CHROMATIC BASES FOR SYMMETRIC FUNCTIONS

SOOJIN CHO AND STEPHANIE VAN WILLIGENBURG

Abstract. In this note we obtain numerous new bases for the algebra of symmetric functions whose generators are chromatic symmetric functions. More precisely, if $\{G_k\}_{k\geq 1}$ is a set of connected graphs such that G_k has k vertices for each k, then the set of all chromatic symmetric functions $\{X_{G_k}\}_{k\geq 1}$ generates the algebra of symmetric functions. We also obtain explicit expressions for the generators arising from complete graphs, star graphs, path graphs and cycle graphs.

1. Introduction

In [7] Stanley defined a symmetric function X_G that was reliant on a finite simple graph G, called the chromatic symmetric function of G. He proved that X_G specializes to the chromatic polynomial of G and generalizes other chromatic polynomial properties, although intriguingly not the deletion-contraction property. Since then the chromatic symmetric function X_G has been the genesis of two long-standing conjectures in algebraic combinatorics. The first of these conjectures that the chromatic symmetric functions of all $(3+1)$-free posets are a positive linear combination of elementary symmetric functions, for which a variety of evidence exists [3, 4, 8]. The second of these conjectures that the chromatic symmetric function distinguishes non-isomorphic trees. This conjecture has been confirmed for spiders [5] plus a variety of caterpillars [1, 5], and towards a different approach a sufficient condition for graphs to have equal chromatic symmetric functions has also been discovered [6].

In this vein of a different perspective on the chromatic symmetric function we provide a potential new tool, namely we determine a myriad of new bases for the algebra of symmetric functions, whose generators are chromatic symmetric functions (Theorem 2.5) and give explicit expansions for the generators arising from well-known graphs (Theorem 3.1).

2. Chromatic symmetric function bases

We begin by recalling concepts that will be useful later, and by defining the algebra of symmetric functions. A *partition* $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n, denoted by $\lambda \vdash n$, is a list of positive integers whose parts λ_i satisfy $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\sum_{i=1}^{\ell} \lambda_i = n$. If λ has exactly m_i parts equal to i for $1 \leq i \leq n$ we will sometimes denote λ by $\lambda = (1^{m_1}, \dots, n^{m_n})$. Also, given partitions of n, $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_m)$ we say $\mu \leq \lambda$ *in lexicographic order* if $\mu = \lambda$ or $\mu_i = \lambda_i$ for $i < j$ and $\mu_j < \lambda_j$ for some $1 \leq j \leq m$.

2010 *Mathematics Subject Classification.* Primary 05E05; Secondary 05C15, 05C25.
Key words and phrases. chromatic symmetric function, complete graph, star graph, path, cycle.
The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01057476). The second author was supported in part by the National Sciences and Engineering Research Council of Canada.

The algebra of symmetric functions is a subalgebra of $\mathbb{Q}[[x_1, x_2, \ldots]]$ and can be defined as follows. We define the *i-th power sum symmetric function* $p_i, i \geq 1$ to be

$$p_i = x_1^i + x_2^i + x_3^i + \cdots$$

and given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, we define the *power sum symmetric function* p_λ to be

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell} .$$

Then the *algebra of symmetric functions*, Λ, is the graded algebra

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots$$

where $\Lambda^0 = \operatorname{span}\{1\} = \mathbb{Q}$ and for $n \geq 1$

$$\Lambda^n = \operatorname{span}\{p_\lambda \mid \lambda \vdash n\} .$$

The power sum symmetric functions in fact form a basis for Λ. Other well-known bases include the basis of Schur functions, the basis of complete homogeneous symmetric functions and the basis of elementary symmetric functions, whose *i-th elementary symmetric function* $e_i, i \geq 1$ is defined to be

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}$$

leading to the celebrated fundamental theorem of symmetric functions, which states that

$$\Lambda = \mathbb{Q}[e_1, e_2, \ldots] .$$

Our object of study is a further symmetric function, known as the chromatic symmetric function, which is reliant on a graph that is *finite* and *simple*, and from here onwards we will assume that all our graphs satisfy these properties. We are now almost ready to define the chromatic symmetric function of a graph, but before we do we recall the notion of a proper coloring. Given a graph G with vertex set V a *proper coloring* κ of G is a function

$$\kappa : V \to \{1, 2, \ldots\}$$

such that if $v_1, v_2 \in V$ are adjacent, then $\kappa(v_1) \neq \kappa(v_2)$.

