The p-modular Descent Algebras of the Hyperoctahedral Group and Dihedral Group

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1 The p-modular Descent Algebra of the Hyperoctahedral Group

Abstract

The p-modular descent algebra of the hyperoctahedral group, ΣB_n, and a homomorphism into the algebra of generalised p-modular characters of the hyperoctahedral group are defined. These are then used to describe both the radical and the irreducible representations of ΣB_n. The nilpotency index of the radical is also determined.

1.1 Introduction

Since their introduction by Louis Solomon in the mid 1970's ([Sol76]), much has been written on what are now known as Solomon's descent algebras of Coxeter groups. Many of these writings have been concerned with the the descent algebra of the symmetric group, ranging from the extensive paper by Garsia and Reutenauer ([GR89]), to compact works by Atkinson ([Atk92], [Atk86]).

In contrast, little has been written on the *descent algebra of the hyperoctahedral group,* B_n, with the exception of the Bergerons who obtained a decomposition of its multiplicative structure, and a complete set of minimal idempotents([BB92], [Ber92]).

Recall that B_n is the group of all *signed permutations* where a signed permutation $\pi = \pi_1\pi_2\dots\pi_n$ ($\pi_i \in \{\pm 1, \pm 2, \dots \pm n\}\forall i$ such that $\sigma(\pi) = \sigma_1\sigma_2\dots\sigma_n$ is a permutation of $\{1, 2, \dots, n\}$ if $\sigma_i = |\pi_i|$) is a shorthand notation for the permutation of the $2n$ symbols $1, 2, \dots, n, -1, -2, \dots -n$ that maps i to π_i, and $-i$ to $-\pi_i$.

The Bergerons define a basis for ΣB_n initially as follows:

Given a signed permutation $\pi = \pi_1\pi_2\dots\pi_n$, and setting $\pi_0 = 0$, we say π has a descent set

$$Des(\pi) = \{i \in [0, n-1] : \pi_i > \pi_{i+1}\}$$

and define

$$A_S = \sum_{Des(\pi)=S} \pi$$

to be a *descent class sum*. The set of all such descent class sums form a basis for ΣB_n of dimension 2^n.

The above works, although widening the knowledge of descent algebras, solely studied the algebra over a field of zero characteristic. However, since the structure constants of the algebra are integers, it is also possible to define ΣB_n over fields $\mathcal{F}_p$ of any prime order p, in a manner similar to that seen in ([vWA]) for the descent algebra of the symmetric group. There we saw that for values of $p > n$, results for characteristic 0 extend virtually unchanged but, $p \leq n$ gives rise to a more complicated situation. We shall see a similar phenomenon in this paper for ΣB_n.

We shall work with a more convenient definition of ΣB_n (justified in [BB92]), in which ΣB_n is spanned by a basis $\{B_q\}$ indexed by compositions of m, where $m \leq n$, and multiplication is defined as follows.

Consider "templates" with the following form

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0t} \\ & b_{11} & b_{12} & \dots & b_{1t} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & b_{s1} & b_{s2} & \dots & b_{st} \\ a_{s0} & a_{s1} & a_{s2} & \dots & a_{st} \end{pmatrix}$$

where

1. $a_{00} = 0$.

2. All entries in a template are non-negative integers.

3. The sum of all entries in a template is no greater than n.

4. The b-lines do not have entries in column 0.

If $q = [q_1, q_2, \dots, q_s]$ is a composition of $k \leq n$, and $r = [r_1, r_2, \dots, r_s]$ is a composition of $m \leq n$, then we define $S(q, r)$ to be the set of templates, T, above, such that

1. $q_i = a_{i0} + \sum_{j=1}^{t}(b_{ij} + a_{ij})$

2. $r_j = a_{0j} + \sum_{i=1}^{s}(b_{ij} + a_{ij})$

Multiplication in ΣB_n is then define by

$$B_q B_r = \sum_{T \in S(q,r)} B_{[a_{01}, a_{02}, \dots, a_{0t}, b_{1k}, \dots, b_{12}, b_{11}, a_{10}, a_{11}, a_{12}, \dots, a_{1t}, \dots, a_{s0}, a_{s1}, a_{s2}, \dots, a_{st}]}$$

where the *reading word*

$[a_{01}, a_{02}, \ldots, a_{0t}, b_{1k}, \ldots, b_{12}, b_{11}, a_{10}, a_{11}, a_{12}, \ldots, a_{1t}, \ldots, a_{s0}, a_{s1}, a_{s2}, \ldots, a_{st}]$

is identified with the composition obtained by omitting zero components.

