

# CHROMATIC CLASSICAL SYMMETRIC FUNCTIONS

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ABSTRACT. In this note we classify when a skew Schur function is a positive linear combination of power sum symmetric functions. We then use this to determine precisely when any scalar multiple of a skew Schur function is the chromatic symmetric function of some graph. From here we are able to prove that of the classical bases for symmetric functions only certain scalar multiples of the elementary symmetric functions can be realised as the chromatic symmetric function of some graph, namely a particular union of complete graphs.

## 1. INTRODUCTION

The areas of symmetric functions and chromatic functions both have a long and distinguished history. More recently a chromatic symmetric function has been defined for any finite simple graph [7]. It naturally generalizes the chromatic function and its positivity when expanded into elementary symmetric functions, for example [3, 4, 6], or Schur functions, for example [2], has been a keen avenue of research due to connections to other areas such as representation theory and algebraic geometry. Other avenues in this area include sufficient conditions for chromatic symmetric function equality [5] and the recent discovery that chromatic symmetric functions give rise to infinitely many bases for the algebra of symmetric functions [1]. Regarding the latter, the question has consequently been frequently raised whether the classical bases for the algebra of symmetric functions can be realised as chromatic symmetric functions, especially the elementary symmetric functions or Schur functions. In this note we answer this question comprehensively (Theorem 2.8) after first establishing which scalar multiples of skew

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Schur functions can be realised as chromatic symmetric functions (Theorem 2.7) that is reliant on establishing when they expand positively in terms of power sum symmetric functions (Proposition 2.6).

## 2. CHROMATIC CLASSICAL BASES

Before we prove our results, for completeness, we first recall classical combinatorial and algebraic concepts, which will be necessary later. Readers familiar with the subject may safely skip to Definition 2.2. A *partition*  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  of  $n$  is a list of weakly decreasing positive integers, or *parts*,  $\lambda_i$  whose sum is  $n$ , which we denote by  $\lambda \vdash n$ . We denote by  $\emptyset$  the empty partition whose sum is 0. Given a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  its corresponding *diagram*, also denoted by  $\lambda$ , is the array of left-justified boxes with  $\lambda_i$  boxes in row  $i$ , where we number the rows from top to bottom and the columns from left to right. Given two partitions  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  and  $\mu = (\mu_1, \dots, \mu_k)$  where  $k \leq \ell$  and  $\mu_i \leq \lambda_i$  for  $1 \leq i \leq k$  we say the *skew diagram*  $\lambda/\mu$  is the array of boxes

$$\lambda/\mu = \{(i, j) \mid (i, j) \in \lambda \text{ and } (i, j) \notin \mu\}.$$

Note that if  $\mu = \emptyset$  then  $\lambda/\mu$  is the diagram  $\lambda$ . The number of boxes in  $\lambda/\mu$  is called its *size* and is denoted by  $|\lambda/\mu|$ . The *transpose* of the skew diagram  $\lambda/\mu$  is the array of boxes

$$(\lambda/\mu)^t = \{(j, i) \mid (i, j) \in \lambda \text{ and } (i, j) \notin \mu\}.$$

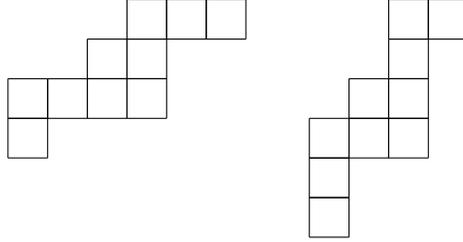
Two skew diagrams that will be of particular interest to us are horizontal and vertical strips. We say a skew diagram is a *horizontal strip* if no two boxes lie in the same column, and is a *vertical strip* if no two boxes lie in the same row. For simplicity we will often denote a skew diagram by  $D$ .

Given a skew diagram  $\lambda/\mu$  we say that a *semistandard Young tableau (SSYT)* of *shape*  $\lambda/\mu$  is a filling of the boxes of  $\lambda/\mu$  with positive integers such that the entries in each row weakly increase when read from left to right, and the entries in each column strictly increase when read from top to bottom. Given an SSYT,  $T$ , with largest entry  $\max_T$  we define the *content* of  $T$  to be

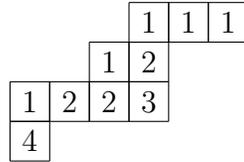
$$c(T) = (c_1(T), \dots, c_{\max_T}(T))$$

where  $c_i(T)$  is the number of times  $i$  appears in  $T$ .