Definition 2.1. For a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E the *chromatic symmetric function of* G is defined to be

$$X_G = \sum_\kappa x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}$$

where the sum is over all proper colorings κ of G.

Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E, and a subset $S \subseteq E$, let $\lambda(S)$ be the partition of n whose parts are equal to the number of vertices in the connected components of the spanning subgraph of G with vertex set V and edge set S. We say a set partition $\pi = \{B_1, \ldots, B_k\}$ of V is *connected* if the subgraph of G determined by B_i is connected for each i, and the *lattice of contractions* L_G *of* G is the set of all connected partitions of V partially ordered by refinement so that the unique minimal element $\hat{0}$ of L_G is the partition into n one element blocks. Lastly, given $\pi = \{B_1, \ldots, B_k\} \in L_G$, the *type* of π, denoted by $\operatorname{type}(\pi)$, is the partition obtained by rearranging $|B_1|, \ldots, |B_k|$ in weakly decreasing order. With this in mind we have the following.

Lemma 2.2. [7, Theorems 2.5 and 2.6] *For a graph G with vertex set V and edge set E we have that*

(1) $X_G = \sum_{S\subseteq E}(-1)^{|S|}p_{\lambda(S)}$,
(2) $X_G = \sum_{\pi\in L_G}\mu(\hat{0},\pi)p_{\mathrm{type}(\pi)}$, *where* μ *is the Möbius function of* L_G, *and* $\mu(\hat{0},\pi)$ *is non-zero for all* $\pi \in L_G$.

The chromatic symmetric function also satisfies the following useful property.

Lemma 2.3. [7, Proposition 2.3] *If a graph* G *is a disjoint union of subgraphs* $G_1, \dots, G_\ell$, *then* $X_G = \prod_{i=1}^{\ell} X_{G_i}$.

We now need one last definition before we can state our theorem.

Definition 2.4. Let $\{G_k\}_{k\geq 1}$ be a set of connected graphs such that G_k has k vertices for each k, and let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition. Then

$$G_\lambda = G_{\lambda_1} \cup \cdots \cup G_{\lambda_\ell}\,,$$

that is, G_λ is the graph whose connected components are $G_{\lambda_1}, \dots, G_{\lambda_\ell}$.

We can now determine a plethora of new bases for Λ.

Theorem 2.5. *Let* $\{G_k\}_{k\geq 1}$ *be a set of connected graphs such that* G_k *has* k *vertices for each* k*. Then*

$$\{X_{G_\lambda} \mid \lambda \vdash n\}$$

is a $\mathbb{Q}$*-basis of* Λ^n*. Plus we have that*

$$\Lambda = \mathbb{Q}[X_{G_1}, X_{G_2}, \dots]$$

and the X_{G_k} *are algebraically independent over* $\mathbb{Q}$.

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and V_i be the sets of vertices in G_{λ_i} for $1 \leq i \leq \ell$. Then

$$V = \biguplus_{i=1}^{\ell} V_i$$

is the set of vertices in G_λ. By the definition of G_λ, we know that if $\pi \in L_{G_\lambda}$, then $\mathrm{type}(\pi) \leq \lambda$ in lexicographic order. Thus by Lemma 2.2 it follows that

$$X_{G_\lambda} = \sum_{\mu\leq\lambda} c_{\lambda\mu}p_\mu$$

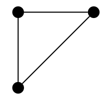
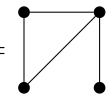
and, moreover, that $c_{\lambda\lambda} = \mu(\hat{0},\pi_\lambda) \neq 0$ where $\pi_\lambda = (V_1, \dots, V_\ell)$ is the unique connected partition of V satisfying $\mathrm{type}(\pi_\lambda) = \lambda$. Hence, $\{X_{G_\lambda} \mid \lambda \vdash n\}$ is a $\mathbb{Q}$-basis of Λ^n.