In order to study the case when the characteristic of the field is p, we define $\mathcal{Z}_n$ to be the subring of ΣB_n that consists of all integral combinations of the basis elements given before, and consider its ideal $\mathcal{P}_n = p\mathcal{Z}_n$. We define the *p-modular descent algebra of the hyperoctahedral group*, $\Sigma B(n,p)$, to be the quotient ring $\mathcal{Z}_n/\mathcal{P}_n$. This is indeed an algebra over a field, $\mathcal{F}_p$, of characteristic p.

Let $\rho_1 : \mathcal{Z}_n \to \Sigma B(n,p)$ be the natural homomorphism with kernel $\mathcal{P}_n$, and write $\overline{B}_q = \rho_1(B_q)$. The set of all $\{\overline{B}_q\}$ clearly forms a basis for $\Sigma B(n,p)$, and the multiplication rule for $\overline{B}_q\overline{B}_r$, is the same as for B_qB_r, except that coefficients are reduced modulo p.

Let $q = [q_1, q_2, \ldots, q_r]$ be a composition of $m \leq n$, and let $H_q = B_{n-m} \times S_{q_1} \times S_{q_2} \times \ldots \times S_{q_r}$. Let 1_q be the principal character of H_q, and let $\chi_q = 1_q^{B_n}$ be the permutation character of B_n induced by 1_q. Then the Z-module, G_n consisting of all integral combinations of $\{\chi_q\}$ has a ring structure, by the Mackey formula. Solomon proved that the linear map $\theta : \mathcal{Z}_n \to G_n$, defined by $\theta(B_q) = \chi_q$, for all compositions q, is a homomorphism of rings.

To extend these results to $\Sigma B(n,p)$, we let ρ_2 be the map defined on generalised characters in G_n, which reduces the character values modulo p, and let $G(n,p)$ denote the image of G_n under this map. The kernel of the map clearly contains $\mathcal{P}_n$, and hence induces an epimorphism of $\mathcal{F}_p$-algebras $\phi : \Sigma B(n,p) \to G(n,p)$ which satisfies

$$\phi(\rho_1(x)) = \rho_2\theta((x)) \ \forall x \in \Sigma B_n$$

from which it follows $\phi(\overline{B}_q) = \tilde{\chi_q}$, (where $\tilde{\chi_q} = \rho_2(\chi_q)$).

We define two relations on the set of compositions $m \leq n$:

If q and r are compositions of some $k \leq n$ which differ only in the order of their components then we write $q \approx r$. The relation $q \approx r$ is an equivalence relation on the compositions of $m \leq n$.

There is also a partial order relation on the set of compositions $m \leq n$. For compositions $r \vDash k \leq n$ and $q \vDash l, k \leq l \leq n$, we say $q \preccurlyeq r$ if the parts of r can be obtained from the components q by deleting components of q to give $q' \vDash k$ and replacing adjacent components of q' by their sum.

Definition 1 *Two templates with $j+1$ columns are said to be* column-row equivalent *if one can be obtained from the other by permuting the columns 1 to j, and exchanging pairs of template entries (a_{ik}, b_{ik}).*

Lemma 1 *Let B_q, B_r be basis elements of ΣB_n, and suppose that the composition r has k parts, and the number of components in it equal to i is denoted by t_i. Then*

1. *If the coefficient of B_s in B_qB_r is non-zero, $s \preccurlyeq q$.*

2. *The coefficient of B_q in B_qB_r is a multiple of $2^k t_1!t_2!\dots t_n!$, and is dependant only on the equivalence class of r. item If $q \approx r$, the coefficient of B_q in B_qB_r is exactly $2^k t_1!t_2!\dots t_n!$.*

PROOF Let $q = [q_1, q_2, \dots, q_l]$ and $r = [r_1, r_2, \dots, r_k]$. If B_s occurs in B_qB_r with non-zero multiplicity, then there exists a template whose reading word gives the composition s. However, since $a_{i0} + \sum_{j=1}^{k}(b_{ij} + a_{ij}) = q_i$, it follows that $s \preccurlyeq q$.

Now consider the template that contributes to the coefficient of B_q in B_qB_r. It satisfies

$$q_i = a_{i0} + \sum_{j=1}^{k}(b_{ij} + a_{ij}) \tag{1}$$

$$r_j = a_{0j} + \sum_{i=1}^{l}(b_{ij} + a_{ij}) \tag{2}$$

It follows that $a_{0j} = 0$ for all j, and that amongst the entries $a_{i0}, b_{i1}, a_{i1}, \dots, b_{ik}, a_{ik}$, there will be only one non-zero entry equal to q_i. The set of templates Q (if any) that satisfy this fall into a number of column-row equivalence classes and each of these has precisely $2^k t_1!t_2!\dots t_n!$ members since the set of columns $1, \dots, k$ of a given template in Q with a common column sum may be permuted arbitrarily, and a non-zero entry equal to q_i appearing in column j may appear in position a_{ij} or b_{ij} (note that the columns are distinct, so the $t_i!$ permutations of a set of t_i columns give distinct templates). If s is some composition equivalent to r then the set of matrices analogous to Q is related to Q by permuting the columns. This proves the second part.