**Example 2.1.** Below on the left is the skew diagram  $D = (6, 4, 4, 1)/(3, 2)$  and on the right its transpose  $D^t$ .



Note that  $D$  is neither a horizontal nor a vertical strip. Below is an SSYT of shape  $D$  and content  $(5, 3, 1, 1)$ .



Given partitions and tableaux we can now turn our attention to symmetric functions, which will be the focus of our study, and whose algebra is a subalgebra of  $\mathbb{C}[[x_1, x_2, \dots]]$ . Given a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n \geq 1$  we define the *monomial symmetric function*  $m_\lambda$  to be

$$m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

where the sum is over all  $\ell$ -tuples  $(i_1, \dots, i_\ell)$  of distinct indices that yield distinct monomials. We define the  $r$ -th *power sum symmetric function* to be  $p_r = m_{(r)}$  from which we define the *power sum symmetric function*  $p_\lambda$  to be

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

Meanwhile, we define the renowned *Schur function*  $s_\lambda$  to be

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$$

where  $K_{\lambda\mu}$  is the number of SSYTs of shape  $\lambda$  and content  $\mu$ . Schur functions can further be generalised to *skew Schur functions*  $s_D$  for a skew diagram  $D$

$$(2.1) \quad s_D = \sum_{\nu \vdash |D|} c_{D\nu} s_\nu$$

where  $c_{D\nu}$  is the number of SSYTs,  $T$ , of shape  $D$  and content  $\nu$  such that when the entries of  $T$  are read from right to left from the top row to the bottom row, the number of  $i$ 's that have been read is always weakly greater than the number of  $(i + 1)$ 's that have been read, for all  $i \geq 1$ . Note that the SSYT in Example 2.1 satisfies this latter criterion.

This rule for computing  $c_{D\nu}$  is called the *Littlewood-Richardson rule*. Note that when  $D = \lambda/\emptyset$  the Schur function  $s_\lambda$  is recovered. We define the  $r$ -th *complete homogeneous symmetric function* to be  $h_r = s_{(r)}$ , and if  $D$  is a horizontal strip with  $\alpha_i$  boxes in row  $i$  for  $1 \leq i \leq \ell$  such that  $\alpha_1, \dots, \alpha_\ell$  in weakly decreasing order yields  $\lambda_1, \dots, \lambda_\ell$  then we define the *complete homogeneous symmetric function*  $h_\lambda$  to be

$$(2.2) \quad h_\lambda = s_D = h_{\alpha_1} \cdots h_{\alpha_\ell} = h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

We define the  $r$ -th *elementary symmetric function* to be  $e_r = s_{(1^r)}$  where  $(1^r)$  is the partition consisting of  $r$  parts equal to 1. Furthermore,  $D^t$  is a vertical strip with  $\alpha_i$  boxes in column  $i$  for  $1 \leq i \leq \ell$  and we define the *elementary symmetric function*  $e_\lambda$  to be

$$(2.3) \quad e_\lambda = s_{D^t} = e_{\alpha_1} \cdots e_{\alpha_\ell} = e_{\lambda_1} \cdots e_{\lambda_\ell}.$$

Lastly, by convention

$$e_\emptyset = h_\emptyset = m_\emptyset = p_\emptyset = s_\emptyset = 1.$$

Note also that

$$e_{(1)} = h_{(1)} = m_{(1)} = p_{(1)} = s_{(1)}$$

and

$$e_{(1^r)} = h_{(1^r)} = p_{(1^r)} \quad e_{(r)} = m_{(1^r)} = s_{(1^r)}$$

for  $r \geq 1$ . The *algebra of symmetric functions* is then the graded algebra

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots$$

where a basis for the  $n$ -th graded part is given by any of  $\{e_\lambda\}_{\lambda \vdash n}$ ,  $\{h_\lambda\}_{\lambda \vdash n}$ ,  $\{m_\lambda\}_{\lambda \vdash n}$ ,  $\{p_\lambda\}_{\lambda \vdash n}$ ,  $\{s_\lambda\}_{\lambda \vdash n}$ . There exists a scalar product on  $\Lambda$  such that the power sum symmetric functions form an orthogonal basis, that is, for partitions  $\lambda$  and  $\mu$