Since for $\lambda = (\lambda_1, \dots, \lambda_\ell)$ we have

$$X_{G_\lambda} = \prod_{i=1}^{\ell} X_{G_{\lambda_i}} \tag{2.6}$$

by Lemma 2.3 and $\{X_{G_\lambda}\}_{\lambda\vdash n\geq 1} \cup \{1\}$ forms a $\mathbb{Q}$-basis for Λ, every element of Λ is expressible uniquely as a polynomial in the X_{G_k} and hence $\Lambda = \mathbb{Q}[X_{G_1}, X_{G_2}, \dots]$ and the X_{G_k} are algebraically independent over $\mathbb{Q}$. ☐

Example 2.7. If

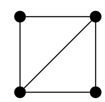
$G_1 =$ $G_2 =$ $G_3 =$  $G_4 =$ 

then $\{X_{G_1}, X_{G_2}, X_{G_3}, X_{G_4}\}$ is a set of generators for Λ^4 and
$X_{G_{(4)}} = X_{G_4} = -2p_{(4)} + 4p_{(3,1)} + p_{(2,2)} - 4p_{(2,1,1)} + p_{(1,1,1,1)}$,
$X_{G_{(3,1)}} = X_{G_3}X_{G_1} = (2p_{(3)} - 3p_{(2,1)} + p_{(1,1,1)})p_{(1)} = 2p_{(3,1)} - 3p_{(2,1,1)} + p_{(1,1,1,1)}$,
$X_{G_{(2,2)}} = X_{G_2}X_{G_2} = (-p_{(2)} + p_{(1,1)})^2 = p_{(2,2)} - 2p_{(2,1,1)} + p_{(1,1,1,1)}$,
$X_{G_{(2,1,1)}} = X_{G_2}X_{G_1}X_{G_1} = (-p_{(2)} + p_{(1,1)})p_{(1)}p_{(1)} = -p_{(2,1,1)} + p_{(1,1,1,1)}$, and
$X_{G_{(1,1,1,1)}} = X_{G_1}X_{G_1}X_{G_1}X_{G_1} = p^4_{(1)} = p_{(1,1,1,1)}$
is a $\mathbb{Q}$-basis of Λ^4.

Alternatively, if A, B, C, D are as below, then $\{X_A, X_B, X_C, X_D\}$ is a set of generators for Λ^4 and

$$\{X_D, X_CX_A, X_BX_B, X_BX_AX_A, X_AX_AX_AX_A\}$$

is a $\mathbb{Q}$-basis of Λ^4.

$A =$ $B =$ $C =$ $D =$ 

Remark 2.8. Observe that the only connected graph on two vertices is G_2 above, and

$$e_2 = \sum_{j_1<j_2} x_{j_1}x_{j_2} = \frac{1}{2}X_{G_2} .$$

Therefore, while every $\{X_{G_\lambda}\}_{\lambda\vdash n\geq 1} \cup \{1\}$ is a $\mathbb{Q}$-basis of Λ it is never a $\mathbb{Z}$-basis of Λ.

3. Chromatic symmetric functions for classes of graphs

In this section, we compute chromatic symmetric functions for some particular connected graphs, whose definitions we include for clarity. The *complete graph* $K_n, n \geq 1$ has n vertices each pair of which are adjacent. The *star graph* $S_{n+1}, (n+1) \geq 1$ has $n+1$ vertices and is the tree with one vertex of degree n and n vertices of degree one. The *path graph* $P_n, n \geq 1$ has n vertices and is the tree with 2 vertices of degree one and $n-2$ vertices of degree 2 for $n \geq 2$ and $P_1 = K_1$. Lastly, the *cycle graph* $C_n, n \geq 1$ is the connected graph with n vertices of degree 2 for $n \geq 3$, $C_2 = K_2$, $C_1 = K_1$. The chromatic symmetric functions of Ferrers graphs, naturally related to Λ via Ferrers diagrams, were computed in [2]. We note that the third formula appears in the second proof of [7, Proposition 5.3] that gives the generating function for X_{P_n}, and the generating function for X_{C_n} is given in [7, Proposition 5.4].

Theorem 3.1. *(1) If K_n is the complete graph with $n \geq 1$ vertices, then*

$$X_{K_n} = n!e_n .$$

(2) If S_{n+1} is the star graph with $(n+1) \geq 1$ vertices, then

$$X_{S_{n+1}} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}p_{(r+1,1^{n-r})} .$$

(3) If P_n is the path graph with $n \geq 1$ vertices, then

$$X_{P_n} = \sum_{\lambda=(1^{m_1},\ldots,n^{m_n})\vdash n} (-1)^{n-\sum_{i=1}^n m_i} \frac{(\sum_{i=1}^n m_i)\,!}{\prod_{i=1}^n (m_i)!}\; p_\lambda.$$

(4) If C_n is the cycle graph with $n \geq 1$ vertices, then

$$X_{C_n} = \sum_{\lambda=(1^{m_1},\ldots,n^{m_n})\vdash n} (-1)^{n-\sum_{i=1}^n m_i} \frac{(\sum_{i=1}^n m_i)\,!}{\prod_{i=1}^n (m_i)!} \left(1 + \sum_{j=2}^{n} (j-1)\frac{m_j}{\sum_{i=1}^n m_i}\right) p_\lambda\,.$$

Proof. For K_n, since every vertex must be colored a different color and this can be done in $n!$ ways, $X_{K_n} = n! \sum_{j_1<\cdots<j_n} x_{j_1}\cdots x_{j_n} = n! e_n$.