When $q \approx r$ we see that Q consists of exactly one column-row equivalence class, since each template will have exactly one non-zero entry in each column $1, \dots, k$, and $a_{i0} = 0$ for all i. □

Note that the conclusions of the above Lemma also hold for basis elements in $\Sigma B(n,p)$, except that coefficients must be reduced modulo p.

1.2 On the form of the radical, and the irreducible representations of $\Sigma B(n,p)$.

Given a permutation Θ in disjoint cycle form, that contains no cycle with both positive and negative entries, we say that it has *positive* (resp. *negative*) *cycle type* θ, where θ is the permutation obtained by concatenating all cycles containing only positive (resp. negative) entries.

Lemma 2 *$G(n,p)$ has dimension $g(n,p)$ over $\mathcal{F}_p$ where*

$$\begin{aligned} g(n,p) &= \textstyle\sum_{m=0}^{n} pp(m) \quad p \neq 2 \\ &= 1 \quad\quad\quad\quad\quad\; p = 2 \end{aligned}$$

and $pp(m)$ is the number of partitions of m containing no part of multiplicity p or more.

PROOF For each composition $q, q \models m \leq n$ and partition π with k parts, $\pi \vdash l \leq n$, we let $m_{q\pi}$ be the value of the character $\tilde{\chi_q}$ on the conjugacy class of elements of B_n of positive cycle shape π, negative cycle shape $[n-l]$ (we can use this description, since we can choose such an element to represent the conjugacy class). Let M be the $2^n \times \sum_{i=0}^{n} p(i)$ matrix

$$[m_{q\pi}]$$

It follows that $\dim G(n,p) = \text{rank}M$.

Recall that if ϕ is a character of U, a subgroup of a group G, then

$$\phi^G(g) = |\mathcal{C}_G(g)| \sum_{g' \sim g} \frac{1}{|\mathcal{C}_U(g')|} \phi(g') \tag{3}$$

where the sum ranges over all representatives g' of conjugacy classes of U conjugate to g in G. Moreover, if $g = g_{n-m} g_{q_1} g_{q_2} \ldots g_{q_r}$, where $g_{n-m} \in B_{n-m}$, $g_{q_1} \in S_{q_1}$, $g_{q_2} \in S_{q_2} \ldots g_{q_r} \in S_{q_r}$, then

$$\chi_q(g) = |\mathcal{C}_{B_n}(g)| \sum_{g' \sim g} \frac{1}{|\mathcal{C}_{H_q}(g')|} \tag{4}$$

where

$$|\mathcal{C}_{H_q}(g)| = |\mathcal{C}_{B_{n-m}}(g_{n-m})| \times |\mathcal{C}_{S_{q_1}}(g_{q_1})| \times |\mathcal{C}_{S_{q_2}}(g_{q_2})| \times \ldots \times |\mathcal{C}_{S_{q_r}}(g_{q_r})| \tag{5}$$

$$|\mathcal{C}_{B_n}(g)| = (n-m) t_1! t_2! \ldots t_n! \prod_i (2i)^{t_i} \tag{6}$$

$$|\mathcal{C}_{S_n}(x)| = t_1! t_2! \ldots t_n! \prod_i i^{t_i} \tag{7}$$

where t_i is the multiplicity of i in the given permutation's (positive) cycle shape, and x is any permutation in S_n.

When $p \neq 2$, list the rows of M such that the first $p(0)$ rows are indexed by the $p(0)$ partitions of 0. The next $p(1)$ rows index by the $p(1)$ partitions of 1 and so on until the first $\sum_{m=0}^{n} p(m)$ rows have been indexed in this way. Consider the $\sum_{m=0}^{n} p(m) \times \sum_{m=0}^{n} p(m)$ sub-matrix N containing these rows. From equations 4 - 7 it follows that these rows can be further arranged to give a lower triangular sub-matrix with diagonal entries $m_{\pi\pi} = 2^k t_1! t_2! \ldots t_n!$ mod p. Hence there are $g(n,p)$ non-zero diagonal entries, and so $\text{rank}M \geq g(n,p)$.

Alternatively, if π_1, π_2 are the positive cycle shapes of two elements of B_n with the same $p-$regular part, then the columns of M corresponding to π_1 and π_2 are equal [CR62] hence $\text{rank}M \leq g(n,p)$.