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$$

while the Schur functions form an orthonormal basis, that is,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

where  $\delta_{\lambda\mu} = 1$  if  $\lambda = \mu$  and 0 otherwise, and  $z_\lambda$  is a positive integer reliant on  $\lambda$ . A further useful tool is the involution  $\omega$  that satisfies

$$\omega(s_D) = s_{D^t}$$

for any skew diagram  $D$ . One last useful fact is that

$$(2.4) \quad h_r = \sum_{\lambda \vdash r} z_\lambda^{-1} p_\lambda.$$

The symmetric functions above are not the only ones of interest to us. In [7] Stanley defined the chromatic symmetric function as follows,

which is reliant on a graph that is finite and simple and the notion of a proper colouring, where given a graph  $G$  with vertex set  $V$  a proper colouring  $\kappa$  of  $G$  is a function

$$\kappa : V \rightarrow \{1, 2, \dots\}$$

such that if  $v_1, v_2 \in V$  are adjacent then  $\kappa(v_1) \neq \kappa(v_2)$ .

**Definition 2.2.** For a finite simple graph  $G$  with vertex set  $V = \{v_1, \dots, v_n\}$  and edge set  $E$  the chromatic symmetric function of  $G$  is defined to be

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}$$

where the sum is over all proper colourings  $\kappa$  of  $G$ .

The chromatic symmetric function can equivalently be defined in terms of power sum or monomial symmetric functions with the aid of some natural partitions. Given a graph  $G$  with vertex set  $V = \{v_1, \dots, v_n\}$ , edge set  $E$ , and a subset  $S \subseteq E$ , let  $\lambda(S)$  be the partition of  $n$  whose parts are equal to the number of vertices in the connected components of the spanning subgraph of  $G$  with vertex set  $V$  and edge set  $S$ . Meanwhile, a stable partition  $\pi$  of the vertices of  $G$  is a partition of  $v_1, \dots, v_n$  such that each block is totally disconnected. Let  $\mu(\pi)$  be the partition of  $n$  whose parts are equal to the number of vertices in each block of  $\pi$ .

**Lemma 2.3.** [7, Proposition 2.4, Theorem 2.5] For a finite simple graph  $G$  with vertex set  $V$  and edge set  $E$  we have the following.

- (1)  $X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}$ .
- (2)  $X_G = \sum_{\pi} (\pi_1! \pi_2! \cdots) m_{\mu(\pi)}$  where the sum is over all stable partitions  $\pi$  of  $G$ , and  $\pi_i$  is the multiplicity of  $i$  in  $\mu(\pi)$ .

The chromatic symmetric function also satisfies the following two crucial properties, recalling that when we say a function is  $p$ -positive (respectively,  $-negative$ ) we mean it is a positive (respectively, negative) linear combination of power sum symmetric functions.

**Lemma 2.4.** [7, Proposition 2.3] If a finite simple graph  $G$  is a disjoint union of subgraphs  $G_1 \cup \cdots \cup G_\ell$  then  $X_G = \prod_{i=1}^{\ell} X_{G_i}$ .

**Lemma 2.5.** [7, Corollary 2.7] For any finite simple graph  $G$ ,  $\omega(X_G)$  is  $p$ -positive.

We are now close to establishing which classical symmetric functions are also the chromatic symmetric function of some graph. To this end we first deduce a natural result on skew Schur functions when expanded as a sum of power sum symmetric functions.

**Proposition 2.6.** *A skew Schur function  $s_D$  is  $p$ -positive if and only if  $D$  is a horizontal strip.*

*Proof.* Let  $D$  be a horizontal strip with  $\ell$  rows and  $\alpha_i$  boxes in row  $i$  for  $1 \leq i \leq \ell$ . Then by Equation (2.2)

$$s_D = h_{\alpha_1} \cdots h_{\alpha_\ell}.$$

We know by Equation (2.4) that  $h_r$  is  $p$ -positive, and hence so is

$$h_{\alpha_1} \cdots h_{\alpha_\ell} = s_D.$$

Conversely, let  $D$  contain two boxes in some column,  $|D| = n$ , and

$$s_D = \sum_{\lambda \vdash n} a_{D\lambda} p_\lambda.$$

By the Littlewood-Richardson rule of Equation (2.1) we have that the coefficient of  $s_{(n)}$  in  $s_D$  is 0, that is,

$$(2.5) \quad \langle s_D, s_{(n)} \rangle = 0.$$

By Equation (2.4) we know that

$$s_{(n)} = h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$$

and substituting into Equation (2.5) we get

$$\left\langle \sum_{\lambda \vdash n} a_{D\lambda} p_\lambda, \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \right\rangle = 0.$$

Since the power sum symmetric functions are an orthogonal basis for  $\Lambda$ , we have that  $a_{D\lambda} \not\equiv 0$  for all  $\lambda \vdash n$ , and the result follows.  $\square$

With this proposition we now classify which skew Schur functions can be realised as the chromatic symmetric function of some graph, and key to our classification is the complete graph  $K_n$ ,  $n \geq 1$  consisting of  $n$  vertices, each pair of which are adjacent.

