of $\mathcal{J}$ be*

$$(\mathcal{J}_{-v}, \ldots, \mathcal{J}_{-1}, \mathcal{J}_0, \mathcal{J}_1, \ldots, \mathcal{J}_v),$$

and $\mathcal{K}$ *be*

$$(\mathcal{K}_{-u},\ldots,\mathcal{K}_{-1},\mathcal{K}_0,\mathcal{K}_1,\ldots,\mathcal{K}_u).$$

Then an ordered presentation of $x^{-1}\mathcal{J}\cap\mathcal{K}$ *is*

$$\begin{gathered}(x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{-u},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{-u},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{-u},\\ x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{-u+1},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{-u+1},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{-u+1},\\ \ldots,x^{-1}\mathcal{J}_0\cap\mathcal{K}_0,\ldots\\ x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{u},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{u},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{u})\end{gathered}$$

or

$$\begin{gathered}(x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{-u},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{-u},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{-u},\\ x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{-u+1},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{-u+1},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{-u+1},\\ \ldots,\{-1\},\{\ \},\{1\},\ldots\\ x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{u},x^{-1}\mathcal{J}_{-v+1}\cap\mathcal{K}_{u},\ldots,x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{u})\end{gathered}$$

if $x^{-1}\mathcal{J}_0\cap\mathcal{K}_0=\{-1,1\}$, *with empty sets removed. However if* $x^{-1}\mathcal{J}_0\cap\mathcal{K}_0$ *is empty then we include it in the list.*

Due to our various correspondences, this lemma gives us a handle, in terms of graphs, of the various summands, $\mathcal{X}_{x^{-1}Jx\cap K}$, in Theorem 1. If we can display this information in terms of filled templates with appropriate reading words, then we shall have all the necessary ingredients for the reformulation of Theorem 1 into Theorem 2. This is what we shall head towards next.

From Lemma 2 [6], which states the first equality; and Lemma 1, it follows that if $x\in X_J^{-1}\cap X_K$ then

$$\begin{aligned}x^{-1}W_Jx\cap W_K &= W_{x^{-1}Jx\cap K}\\ &= \mathcal{W}_{x^{-1}\mathcal{J}\cap\mathcal{K}}\\ &= [\mathcal{W}_{(x^{-1}\mathcal{J}_{-v}\cap\mathcal{K}_{-u})}\times\ldots\times\mathcal{W}_{(x^{-1}\mathcal{J}_{v}\cap\mathcal{K}_{u})}]|_D\\ &= [(x^{-1}\mathcal{W}_{\mathcal{J}_{-v}}x\cap\mathcal{W}_{\mathcal{K}_{-u}})\times\ldots\times(x^{-1}\mathcal{W}_{\mathcal{J}_{v}}x\cap\mathcal{W}_{\mathcal{K}_{u}})]|_D\\ &= [(x^{-1}S^-_{\kappa_v}x\cap S^-_{\nu_u})\times\ldots\times(x^{-1}S^+_{\kappa_v}x\cap S^+_{\nu_u})]|_D\end{aligned}$$

where $\kappa=[\kappa_1,\ldots,\kappa_v]$ and $\nu=[\nu_1,\ldots,\nu_u]$ are suitable compositions in $\mathcal{C}(n)$, determined by the correspondence between $\mathcal{G}(n)$ and $\mathcal{C}(n)$. The superscript $-$ or $+$ on a symmetric group S_{κ_i} simply denotes that for the given $i>0$, S_{κ_i} acts on $\mathcal{J}_{-i}$ or $\mathcal{J}_i$ respectively. Observe that, say, $x^{-1}S^-_{\kappa_i}x\cap S^-_{\nu_j}$ is the group of all permutations on

$$x^{-1}\mathcal{J}_{-i}\cap\mathcal{K}_{-j}.$$

Let $z_{ij}=|x^{-1}\mathcal{J}_i\cap\mathcal{K}_j|$, then we have a mapping

$$\zeta : x\mapsto(z_{ij})$$

from $X_J^{-1} \cap X_K$ into a subset of all $(2u+1) \times (2v+1)$ matrices

$$\begin{pmatrix} z_{(-u)(-v)} & \cdots & z_{(-u)0} & \cdots & z_{(-u)v} \\ \vdots & & \vdots & & \vdots \\ z_{0(-v)} & \cdots & z_{00} & \cdots & z_{0v} \\ \vdots & & \vdots & & \vdots \\ z_{u(-v)} & \cdots & z_{u0} & \cdots & z_{uv} \end{pmatrix}$$

with non-negative integer entries that satisfy

$$\begin{array}{ll} \sum_{i=-u}^{u} z_{ij} = \kappa_{|j|}, j \neq 0, & \sum_{j=-v}^{v} z_{ij} = \nu_{|i|}, i \neq 0 \\ \sum_{i=-u}^{u} z_{i0} = 2\kappa_0, & \sum_{j=-v}^{v} z_{0j} = 2\nu_0, \end{array}$$

$$z_{ij} = z_{(-i)(-j)},$$

and

1. If $\mathcal{J}, \mathcal{K} \in \mathcal{G}_1 \cup \mathcal{G}_n$; or $\mathcal{J}, \mathcal{K} \in \mathcal{G}_n'$ then $\sum_{i,j>0} z_{i(-j)}$ must be even.
2. If $\mathcal{J} \in \mathcal{G}_1 \cup \mathcal{G}_n$, and $\mathcal{K} \in \mathcal{G}_n'$; or $\mathcal{J} \in \mathcal{G}_n'$ and $\mathcal{K} \in \mathcal{G}_1 \cup \mathcal{G}_n$ then $\sum_{i,j>0} z_{i(-j)}$ must be odd.