When $p = 2$ by equations 4 - 7, it follows that the only non-zero permutation character in $G(n,p)$ will be $\tilde{\chi_0}$.

This completes the proof. □

Lemma 3 $\mathcal{R}(n,p) \subseteq \ker\phi$.

Proof Since the image of ϕ is a space of functions defined over a field, it is semi-simple. Hence the two-sided nilpotent ideal $\phi(\mathcal{R}(n,p))$ must be zero. $\square$

Theorem 1 $\Sigma B(n,p)/\mathcal{R}(n,p)$ *is commutative.*

Proof Since the radical of $\mathcal{Z}_n$, $\mathcal{R}_n$, is a nilpotent ideal, $\rho_1(\mathcal{R}_n)$ is a nilpotent ideal of $\Sigma B(n,p)$, and therefore $\rho_1(\mathcal{R}_n) \subseteq \mathcal{R}(n,p)$. Hence there exists an ideal $\mathcal{S}_n$ of ΣB_n, the pre-image of $\mathcal{R}(n,p)$, such that $\mathcal{R}_n \subseteq \mathcal{S}_n$ and $\mathcal{S}_n/\mathcal{P}_n \cong \mathcal{R}(n,p)$. Since $\Sigma B(n,p) \cong \mathcal{Z}_n/\mathcal{P}_n$, $\Sigma B(n,p)/\mathcal{R}(n,p) \cong \mathcal{Z}_n/\mathcal{S}_n$ is a homomorphic image of $\mathcal{Z}_n/\mathcal{R}_n \cong G_n$. Since the latter ring is commutative the theorem follows. $\square$

Lemma 4 *Let $\overline{B}_r$ be a basis element of $\Sigma B(n,p)$. If $p \neq 2$ then $\overline{B}_r$ is nilpotent if and only if r has a component of multiplicity p or more. If $p = 2$, $\overline{B}_r$ is nilpotent if and only if $r \neq 0$.*

Proof Suppose that r has t_i components equal to i, and k parts. Set

$$I_r = \langle \overline{B}_q | q \preceq r \rangle$$

By Lemma 1 I_r is a right ideal of $\Sigma B(n,p)$ and so right multiplication by $\overline{B}_r$ induces a linear transformation on I_r. We consider the matrix of this transformation with respect to the given basis $\overline{B}_{q_1} \dots \overline{B}_{q_w}$ of I_r ordered so that $q_i \preceq q_j$ implies $i \leq j$. This matrix is, by Lemma 1, lower triangular with diagonal elements all equal to a multiple of $2^k t_1! t_2! \dots t_n! \mod p$. Therefore the matrix is nilpotent if and only if at least one of the multiplicities t_i is p or more. If the matrix is not nilpotent then certainly $\overline{B}_r$ is not nilpotent. On the other hand, if the matrix is nilpotent then $I_r \overline{B}_r^t = 0$ for some t and so, as $\overline{B}_r \in I_r$, $\overline{B}_r^{t+1} = 0$.

By the above argument the result for $p = 2$ follows immediately. $\square$

Lemma 5 $\dim \mathcal{R}(n,p) \geq \dim \ker \phi$.

Proof $\Sigma B(n,p)/\mathcal{R}(n,p)$ is a commutative semi-simple algebra and so contains no non-zero nilpotent elements. Therefore all nilpotent elements of $\Sigma B(n,p)$ belong to $\mathcal{R}(n,p)$.Thus to obtain our result we must identify $\dim \ker \phi$ linearly independent nilpotent elements in $\mathcal{R}(n,p)$.

1. $p \neq 2$

   The elements $B_q - B_r$ with $q \approx r$ of ΣB_n lie in the radical of ΣB_n ([BB92]) and so are all nilpotent. Hence their images $\overline{B}_q - \overline{B}_r$ are also nilpotent; they span a subspace U of $\mathcal{R}(n,p)$ of dimension $2^n - \sum_{m=0}^{n} p(m)$.

   If q is a composition of $m \leq n$ with a component of multiplicity p or more then every composition r with $q \approx r$ also has this property. We choose a complete set A of inequivalent compositions with this property; clearly the members of A can be put in 1-1 correspondence with the set of *partitions* of $m \leq n$ which have a part of multiplicity p or more. However, it is known that the number of such partitions is the same as the number of partitions which have a part divisible by p ([JK81], p.41), and this number is $\sum_{m=0}^{n} p(m) - g(n,p)$.