We now use the first part of Lemma 2.2, which states that $X_G = \sum_{S\subseteq E}(-1)^{|S|}p_{\lambda(S)}$, for the remainder of the proof, where E is the set of edges of our graph G.

For S_{n+1}, if we choose any r edges, then $(r+1)$ vertices will make a connected component and the remaining $(n-r)$ vertices will be isolated. Hence the second part of the theorem follows.

For P_n, draw this graph on a horizontal axis. Now consider the spanning subgraph of P_n with n vertices and edge set $S \subseteq E$, P_n^S. Counting the number of vertices in each connected component of P_n^S from left to right yields a list of positive integers when rearranged into weakly decreasing order yield a partition, say $\lambda = (1^{m_1},\ldots,n^{m_n}) \vdash n$. Since the number of edge sets, S, that will yield λ is $\frac{(\sum_{i=1}^n m_i)\,!}{\prod_{i=1}^n (m_i)!}$, and $|S| = \sum_{i=1}^n (i-1)m_i = n - \sum_{i=1}^n m_i$ for such S, the third part is now proved.

For C_n and a partition $\lambda = (1^{m_1},\ldots,n^{m_n}) \vdash n$, we look for all subsets S of the edge set E that contribute to p_λ in the expansion of X_{C_n}; that is, subsets S satisfying $\lambda(S) = \lambda$. To this end, label the vertices of C_n with $v_1,\ldots,v_n$ in a clockwise direction, choosing v_1 arbitrarily, and let ϵ_i be the edge connecting v_i and v_{i+1} for $i = 1,\ldots,n$, where $v_{n+1} = v_1$. We first consider the possible S that do not contain ϵ_n: Since $\epsilon_n \notin S$, such S can be understood as a subset of the vertex set of P_n, and the contribution of such S to p_λ in the expansion of X_{C_n} is the same as the coefficient of p_λ in the expansion of X_{P_n}, which is

$$(-1)^{n-\sum_{i=1}^n m_i} \frac{(\sum_{i=1}^n m_i)\,!}{\prod_{i=1}^n (m_i)!}\,. \tag{3.2}$$

We now consider the possible S that do contain ϵ_n: There are cases to consider, depending on the number of vertices j, where $j \geq 2$, in the connected component of S that contains ϵ_n. For each j, there are $(j-1)$ possible connected components depending on the smallest labeled vertex of the component, which can be $v_{n-j+2},\ldots,v_n$. After we identify the connected component containing ϵ_n in C_n, the remainder of the graph is the path graph with $(n-j)$ vertices to which we can apply the third part of this theorem. The overall contribution of such S is thus

$$\sum_{j=2}^{n} (-1)^{j-1}(j-1)(-1)^{n-j-(\sum_{i=1}^n m_i)+1} \frac{((\sum_{i=1}^n m_i)-1)\,!(m_j)}{\prod_{i=1}^n (m_i)!}. \tag{3.3}$$

We now add (3.2) and (3.3) to obtain that the coefficient of p_λ is

$$(-1)^{n-\sum_{i=1}^n m_i} \frac{(\sum_{i=1}^n m_i)\,!}{\prod_{i=1}^n (m_i)!} \left(1 + \sum_{j=2}^{n} (j-1)\frac{m_j}{\sum_{i=1}^n m_i}\right)\,.$$

□

Remark 3.4. A natural question to ask is whether any of these bases is Schur positive, that is, a positive linear combination of Schur functions. The answer is not at all obvious, since the basis whose generators stem from complete graphs is trivially Schur positive, whereas the basis whose generators stem from star graphs is not as X_{S_4} is not.

Acknowledgements

The authors would like to thank Boram Park for helpful conversations, the referee for helpful comments, and Ajou University where the research took place.

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Department of Mathematics, Ajou University, Suwon 443-749, Korea
E-mail address: chosj@ajou.ac.kr

Department of Mathematics, University of British Columbia, Vancouver BC V6T 1Z2, Canada
E-mail address: steph@math.ubc.ca