By doing a simple case study on the possible combinations of $\mathcal{J}, \mathcal{K}$, depending on whether they belong to $\mathcal{G}_1, \mathcal{G}_n, \mathcal{G}_n'$ or $\mathcal{G}_{<n}$ we can prove that ζ is surjective, and injective apart from when $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$ and $x^{-1}\mathcal{J}_0 \cap \mathcal{K}_0 = \{\ \}$. In this case $x, (-1,1)x \in X_J^{-1} \cap X_K$ have the same image under ζ. So now we have a handle in terms of matrices for the summands of Theorem 1.

We now require, for each $x \in X_J^{-1} \cap X_K$, that the matrix $(z_{ij}) = \zeta(x)$ should yield a composition $\eta \in \mathcal{C}(n)$ such that $\boldsymbol{D_\eta}$ is isomorphic to $W_{x^{-1}Jx \cap K}$. This will give us our desired reading word. To do this we need to know the components of η and whether η belongs to $\mathcal{C}_1, \mathcal{C}_n, \mathcal{C}_n'$ or $\mathcal{C}_{<n}$. After an analysis of the matrix entries of $\zeta(x)$ we obtain that the components of η are, in order, all the non-zero entries of $\zeta(x)$ read by row after z_{00}, or 1 followed by all the non-zero entries of $\zeta(x)$ read by row after z_{00} if $z_{00} = 1$.

As to whether η belongs to $\mathcal{C}_1, \mathcal{C}_n, \mathcal{C}_n'$ or $\mathcal{C}_{<n}$, since the graph of η is a subgraph of $\mathcal{K}$ we can deduce the following.

1. If $\mathcal{K} \in \mathcal{G}_1$ then $\eta \in \mathcal{C}_1$,
2. If $\mathcal{K} \in \mathcal{G}_n$ then $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$,
3. If $\mathcal{K} \in \mathcal{G}_n'$ then $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n'$,
4. If $\mathcal{K} \in \mathcal{G}_{<n}$ and
    (a) $\mathcal{J} \in \mathcal{G}_1 \cup \mathcal{G}_n$ then if $\sum_{i,j>0} z_{i(-j)}$ is even then $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$, otherwise $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n'$.

(b) $\mathcal{J} \in \mathcal{G}'_n$ then if $\sum_{i,j>0} z_{i(-j)}$ is even then $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$, otherwise $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$.

(c) $\mathcal{J} \in \mathcal{G}_{<n}$ then if $z_{00} > 2$ then $\eta \in \mathcal{C}_{<n}$, if $z_{00} = 2$ then $\eta \in \mathcal{C}_1$, if $z_{00} = 0$ then if $1 \in \bigcup_{i>0} x^{-1}\mathcal{J}_i \cap \mathcal{K}_0$ then $\eta \in \mathcal{C}_1 \cup \mathcal{C}_n$ otherwise $\eta \in \mathcal{C}_1 \cup \mathcal{C}'_n$.

We now observe that each matrix in the image of ζ corresponds to one $x \in X_J^{-1} \cap X_K$, that is one summand in Theorem 1, unless $z_{00} = 0$ and $\mathcal{J}, \mathcal{K} \in \mathcal{G}_{<n}$, in which case the matrix involved corresponds to both $x, x' \in X_J^{-1} \cap X_K$, where $x' = (-1, 1)x$.

If we now recode the basis elements $\mathcal{X}_J$, given in Solomon's Theorem, to B_κ, where κ and $\mathcal{J}$ correspond via the correspondence between $\mathcal{C}(n)$ and $\mathcal{G}(n)$; biject the matrices into filled templates; and summarize what we have deduced as clearly as possible, we arrive at the matrix interpretation of Solomon's Theorem for the descent algebra of D_n given in the Introduction.

References

[1] F Bergeron and N Bergeron. A decomposition of the descent algebra of the hyperoctahedral group 1. *Journal of Algebra*, 148:86-97, 1992.

[2] N Bergeron. A decomposition of the descent algebra of the hyperoctahedral group 2. *Journal of Algebra*, 148:98-122, 1992.

[3] AM Garsia and J Remmel. Shuffles of permutations and the Kronecker product. *Graphs and Combinatorics*, 1:217-263, 1985.

[4] AM Garsia and C Reutenauer. A decomposition of Solomon's descent algebra. *Advances in Mathematics*, 77:189-262, 1989.

[5] JE Humphrey. *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[6] L Solomon. A formula in the group ring of a Coxeter group. *Journal of Algebra* 41:255-268, 1976.