Finally we note that $\{\overline{B}_q | q \in A\}$, a set of nilpotent elements, is contained in $\mathcal{R}(n,p)$ and is linearly independent of the subspace U. Therefore

$$\begin{aligned} \dim \mathcal{R}(n,p) &\geq 2^n - \sum_{m=0}^{n} p(m) + \sum_{m=0}^{n} p(m) - g(n,p) \\ &= 2^n - g(n,p) \\ &= \dim \Sigma B(n,p) - \dim G(n,p) \\ &= \dim \ker \phi \end{aligned}$$

2. $p = 2$

   If we choose a set B of distinct compositions of m, $0 < m \leq n$, then $\{\overline{B}_q | q \in B\}$ is a set of linearly independent nilpotent elements contained in $\mathcal{R}(n,2)$. Hence

$$\begin{aligned} \dim \mathcal{R}(n,2) &\geq 2^n - 1 \\ &= 2^n - g(n,2) \\ &= \dim \Sigma B(n,2) - \dim G(n,2) \\ &= \dim \ker \phi \end{aligned}$$

Our result now follows. □

We can now describe $\mathcal{R}(n,p)$ exactly.

Theorem 2 *$\mathcal{R}(n,p) = \ker \phi$. When $p \neq 2$ it is spanned by all $\overline{B}_q - \overline{B}_r$ with $q \approx r$ together with all $\overline{B}_q$ where q has a component of multiplicity p or more. However, when $p = 2$, it is spanned by all $\overline{B}_q$ with $q \neq 0$.*

PROOF $\mathcal{R}(n,p) = \ker \phi$ follows from Lemma 3 and Lemma 5. The proof of Lemma 5 then shows that when $p \neq 2$, $\mathcal{R}(n,p)$ not only contains but is actually spanned by all $\overline{B}_q - \overline{B}_r$ with $q \approx r$ together with all $\overline{B}_q$ where q has a component of multiplicity p or more, whereas when $p = 2$, it is spanned by all $\overline{B}_q$ with $q \neq 0$. □

It follows from Theorem 2 that $\dim \Sigma B(n,p)/\mathcal{R}(n,p) = g(n,p)$ and so, by Theorem 1, $\Sigma B(n,p)$ has $g(n,p)$ irreducible representations which are all 1-dimensional. We can describe them as follows.

Let π be any partition of $m \leq n$ and x any element of $\Sigma B(n,p)$. Then $\phi(x)$ is a p-modular character of B_n and we let $\phi(x)^\pi$ be the value of this character on the conjugacy class which corresponds to π. We define $\lambda_\pi : \Sigma B(n,p) \to \mathcal{F}_p$ by

$$\lambda_\pi(x) = \phi(x)^\pi \text{ for all } x \in \Sigma B(n,p)$$

Since ϕ is a homomorphism and characters of B_n are added and multiplied pointwise, it follows that λ_π is a (1-dimensional) representation of $\Sigma B(n,p)$.

λ_π is determined by its values $\phi(\overline{B}_q)^\pi = \tilde{\chi}^\pi$ on the basis of $\Sigma B(n,p)$ and, by ordering the basis, we may define a column vector D^π of these values. By the proof of Lemma 2 the matrix whose columns are the vectors D^π has rank $g(n,p)$. This lemma also shows that for $p \neq 2$ the set of partitions with no part of multiplicity p or more provides a suitable set of distinct columns that may be taken to define the $g(n,p)$ distinct irreducible representations of $\Sigma B(n,p)$, and that in the case $p = 2$, the empty partition is satisfactory for this.

1.3 On the Nilpotency index of the Radical

Using an idea similar to that seen in [vWA], we shall let Y_m be the subspace of $\Sigma B(n,p)$ spanned by all $\overline{B}_q$ where q has m or more components (for simplicity of notation we shall omit the reference to the dependency of Y_n on n and p). Then it follows

$$\Sigma B(n,p) = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_n \supseteq Y_{n+1} = 0$$

We also let $X(m,i,k,l)$ be the subspace of $\Sigma B(n,p)$ spanned by all $\overline{B}_q$ where q has $m+ik$ components such that there exist i disjoint component subsets, with k elements, each being in $1-1$ correspondence with some composition of j where $l \leq j \leq n$, under the natural mapping (again we omit the reference to the dependency on n and p).