**Theorem 2.7.**  *$X_G = cs_D$  for some graph  $G$  and  $c \neq 0$  if and only if  $D$  is a vertical strip. In particular, if  $D$  is a vertical strip consisting of  $\ell$  columns with  $\alpha_i$  boxes in column  $i$  for  $1 \leq i \leq \ell$  then*

$$X_G = \left( \prod_{i=1}^{\ell} \alpha_i! \right) s_D$$

where

$$G = K_{\alpha_1} \cup \cdots \cup K_{\alpha_\ell}.$$

*Proof.* Let  $D$  be a vertical strip with  $\ell$  columns containing  $\alpha_i$  boxes in column  $i$  for  $1 \leq i \leq \ell$ . Then, since for  $K_n$ ,  $n \geq 1$  we have that [1, Theorem 8]

$$X_{K_n} = n!e_n,$$

by Equation (2.3) and Lemma 2.4 we obtain

$$s_D = \prod_{i=1}^{\ell} e_{\alpha_i} = \prod_{i=1}^{\ell} \frac{1}{\alpha_i!} X_{K_{\alpha_i}} = \left( \prod_{i=1}^{\ell} \frac{1}{\alpha_i!} \right) X_{K_{\alpha_1 \cup \dots \cup K_{\alpha_\ell}}}.$$

Now let  $D$  contain two boxes in some row, and assume that there exists a graph  $G$  such that  $X_G = cs_D$  for  $c \neq 0$ . Then by Lemma 2.5

$$\omega(X_G) = \omega(cs_D) = cs_{D^t}$$

is  $p$ -positive. Hence  $s_{D^t}$  is either  $p$ -positive or  $p$ -negative and moreover  $D^t$  contains two boxes in some column. By Proposition 2.6 this is not possible, and hence no such  $G$  exists.  $\square$

We are now ready to establish which classical symmetric functions can be realised as the chromatic symmetric function of some graph.

**Theorem 2.8.** *Of the classical symmetric functions  $\{e_\lambda\}_{\lambda \vdash n \geq 1}$ ,  $\{h_\lambda\}_{\lambda \vdash n \geq 1}$ ,  $\{m_\lambda\}_{\lambda \vdash n \geq 1}$ ,  $\{p_\lambda\}_{\lambda \vdash n \geq 1}$  and  $\{s_\lambda\}_{\lambda \vdash n \geq 1}$  only those that are scalar multiples of the elementary symmetric functions  $\{e_\lambda\}_{\lambda \vdash n \geq 1}$  can be realised as the chromatic symmetric function of some graph. In particular, if  $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n \geq 1$  then*

$$e_\lambda = \left( \prod_{i=1}^{\ell} \frac{1}{\lambda_i!} \right) X_{K_{\lambda_1 \cup \dots \cup K_{\lambda_\ell}}}.$$

*Proof.* Since  $e_{(1^n)} = h_{(1^n)} = p_{(1^n)}$  for  $n \geq 1$  we consider this to be  $e_{(1^n)}$  for convenience, and likewise consider  $e_{(n)} = m_{(1^n)} = s_{(1^n)}$  for  $n \geq 1$  to be  $e_{(n)}$ . The result for  $\{p_\lambda\}_{\lambda \vdash n \geq 1}$  and  $\{m_\lambda\}_{\lambda \vdash n \geq 1}$  follows immediately from Lemma 2.3. Meanwhile, the result for  $\{s_\lambda\}_{\lambda \vdash n \geq 1}$  and  $\{h_\lambda\}_{\lambda \vdash n \geq 1}$  follows from Equations (2.1) and (2.2) combined with Theorem 2.7. Equation (2.3) combined with Theorem 2.7 then yields the result for  $\{e_\lambda\}_{\lambda \vdash n \geq 1}$ .  $\square$

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