Lemma 6 *1. If $p = 2$ then $Y_m\mathcal{R}(n,2) \subseteq Y_{m+1}$.*

2. If $p \neq 2$ then $Y_m\mathcal{R}(n,p) \subseteq Y_{m+3} + X(m,1,2,3)$.

Proof Let $s = [s_1, s_2, \ldots, s_r]$ be a composition with at least m components, so that $\overline{B}_s \in Y_m$. Consider the product $\overline{B}_s X$ for each of the spanning elements of $\mathcal{R}(n,p)$ given in Theorem 2. Such a product is, by Lemma 1, a linear combination of $\overline{B}_t$ with $t \preceq s$ but we shall now prove, the term $\overline{B}_s$ itself occurs with coefficient zero. There are two cases to consider:

1. $X = \overline{B}_q - \overline{B}_r, q \approx r$. By Lemma 1, the coefficients of $\overline{B}_s$ in both $\overline{B}_s\overline{B}_q$ and $\overline{B}_s\overline{B}_r$ are equal; thus, in $\overline{B}_s(\overline{B}_q - \overline{B}_r)$, the coefficient of $\overline{B}_s$ is zero.
2. $X = \overline{B}_r$ where r has t_i components equal to i with at least one t_i being p or more (unless $p = 2$), and k parts. By Lemma 1 since $2^k t_1! \ldots t_n!$ is zero in $\mathcal{F}_p$, irrespective of p, the coefficient of $\overline{B}_s$ in $\overline{B}_s\overline{B}_r$ is zero.

This proves the first part.

To prove part two, we must do some further analysis.

Consider again $\overline{B}_s X$. We have already shown that this is a linear combination of $\overline{B}_t, t \precnsim s$, such that $\overline{B}_s$ occurs with coefficient zero. We shall now show that all such $\overline{B}_t$ where $\overline{B}_t \in Y_{m+1} \cap \overline{B}_s X$ also has coefficient zero.

First we must note what sort of template would contribute to such a $\overline{B}_t$. Without loss of generality there are two to consider:

TEMPLATE 1 $a_{0j}=0, a_{i0}=0$ for all i, j, and there is one non zero entry in each pair of rows contributing to a given s_i, except one pair, which has $s_{i'_1}$ in position $b_{i'j'}$ and $s_{i'_2}$ in position $a_{i'j'}$ such that $s_{i'_1}+s_{i'_2}=s_{i'}$.

TEMPLATE 2 $a_{0j}=0$ for all j bar j' where $a_{0j'}=s_{i'}$ and $a_{i0}=0$ for all i bar i' where $a_{i'0}=s_{i'}$, and there is one non-zero entry in every other pair of rows contributing to a given s_i.

For each of these types of template we have two cases to consider:

1. $X=\overline{B}_q-\overline{B}_r, q\approx r$.
2. $X=\overline{B}_r$ where r has t_i components equal to i with at least one t_i being p or more.

Considering the four cases in turn in a way similar to that seen in the proof of part one, it follows easily that the coefficient of $\overline{B}_t$ in each of these cases is zero.

Hence we have that $Y_m\mathcal{R}(n,p)\subseteq Y_{m+2}$.

To finish this proof, we must show that if there exists a $\overline{B}_u\in Y_{m+2}\cap\overline{B}_sX$ then $\overline{B}_u\in X(m,1,2,3)$. This is easily done by considering the three templates that could contribute to the coefficient of $\overline{B}_u$, which (without loss of generality) can be described as follows:

1. The template resembles TEMPLATE 1, except that there are now two such pairs of rows such that $s_{i'_1}$ in position $b_{i'j'}$ and $s_{i'_2}$ in position $a_{i'j'}$ satisfy $s_{i'_1}+s_{i'_2}=s_{i'}$.
2. The template resembles TEMPLATE 2, except that there are now two non-zero entries a_{0j} and $a_{0j'}$ in row 0, and two non-zero entries $a_{i0}=a_{0j}$ and $a_{i'0}=a_{0j'}$ in column 0.
3. $a_{i0}=0$ for all i, and each pair of rows contributing to a given s_i have one non-zero entry, except two, which can be visualised as follows:

$$\begin{array}{ccccccccc} \dots & 0 & a_{ij} & 0 & \dots & 0 & a_{ij'} & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \\ \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & a_{i'j} & 0 & \dots & 0 & a_{i'j'} & 0 & \dots \end{array}$$

where $a_{ij}, a_{i'j}, a_{ij'}, a_{i'j'}\neq 0$, $a_{i'j}=a_{ij'}$, $a_{ij}+a_{ij'}=s_k$, $a_{i'j}+a_{i'j'}=s_l$.

Performing analysis used to prove that $Y_m\mathcal{R}(n,p)\subseteq Y_{m+2}$, it follows that contributions to the coefficient of $\overline{B}_u$ in $\overline{B}_sX$ by templates of type 1 or 3 will be zero. This will also be true of templates of type 2, unless entry $a_{0j}\neq$ entry $a_{0j'}$, in which case $\overline{B}_u\in X(m,1,2,3)$.

This completes the proof of the lemma. □

Let $\mathcal{T}$ denote the subspace of $\mathcal{R}(n,p)$ generated by all $\overline{B}_q-\overline{B}_r$ with $q\approx r$. Since $\mathcal{T}$ is the image of the radical of ΣB_n under the homomorphism ρ_1, $\mathcal{T}$ is a nilpotent ideal and is hence contained in $\mathcal{R}(n,p)$.

Lemma 7 *1. If $p = 2$ then $\mathcal{R}(n,2) \subseteq Y_1$.*

2. If $p \neq 2$ then $\mathcal{R}(n,p) \subseteq Y_2 \cap \mathcal{T} + Y_3$.

Proof Clearly the first part of the lemma holds since $\overline{B}_1 \in \mathcal{R}(n,2)$, but $\overline{B}_0 \notin \mathcal{R}(n,2)$. For the second part consider the spanning set for $\mathcal{R}(n,p)$ given in Theorem 2. An element $\overline{B}_q - \overline{B}_r$ with $q \approx r$ is non-zero only if q and r have at least 2 components and so such an element belongs to $Y_2 \cap \mathcal{T}$.

Consider an element $\overline{B}_q$ where the composition q has a component which occurs p or more times. q will have at least 3 components, and hence $\overline{B}_q \in Y_3$. The lemma now follows. □

With these Lemmas, we can now prove the following Theorem

Theorem 3 *1. If $p = 2$, the nilpotency index of $\mathcal{R}(n,2)$ is $n+1$.*

2. If $p \neq 2$, the nilpotency index of $\mathcal{R}(n,p)$ is $[\frac{n}{3}]+1$.

Proof A simple induction shows that in $\Sigma B(n,2)$, $\overline{B}_1^k = \overline{B}_{1^k}, k \leq n$. Hence $\overline{B}_1^n \neq 0$, and so the nilpotency index of $\mathcal{R}(n,2)$ is not less than $n+1$.

To prove it is no more than $n+1$, we see that from Lemma 6 and Lemma 7

$$\mathcal{R}(n,2)^{n+1} \subseteq Y_{n+1}$$

and since $Y_{n+1} = 0$ part one follows.

When $p \neq 2$, consider the strings 21 and 12, and consider the product $(21-12)^k$, where multiplication is given by concatenation of strings. When multiplied out, an easy induction shows that this product can be re-written as a sum $\sum_{i=1, a_i=\pm}^{2^k} a_i x_i$, where the 2^k x_i are all distinct. Let

$$\mathcal{B}((21-12)^k) = \sum_{x_i} a_i \overline{B}_{x_i}$$

A simple induction gives that $(\overline{B}_{21} - \overline{B}_{12})^k = \mathcal{B}((21-12)^k)$. Hence $(\overline{B}_{21} - \overline{B}_{12})^{[\frac{n}{3}]} \neq 0$, and so the nilpotency index of $\mathcal{R}(n,p)$ is no less than $[\frac{n}{3}]+1$. Again by Lemma 6 and Lemma 7, we have that

$$\begin{aligned}\mathcal{R}(n,p)^{[\frac{n}{3}]+1} \subseteq\ & Y_{3+3([\frac{n}{3}])} + Y_{2+3([\frac{n}{3}])} \cap \mathcal{T} \\ & + X(3,[\frac{n}{3}],2,3) + X(2,[\frac{n}{3}],2,3) \cap \mathcal{T}\end{aligned}$$

and since each of the terms on the right hand side is equal to zero, the Theorem is proved. □

2 Results on the p-modular Descent Algebra of $I_2(n)$

Abstract

Here follows, in note form, findings on the p-modular descent algebra of $I_2(n)$.

2.1 Introduction

Let $I_2(n)$ be the dihedral group with order $2n$, generators a,b, and relations

$$a^2 = b^2 = (ab)^n = 1.$$

If we denote the p-modular descent algebra of this by $\Phi(n,p)$, then due to results by [BBHT92] it can be verified that the only field characteristics that cause a change in structure are all proper factors of $2n$, and that if p satisfies this criteria then the results in the following sections hold.

2.1.1 Notation

$$\begin{aligned} 1 &= 1 \\ x_b &= 1 + a + ab + aba \dots \\ x_a &= 1 + b + ba + bab \dots \\ x_\emptyset &= \sum \textit{all elements} \end{aligned}$$

2.2 $\Phi(n,p)$, $p \neq 2$

The nature of these descent algebras fall into 2 categories, when n is even, and when n is odd.They are as follows.

2.2.1 n Even

2.2.2 Multiplication Table

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	$2x_b + (p-1)x_\emptyset$	0	0
x_a	x_a	0	$2x_a + (p-1)x_\emptyset$	0
$x_\emptyset$	$x_\emptyset$	0	0	0

2.2.3 Radical

Consists of $x_\emptyset$.

2.2.4 Idempotents

These are given by:

$$\begin{aligned} e_1 &= 1 + \frac{p-1}{2}(x_b + x_a) + \frac{p+1}{2}x_\emptyset \\ e_b &= \frac{p+1}{2}x_b + \frac{p-1}{4}x_\emptyset \\ e_a &= \frac{p+1}{2}x_a + \frac{p-1}{4}x_\emptyset \end{aligned}$$

2.2.5 Modular Structure

The principal indecomposable modules are structured as follows:

x_b	x_a	1
$i1 = 1$	$j1 = j$	$k1 = k$
$ix_b = i$	$jx_b = 0$	$kx_b = 0$
$ix_a = 0$	$jx_a = j$	$kx_a = 0$
$ix_\emptyset = 0$	$jx_\emptyset = 0$	$kx_\emptyset = 0$

(See structure at end)

2.2.6 n Odd

2.2.7 Multiplication Table

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	$x_b + \frac{p-1}{2}x_\emptyset$	$x_b + \frac{p-1}{2}x_\emptyset$	0
x_a	x_a	$x_a + \frac{p-1}{2}x_\emptyset$	$x_a + \frac{p-1}{2}x_\emptyset$	0
$x_\emptyset$	$x_\emptyset$	0	0	0

2.2.8 Radical

Consists of x_b- x_a, $x_\emptyset$.

2.2.9 Idempotents

These are given by:

$$\begin{aligned} e_1 &= 1 + (p-1)x_b + \frac{p+1}{2}x_\emptyset \\ e_b &= x_b + \frac{p-1}{2}x_\emptyset \end{aligned}$$

2.2.10 Modular Structure

The principal indecomposable modules are structured as follows:

x_b	1
$i1 = i$	$j1 = j$
$ix_b = i$	$jx_b = 0$
$ix_a = i$	$jx_a = 0$
$ix_\emptyset = 0$	$jx_\emptyset = 0$

(See structure at end)

2.3 $\Phi(n,p)$, $p = 2$

When $p = 2$, the results change somewhat, again due to the multiplication table of [BBHT92].But in this case we can derive some general results for the four sub-cases n even $n/2$ even,n even $n/2$ odd, n odd $n/2$ even, n odd $n/2$ odd.

2.3.1 n Even

2.3.2 Multiplication Table

n/2 even

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	$x_\emptyset$	0	0
x_a	x_a	0	$x_\emptyset$	0
$x_\emptyset$	$x_\emptyset$	0	0	0

n/2 odd

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	0	$x_\emptyset$	0
x_a	x_a	$x_\emptyset$	0	0
$x_\emptyset$	$x_\emptyset$	0	0	0

There two cases are not isomorphic, since the former contains a nilpotent element of order three, whereas the latter does not.

2.3.3 Radical

Consists of x_a, x_b, and $x_\emptyset$.

2.3.4 Idempotents

Due to the radical the only idempotent is 1.

2.3.5 Modular Structure

The principal indecomposable module is structured as follows

$$\frac{1}{\begin{array}{c} i1 = i \\ ix_b = 0 \\ ix_a = 0 \\ ix_\emptyset = 0 \end{array}}$$

(See structure at end)

2.3.6 n Odd

2.3.7 Multiplication Table

n/2 even

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	x_b	x_b	0
x_a	x_a	x_a	x_a	0
$x_\emptyset$	$x_\emptyset$	0	0	0

n/2 odd

	1	x_b	x_a	$x_\emptyset$
1	1	x_b	x_a	$x_\emptyset$
x_b	x_b	$x_b + x_\emptyset$	$x_b + x_\emptyset$	0
x_a	x_a	$x_a + x_\emptyset$	$x_a + x_\emptyset$	0
$x_\emptyset$	$x_\emptyset$	0	0	0

If we change the basis in the latter case to $\{1, x_b + x_\emptyset, x_a + x_\emptyset, x_\emptyset\}$, then the multiplication tables for each case are isomorphic, and so by linearity it follows that these two cases are isomorphic.

2.3.8 Radical

Consists of $x_b + x_a$, $x_\emptyset$.

2.3.9 Idempotents

n/2 even

These are given by:

$$\begin{aligned} e_1 &= 1 + x_b \\ e_b &= x_b \end{aligned}$$

and due to the isomorphic nature of these two cases it follows that for n/2 odd

These are given by:

$$
\begin{aligned}
e_1 &= 1 + x_b + x_\emptyset \\
e_b &= x_b + x_\emptyset
\end{aligned}
$$

2.3.10 Modular Structure

The principal indecomposable modules are structured as follows:

x_b	1
$i1 = i$	$j1 = j$
$ix_b = i$	$jx_b = 0$
$ix_a = i$	$jx_a = 0$
$ix_\emptyset = 0$	$jx_\emptyset = 0$

(See structure at end)

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